PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

Series 2.—VOL. 2.—Part 1.

CONTENTS.

Issued June 27, 1904.

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 Newton Street, High Holborn, W.C.;
and published for the society by
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. Love. W. Burnside.

34 St. Margaret's Road, Oxford.—A. E. H. Love.

The Croft, Bromley Road, Catford, S.E.—W. Burnside.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 2

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1905

LONDON:
PRINTED BY C. F. HODGSON & SON,
NEWTON STREET, HIGH HOLBORN, W.C.

CONTENTS.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1903–JUNE, 1904.

Thursday, November 12th, 1903.

ANNUAL GENERAL MEETING.

Prof. H. LAMB, President, in the Chair.

Present fifteen members and a visitor.

Mr. B. S. Wolfe was elected a member.

Miss C. I. Marks was admitted into the Society.

The Treasurer read his Report. On the motion of Mr. Dallas, seconded by Mr. C. S. Jackson, the Treasurer's Report was received.

Dr. J. G. Leathem was appointed Auditor.

The Secretaries reported that during the year the Society had lost by death two honorary members and six members, and that the number of members (260) at the beginning of this Session was the same as that at the beginning of the previous Session. During the year the Upsala Society of Sciences had been added to the list of Societies with which the Society exchanges publications.

The President referred to the loss sustained by the Society, and by mathematicians in general, by the death of Luigi Cremona, and gave an account of his scientific work. He also referred to the loss sustained by the Society by the early death of Augustus Perronet Thompson.

The Council and Officers for the ensuing Session were elected. They are as follows: — President, Prof. H. Lamb; Vice-Presidents, Prof. E. B. Elliott, Dr. E. W. Hobson, Dr. H. F. Baker; Treasurer, Prof. J. Larmor; Secretaries, Prof. A. E. H. Love and Prof. W. Burnside; other members of Council, Mr. J. E. Campbell, Dr. J. W. L. Glaisher, Mr. J. H. Grace, Mr. H. M. Macdonald, Major P. A. MacMahon, Mr. G. B. Mathews, Mr. A. E. Western, Mr. E. T. Whittaker, Mr. A. Young.

The following papers were communicated :—

Note on Borgnet's Method of Dividing an Angle in an Arbitrary Ratio: Prof. J. D. Everett.

The Propagation of Wave-motion in an Isotropic Elastic Solid Medium: Prof. A. E. H. Love.*

On Spherical Curves: Mr. H. Hilton.*

On Sequences of Sets of Intervals containing a given Set of Points: Mr. W. H. Young.*

A Formal Generalization of Maclaurin's Theorem: Rev. F. H. Jackson.*

On the Weddle Quartic Surface: Dr. H. F. Baker.*

The Theory of Diffraction: Mr. W. H. Jackson.*

A General Theorem concerning Absolutely Convergent Series: Mr. G. H. Hardy.*

Notes on Quaternions, including a Geometrical Interpretation of $V\alpha\beta\gamma$: Prof. R. W. Genese.

On the Expression of the Electro-magnetic Field by means of Two Scalar Potential Functions: Mr. E. T. Whittaker.*

Analogue of the Jordan Lemma for Four Variables: Mr. P. W. Wood.*

Thursday, December 10*th*, 1903.

Prof. H. LAMB, President, in the Chair.

Fourteen members present.

The following were elected members:—Miss A. E. Bennett, Rev. M. F. Egan, Major Close, Messrs. W. H. Jackson, T. H. Havelock, H. Bateman, Z. U. Ahmad.

The President referred to the loss sustained by the Society by the death of Mr. G. H. Stuart.

The following papers were communicated :—

Proof of a Formula in Elliptic Functions: Mr. R. G. Dallas.

Modes of Convergence of an Infinite Series of Functions of a Real Variable: Dr. E. W. Hobson.*

Many-valued Newtonian Potentials: Prof. A. C. Dixon.*

A Generalization of Neumann's Expansion of an Arbitrary Function in a Series of Bessel's Functions: Rev. F. H. Jackson.*

On Normal and Anti-Normal Piling: Prof. J. D. Everett.*

---

* Printed in *Proceedings*, Ser. 2, Vol. 1.

On the Distribution of Points of Uniform Convergence of a Series of Functions: Mr. W. H. Young.*

On Functions all of whose Singularities are Non-essential: Mr. P. E. B. Jourdain.

Lt.-Col. A. Cunningham announced a property of Fermat's numbers.

Thursday, January 14th, 1904.

Dr. E. W. HOBSON, Vice-President, in the Chair.

Present nineteen members and a visitor.

Mr. W. M. Roberts was elected a member.

Miss A. E. Bennett was admitted into the Society.

The following papers were communicated:—

On Various Systems of Piling: Prof. J. D. Everett.*

Electric Radiation from Conductors: Mr. H. M. Macdonald.*

The Notion of Lines of Curvature in the Theory of Surfaces: Dr. G. Prasad.

Groups of Order $p^{\alpha}q^{\beta}$: Prof. W. Burnside.

The Solution of Partial Differential Equations by means of Definite Integrals: Mr. H. Bateman.*

Open Sets and the Theory of Content: Dr. W. H. Young.†

Upper and Lower Integration: Dr. W. H. Young.†

List of Primes of the form $4n+1$ between 10^8 and 10^8+10^3: Dr. T. B. Sprague.

The Treasurer presented the Auditor's Report. On the motion of Mr. Dallas, seconded by Lt.-Col. A. Cunningham, the Report of the Treasurer, read in November, 1903, and the Report of the Auditor were adopted, and the thanks of the Society were given to the Treasurer and Auditor.

Thursday, February 11th, 1904.

Prof. H. LAMB, President, in the Chair.

Eighteen members present.

Messrs. R. C. Maclaurin, E. M. Radford, P. W. Wood, J. W. Sharpe, were elected members.

Mr. Z. U. Ahmad was admitted into the Society.

---

* Printed in *Proceedings*, Ser. 2, Vol. 1.

† Printed in *Proceedings*, Ser. 2, Vol. 2.

The President referred to the loss sustained by the Society and by mathematicians in general by the death of Dr. Salmon, and gave an account of his scientific work.

The following papers were communicated:—

On the Roots of the Equation $\frac{1}{\Gamma(x+1)} \doteq c$: Mr. G. H. Hardy.†

Some Extensions of Abel's Theorem on Power Series on the Circle of Convergence: Mr. G. H. Hardy.†

On Group-Velocity: Prof. H. Lamb.*

On a certain Double Integral: Prof. A. C. Dixon.†

On an Appropriate Form of Conductor for a Moving Point Singularity: Prof. A. W. Conway.

On the Irreducibility of Perpetuant Types: Mr. P. W. Wood.*

The Expression of $\int_0^\infty \operatorname{cn} xt\, e^{-t}\, dt$ and other Integrals by means of Continued Fractions: Prof. L. J. Rogers.

Thursday, March 10th, 1904.

Dr. E. W. HOBSON, Vice-President, and, temporarily, Prof. ELLIOTT, Vice-President, in the Chair.

Fourteen members present.

Mr. S. T. Shovelton was elected a member.

The following papers were communicated:—

On Inner Limiting Sets of Points in a Linear Interval: Dr. E. W. Hobson.†

Illustrations of Modes of Decay of Vibratory Motions: Prof. A. E. H. Love.†

The Unique Expression of a Quantic of any Order in any Number of Variables, with an application to Binary Perpetuants: Mr. P. W. Wood.†

The Derivation of Generalized Bessel Coefficients from a Function analogous to the Exponential: Rev. F. H. Jackson.†

Transformation of Generalized Legendre Functions: Rev. F. H. Jackson.†

Singularities of Functions determined by Taylor's Series: Mr. H. M. Macdonald.

---

* Printed in *Proceedings*, Ser. 2, Vol. 1.

† Printed in *Proceedings*, Ser. 2, Vol. 2.

Thursday, April 14th, 1904.

Dr. E. W. HOBSON, Vice-President, in the Chair.

Twelve members present.

The following papers were communicated :—

On a Plane Quintic Curve : Prof. F. Morley.*

Note on a System of Linear Congruences : Rev. J. Cullen.*

The Tile Theorem : Dr. W. H. Young.*

Note in addition to a Former Paper on Conditionally Convergent Multiple Series : Mr. G. H. Hardy.*

On Functions generated by Linear Difference Equations of the First Order : Rev. E. W. Barnes.*

Mathematical Analysis of Wave-Propagation in Isotropic Space of p Dimensions : Mr. T. H. Havelock.*

On Spherical Curves—Part II. : Mr. H. Hilton.*

Perpetuant Syzygies of Degree Four : Mr. P. W. Wood.*

Extension of Sylow's Theorem : Prof. G. A. Miller.*

Transformation of the Function $F([a][\beta][\gamma]x)$, and the Extension of Neumann's Addition Theorem for Bessel Functions : Rev. F. H. Jackson.*

Informal communications were made as follows :—

The Singularities of Functions determined by Taylor's Series : Mr. H. M. Macdonald.

Behaviour of a Power Series near a Point on the Circle of Convergence at which the Series diverges : Dr. H. F. Baker.

Transvectant Operators in connexion with Binary Forms : Mr. R. J. Dallas.

Factorization of $13^{39}-1$: Lt.-Col. A. Cunningham.

Thursday, May 12th, 1904.

Dr. E. W. HOBSON, Vice-President, in the Chair.

Seventeen members present.

Mr. G. Birtwhistle was admitted into the Society.

The following papers were communicated :—

On Perpetuant Syzygies : Messrs. A. Young and P. W. Wood.*

On the Evaluation of certain Definite Integrals by means of Gamma Functions, and Generalizations of Legendre's Formula $KE'-(K-E)K' = \frac{1}{2}\pi$: Mr. A. L. Dixon.

---

* Printed in *Proceedings*, Ser. 2, Vol. 2.

Note on the Integration of Linear Differential Equations: Dr. H. F. Baker.*

Some Properties of the Function Γ_p: Rev. F. H. Jackson.

Informal communications were made as follows :—

The Geometrical Representation of Imaginary Points: Mr. G. B. Mathews.

A Collation of Kessler's and Hertzer's Tables of the Residue Index with Shanks' Table of the *Hauptexponent*: Lt.-Col. A. Cunningham.

Thursday, June 9th, 1904.

Prof. H. LAMB, President, in the Chair.

Fourteen members present.

The President referred to the death, on May 30th, of Thomas Savage, who was Second Wrangler and First Smith's Prizeman in Cambridge in 1857, and a Fellow of Pembroke College, Cambridge. He had been elected a member of the Society in June, 1865, the year of its foundation.

The following papers were communicated :—

Note on the Application of Poisson's Formula to Discontinuous Disturbances: Lord Rayleigh.*

Wave Fronts considered as the Characteristics of Partial Differential Equations: Mr. T. H. Havelock.*

Illustrations of Perpetuants: Mr. J. H. Grace.†

Types of Covariants of any Degree in the Coefficients of each of any Number of Binary Quantics: Mr. P. W. Wood.*

Some Expansions for the Periods of the Jacobian Elliptic Functions: Mr. H. Bateman.

---

* Printed in *Proceedings*, Ser. 2, Vol. 2.

† See p. 478, footnote.

LIBRARY

Presents.

BETWEEN June, 1903, and January, 1905, the following presents were made to the Library :—

From the Royal Society of Sciences of Upsala :—

"Arsskrift K. Vetenskaps-Soc.," Vols. I., II., 1860–1.

"Essai sur la Société, en mémoire du 400e Anniversaire de l'Université Royale d'Upsal," 1877.

And numerous Dissertations presented to the University of Upsala and to other Universities.

From the late R. Tucker, Esq. :—

H. J. S. Smith.—"Report on the Theory of Numbers" (British Association, 1859–1865).

J. J. Sylvester.—"The Syzygetic Relations of two Rational Integral Functions" (extract from *Phil. Trans.*, 1853).

An Adams Prize Essay for 1870.

From the Académie Polytechnique de Porto :—

Teixeira, F. Gomes.—"Obras sobre Mathematica," Vol. I. ; Coimbra, 1904.

From Thomas Colby, Esq. :—

Briggs and Gellibrand.—"Trigonometria Britannica ; sive de Doctrina Triangulorum libri duo" ; Gouda, 1633. [Containing Briggs' Tables of the Logarithms of Trignometrical Functions to 14 Places of Decimals.]

"Apollonii Pergæi Conicorum libri IV., cum Commentariis R. P. Claudii Richardi" ; Antwerp, 1655.

Taylor, Michael.—"Tables" ; London, 1780.

Faraday, M.—"Bakerian Lecture," 1829.

Cater, Capt. H.—"New Standards of Weights and Measures," 1826.

Airy, G. B.—"Figure of the Earth," 1830.

Hamilton, W. R.—"Second and Third Supplements on the Theory of Systems of Rays" (*Phil. Trans.*, 1830, 1833).

Babbage, C.—"The application of Machinery to the Calculation and Printing of Tables" ; London, 1822.

Gompertz, B.—"Hints on Porisms" ; London, 1850.

"Méchanique Céleste," Books XIII., XIV., 1824.

And eight other pamphlets.

From the Helwingsche Verlagsbuchhandlung :—

Kiepert, L.—"Grundriss der Differential- und Integral-Rechnung," Theil II. ; Hannover, 1903.

Cunningham, Allan.—"Quadratic Partitions"; London, 1904.
Grace, J. H., and A. Young.—"Algebra of Invariants"; Cambridge, 1903.
Whittaker, E. T.—"A Course of Modern Analysis"; Cambridge, 1902.
Sprague, T. B.—"The Singular Points of Plane Curves."
Ball, Sir R.—"The Reflection of Screw Systems."
Teixeira, F. G.—"La Convergence des Formules d'Interpolation."
"Scientia," Nos. 22, 23, edit. C. Naud; Paris, 1903.
South African Association for the Advancement of Science—Report, 1903.
Muir, T.—"Third List of Writings on Determinants."
Klein, F.—"Uber Umgestaltung des Math.-Unterrichts"; Leipzig, 1904.
Peirce, C. S.—"The Century's Great Men of Science"; Washington, 1901.
Bashforth, F.—"Historical Sketch of the Resistance of the Air"; Cambridge, 1903.
Mukerjee, C.—"Elementary Algebra," Pt. 1; Allahabad, 1903.
"Trigonometrical Survey of India," Vol. XVII.; Dehra Dun, 1901.
Lorenz, L.—"Œuvres Scientifiques," Tome II., Fasc. 2, 1904.
Riecke, E.—"Zur Frage des Unterrichts in Physik"; Leipzig, 1904.
Brioschi, F.—"Opere Matematiche," Tomo III.; Milan, 1904.
Anthony, E.—"Decimal Coinage and the Metric System"; London, 1904.
Mittag-Leffler, G.—"La Réprésentation Analytique d'une Branche Uniforme" (*Acta Math.*, 1904).
Polignac, Prince C. de.—"Elements connected Each to Each" (*Amer. Jour. of Math.*, 1904).
Darboux, G.—"Le Développement des Méthodes Géométriques," 1904.
Fubini, G.—"Il Parallelismo di Clifford," Pisa, 1900; and "Teoria delle Funzioni Armoniche," Pisa, 1902.
Grassi, U.—"Studii d'Idrodinamica"; Pisa, 1902.
Vitali, G.—"Equazioni Differenziali Lineari Omogenee"; Pisa, 1903.
Educational Times, vol. 56, nos. 507–512, 1903, and vol. 57, nos. 513–524, 1904.
Educational Times, Math. Questions and Solutions, vol. 4, 1903, and vols. 5 and 6, 1904.
Hamburg: Math. Gesellschaft Mittheilungen, bd. 4, heft 4, 1904.
Indian Engineering, vol. 33, nos. 21–26, 1903, vol. 34, 1903, vol. 35, 1904, and vol. 36, 1904.
Kansas: Univ. Science Bulletin, vol. 1, nos. 10–12, 1902, and vol. 2, nos. 1–15, 1903–4.
Mathematical Gazette, vol. 2, nos. 40–42, 1903, and vol. 3, nos. 43–49, 1904–5.
Nautical Almanac for 1907 (presented by the Admiralty).
Paris: L'Enseignement Math., ann. 5, nos. 4–6, 1903, and ann. 6., nos. 1–6, 1904.
Tokoyo: Sūgaku-Buturigakkwai, vol. 1, nos. 16–20, and vol. 2, nos. 1–12, 1904.
Tokoyo: College of Science Journal," vol. 19, no. 5, 1903.
Warsaw: Wiadomosci Matem., tom 7, zeszyt 3–6, 1904, and tom 8, zeszyt 1–3, 1904.

Exchanges.

Between June, 1903, and January, 1905, the following exchanges were received:—

American Journal of Mathematics, vol. 25, nos. 3, 4, 1903, and vol. 26, nos. 1, 2, 4, 1904.
American Mathematical Society, Transactions, vol. 4, nos. 3, 4, 1903, and vol. 5, 1904.
American Mathematical Society, Bulletin, vol. 9, no. 10, 1903; vol. 10, 1904; vol. 11, nos. 1–3, 1904; and General Index, 1891–1904, of the Bulletin.
American Philosophical Society, Proceedings, vol. 42, nos. 172–177, 1904.

Amsterdam : Nieuw Archiev, deel 6, stuk 1-3, 1903-4.
Amsterdam : Revue Semestrielle, tome 11, pt. 2, 1903, and tome 12, pts. 1, 2, 1904.
Amsterdam : Wiskundige Opgaven, deel 9, stuk 1, 2, 1904.
Belgique : Académie Royale des Sciences, Annuaire, 1904.
Belgique : Académie Royale des Sciences, Bulletin, 1903, nos. 5-12, and 1904, nos. 1-11.
Berlin : Jahrbuch über die Fortschritte, bd. 32, 1903, and bd. 33, hefte 1, 2, 1904.
Berlin : Journal für die Mathematik, bd. 126, hefte 2-4, 1903 ; bd. 127, heft 2, and bd. 128, 128, hefte 1, 2, 1904.
Berlin : Sitzungsberichte der K. Preuss. Akademie, 1903, nos. 25-53, and 1904, nos. 1-40.
Bordeaux : Société des Sciences, Mémoires, tome 2, cah. 1, 1903, and tome 3.
Bordeaux : Société des Sciences, Procès-Verbaux, 1902 and 1903.
Bordeaux : Société des Sciences, Observations Pluviométriques, 1902, 1903.
Cambridge Philosophical Society, Proceedings, vol. 12, pts. 3-6, 1903-4.
Cambridge Philosophical Society, Transactions, vol. 19, pt. 3, 1904.
Cambridge, Mass. : Annals of Mathematics, vol. 4, no. 4, 1903, vol. 5 and vol. 6, no. 1, 1904.
Canadian Institute, Transactions, no. 15, 1904.
Canadian Institute, Proceedings, no. 12, 1904.
Coimbra : Jornal de Sciencias Mathematicas, vol. 15, nos. 3, 4, 1903-4.
Connecticut Academy, Transactions, vol. 11, pt. 2, 1903.
Deutschen Math.-Vereinigung, Jahresbericht, bd. 11, hefte 1-4, 1902.
Deutschen Math.-Vereinigung, Geschichte, 1904.
Dublin : Royal Irish Academy, Section A, Proceedings, vol. 24, nos. 2, 4, 1903, and vol. 25, nos. 1, 2.
Dublin : Royal Irish Academy, Transactions, vol. 32, pts. 6-10, 1903-4.
Edinburgh : Mathematical Society, Proceedings, vol. 20, 1902.
Edinburgh : Royal Society, Proceedings, vol. 23, 1902, and vol. 24, nos. 1-3, 1902.
Erlangen : Physik.-medicin. Societät, Sitzungsberichte, heft 35, 1904.
France : Société Mathématique, Bulletin, tome 31, fasc. 2-4, 1903, and tome 32, fasc. 1-3, 1904.
Göttingen : Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1903, hefte 3-6, and 1904, hefte 1-5.
Göttingen : Königl. Gesell. der Wissenschaften, Mittheilungen, 1903, hefte 1, 2, and 1904, heft 1.
La Haye : Archives Néerlandaises, tome 8, liv. 3-5, 1903, and tome 9, liv. 1-5, 1904.
Leipzig : Beiblätter zu den Annalen der Physik, bd. 27, hefte 7-12, 1903, and bd. 28, hefte 1-24, 1904.
Leipzig : K. Sächsische Gesell., Math. Klasse, Berichte, 1902, nos, 6, 7, 1903, nos. 1-6, and 1904, nos. 1-3.
Leipzig : K. Sächsische Gesell., Math. Klasse, Abhandlungen, bd. 28, nos. 1-7, 1904, and bd. 29, nos. 1, 2, 1904.
Livorno : Periodico di Matematica, anno 18, fasc. 6, 1903, anno 19, 1904, and anno 20, fasc. 1-3, 1904.
Livorno : Periodico di Matematica, Supplemento, anno 6, fasc. 8, 9, 1903, anno 7, fasc. 1-9, 1904, and anno 8, fasc. 1, 2, 1904.
Lombardo : Reale Istituto—Rendiconti, vol. 36, fasc. 9-20, 1903, and vol. 37, fasc. 1-16, 1904.
Lombardo : Reale Istituto—Memorie, vol. 19, fasc. 9-13, 1903, and vol. 20, fasc. 1, 2, 1904.
London : Royal Society, Proceedings, nos. 476-502, 1903-4.
London : Royal Society, Obituary Notices, pts. 1-3, 1904.
London : Royal Society, Transactions, Series A, vol. 199, 1902, vols. 200 and 201, 1903, and vols. 202, 203, 1904.
London : Physical Society, Proceedings, vol. 18, pts. 5-7, 1903, and vol. 19, pts. 1-4, 1904.
London : Institute of Naval Architects, Transactions, vol. 44, 1902.

London: Institute of Actuaries, Journal, vol. 37, pts. 1–4, 1902–3.

Manchester Literary and Philosophical Society, Memoirs, vol. 46, pts. 5, 6, 1902, and vol. 47, pts. 1, 2, 1903.

Marseille: Annales de la Faculté des Sciences, tome 12, 1902, and tome 13, 1903.

Milano: Annali di Matematica, tomo 8, fasc. 4, 1903, and tomo 9, 1904.

Modena: Regia Accademia, Memorie, vol. 4, 1902.

Napoli: Accademia delle Scienze—Rendiconti, vol. 9, fasc. 5–12, 1903, and vol. 10, fasc. 1–7, 1904.

National Physical Laboratory, Reports, 1901, 1902, and 1903.

Nature, vol. 68, nos. 1754–1775, 1903; vol. 69, 1904; vol. 70, 1904; and vol. 71 to 5 Jan. 1905.

Odessa: Société des Naturalistes, tome 20, 1902, and tome 25, pts. 1, 2, 1903–4.

Palermo: Rendiconti del Circolo Matematico, tomo 17, fasc. 4–6, 1903, and tomo 18, fasc. 1–6, 1904.

Paris: Bulletin des Sciences Mathématiques, tome 27, Juin–Déc., 1903, and tome 28, 1904.

Paris: Journal de l'École Polyt., cah. 8, 1903, and cah. 9, 1904.

Pisa: Annali della R. Scuola Normale, vols. 1–9, 1871–1904.

Roma: Reale Accademia dei Lincei—Rendiconti, vol. 12, sem. 1, fasc. 10–12, 1903; and sem. 2, fasc. 1–12, 1903; vol. 13, sem. 1, fasc. 1–12, and sem. 2, fasc. 1–11, 1904.

Roma: Reale Accademia dei Lincei—Rendiconto dell' Adunanza Solenne, vol. 2, 1904.

Stockholm: Acta Mathematica, bd. 28, and bd. 29, pt. 1, 1904.

Toulouse: Faculté des Sciences, Annales, tome 5, 1903, and tome 6, fasc. 1, 2, 1904.

United States Naval Observatory, Publications, vols. 3, 5, 1903.

Upsala: Nova Acta R. Societatis Scient., series 2, parts of vol. 2, 1775, and of vol. 4, 1784, to vol. 14, 1850; and volume extraordinaire, 1877. And series 3, vol. 1, 1855, to Vol. 18, fasc. 2, 1900, and vol. 20, fasc. 2, 1904.

Venezia: Atti del R. Istituto, tomo 61, disp. 10, 1902, and tomo 62, 1903.

Warsaw: Prace Matematyczno-Fizyczne, tome 15, 1904.

Wien: Monatshefte für Mathematik, jahr. 14, viertel. 4, 1903, and jahr. 15, 1904.

Zurich: Vierteljahrsschrift, 1902, hefte 3, 4, 1903, and 1904, hefte 1, 2.

International Catalogue of Scientific Literature.

In the year April, 1903, to March, 1904 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature :—

"Proceedings of the Edinburgh Mathematical Society," Vol. XXI., 1903.

"Proceedings of the Royal Society of Edinburgh," Vol. XXIV., Nos. 4–6, and Vol. XXV., No. 1, 1903–4.

"Transactions of the Institute of Naval Architects," London, 1903.

"Journal of the Institute of Actuaries," Vol. XXXVIII.; London, 1903–4.

"Proceedings of the Manchester Literary and Philosophical Society," Vol. XLVII., Pts. 3–6, and Vol. XLVIII., Pt. 1, 1903.

The following were also sent especially for the purposes of the Catalogue :—

"Mathematical Gazette," Nos. 38–43 ; London, 1903–4.

"Educational Times," Nos. 505–515 ; London, 1903–4.

"Journal of the Royal Statistical Society," Vol. XLVI., Pts. 1–4 ; London, 1903.

"Transactions of the Royal Society of Edinburgh," Vol. XL., Pt. 3, 1903.

"Transactions of the Insurance and Actuarial Society of Glasgow," Series 5, Nos. 11–14, 1903.

OBITUARY NOTICES

R. W. H. T. HUDSON

[For this notice the Council is indebted to Mr. J. F. Cameron.]

Ronald William Henry Turnbull Hudson belonged to a mathematical family which had many ties with Cambridge, and from his earliest days he looked forward to the Mathematical Tripos as lying before him as a matter of course. He gave the surest signs of great mathematical ability, and while still at St. Paul's School he was looked on as an inevitable Senior Wrangler. From the day when he did phenomenal entrance scholarship papers onwards, his academic career was one of unbroken success. He was Senior Wrangler in 1898, and, after taking the Second Part of his Tripos in 1899, he gained a Smith's Prize for an essay on differential equations in 1900, and was elected a Fellow of St. John's—his own and his father's college—in the same year.

He was always devoted to mathematics, but it was with renewed enthusiasm that he read for his Second Part, and the following years saw a rapid increase both in his mathematical knowledge and in the manifestations of his ability. From this time onward his work was productive, and, though only a few years of life remained, he contributed articles* to the *Quarterly Journal*, the *Messenger of Mathematics*, and the *American Bulletin*, as well as to the *Proceedings* of this Society; and he was the author of many notices and reviews of books in the *Mathematical Gazette* and in *Nature*. One other paper, on "The Geometry of Rotation about an Axis," was published in the *Mineralogical Magazine*, and was afterwards translated and printed in the *Zeitschrift für Krystallographie*. All his original papers, though differing in importance, were alike characterised by the neatness and finish of his methods. His mind was particularly fond of a geometrical investigation, and it always pleased him to give a geometrical interpretation to an analytical result—a tendency which was probably largely fostered by

---

* A list of his papers, with references, will be found in the *Mathematical Gazette*, Vol. III., p. 74.

the careful study he made of Darboux's *Théorie des Surfaces* during his fourth year of residence at Cambridge.

His most serious contribution to mathematics was, however, the *Treatise on Kummer's Quartic Surface* which occupied the last two years of his life. The manuscript was fortunately finished before his untimely end, and it is now passing through the Cambridge University Press under the care of Dr. Baker. In this book he discusses line geometry, the theory of surfaces in general, as well as Kummer's surface more particularly, and the last few chapters are devoted to Abelian functions. The book is brief, perhaps too brief, but it is written with a very broad knowledge and with the widest sympathy with all branches of pure mathematics. Again his love of geometry is apparent, and perhaps the most valuable portions of the book are those in which he succeeds by geometrical methods in establishing results which had previously depended on higher analysis.

He was interested in model-making and might be said to have made it a hobby. In his book there will be found drawings of wire models of Kummer's surface which he had himself made, and the exhibition of models at the recent meeting of the British Association in Cambridge was due to his initiative as secretary.

He joined the University staff at Liverpool in 1902, and it was during term time there, and the vacations, mostly spent in Cambridge, that his book was written. His life in Liverpool and his work there were a great pleasure to him, and he threw himself into his teaching with his usual enthusiasm, and success followed it accordingly.

His life was a short one, but a full one; and the terrible accident on the Welsh hills, which cut him off at the age of twenty-eight in the midst of his early activities, left his friends with a sense of the great things which might have lain before him. His zeal for mathematics was great, and he entered with energy into all movements for its advancement. At the time of his death he had a full programme of work before him: he had already begun a text-book on analytical geometry and he was thinking over the possibility of another on pure geometry, and there is evidence among his papers that he might have competed for the Italian prize offered for a discussion of "Curves in Space."

But no sketch of Ronald Hudson can omit his other interests. His mind was not that of the specialist merely, and to everything he undertook he devoted the same energy which characterised his work. Indeed, his chief characteristic was his many-sided mental activity. He was a serious student of music, and he played several games with

skill. In his private life his tastes were simple. He could make friends and enter into their interests easily, and to his friends he was ever loyal and true. He was perfectly straightforward and open and he would express his opinions frankly, never withholding praise where praise was due. Once, when discussing a literary article in one of the magazines, he said that he often wondered at the excellence with which other people could do their own work. He was not fully conscious of it, but other people wondered at the excellence with which he did his.

GEORGE PIRIE.

[For this notice the Council is indebted to Prof. W. L. Davidson.]

PROFESSOR PIRIE, who died suddenly at Braemar on the 21st August, 1904, was born at the Manse of Dyce, Aberdeenshire, on the 19th July, 1843. He was the eldest son of the Very Reverend Principal Pirie, D.D., whose name will be ever associated with the abolition of patronage in the Church of Scotland. His early education was received at the Grammar School of Aberdeen, and afterwards at the University. His University course coincided with the fusion of the two Universities—King's and Marischal—into one; so that, although he began as a student of Marischal College, he completed his curriculum in the combined University. He thus just missed having Clerk Maxwell as his teacher in natural philosophy (for Maxwell ceased being Professor at Marischal College at the union, in 1860); but he was a student in Bain's first class of logic. His Arts course was a very distinguished one, and he graduated M.A., with highest Honours in Mathematics, in 1862; carrying off also the Natural Science Prize. From Aberdeen he proceeded to Cambridge, where he continued his studies in Queens' College, and, in due time, achieved high distinction, being Fifth Wrangler in the Mathematical Tripos in 1866. Thereafter, he was elected a Fellow of Queens' College, and discharged the duties of Mathematical Lecturer and Tutor. There he wrote his two scientific works, *Lessons on Rigid Dynamics* and *A Short Account of the Principal Geometrical Methods of Approximating to the Value of* π; one in 1875, and the other in 1877. These may be taken as exactly expressive of his distinctive qualities. On the one hand, they show him eager to render natural philosophy and the higher mathematics as comprehensible as possible, having ever in view their practical applications, and, on the other hand, they give evidence of his intense appreciation of the historical side of mathematical problems and his delight in tracing their evolution, and in according to each great name in the history of the subject its due. These characteristics marked him off as essentially a teacher in mathematical science. Consequently, when, in 1878, a vacancy occurred, through the resignation of Professor Fuller, in the Chair of Mathematics in Aberdeen University, the University Court elected him to the position, thus necessitating his return to the city of his early education, where he remained energetically doing his work till his

life ended. Outside recognition of his merits came when, a few years ago, St. Andrews University conferred upon him the honorary degree of LL.D.

Of Professor Pirie as a man it is easy to write. If manliness consists in nobility of character, devotion to friends, and strenuous regard to duty, few have been manlier than he. His ideals were high, and he succeeded in embodying them in his life and conduct. He was, further, a man of business capacity, and had great administrative power. In this way, both his aid and his counsel were of the greatest use to his colleagues in the University, and many institutions in the city—educational and other—benefited by his wisdom and his services.

Dr. Pirie's characteristics as a professor were unvarying courtesy, remarkable lucidity of exposition, and never-failing interest in his students—an interest none the less real that it was never either obtrusive or effusive. His courtesy was no mere mannerism, but a part of his life, and an index of the relationship which he assumed to exist between himself and his audience. It met with a willing response; no class-room was quieter than his, no prælections were more respectfully received; nor did any one ever fail to realize that behind the Professor's imperturbable courtesy lay a strong, if also a self-restrained, character. His personal influence in this respect was aided by the nature of his subject and by his treatment of it. The tyros, who composed the large proportion of his class, felt that, if they failed to listen to him, no subsequent study would replace what they lost. The effort to follow was not too great even for the backward student—so clear and simple was the exposition, and so conscious was the Professor of the limitations of some of his hearers. The best men in the class may have sometimes felt the reiteration of the non-important steps in what was to them a very simple affair; but the weaker brethren, as they revised their note-books, felt that the mathematical degree was capable of attainment after all. This side of his teaching was the most characteristic as far as his expositions to his general class are concerned; but his activity was so great that he also conducted, not one, but several higher classes, and those who attended them speak to his stimulating power as a teacher here too. One ought also to add that, to the last, he strongly insisted upon the practical usefulness of his subject, and delighted in historical references and the evolution of his science.

The greatest reward that came to Professor Pirie as a teacher was the success of his students in the sphere of mathematics after they left him, particularly at Cambridge and in the Civil Service Competition. This was always a real source of pleasure to him.

CORRECTIONS.

Sir Robert Ball has sent the following corrections in his Obituary Notice of Dr. Salmon (Ser. 2, Vol. 1, pp. xxii–xxviii) :—

P. xxii, line 13 from bottom, *for* "three" *read* "those."
P. xxviii, line 9 from bottom, *for* "Murphy" *read* "Morphy."

The Rev. F. H. Jackson has sent the following corrections of his paper in the present volume :—

P. 198, line 5 from bottom, in the third term of the series, *for* "$\frac{\mu}{2}$" *read* "$\left(\frac{\mu}{2}\right)^2$."
P. 199, in the last term of (17), *for* "λ" *read* "$\kappa\lambda$."
P. 215, in the second term of (71), *for* "$[n][2n-1]$" *read* "$[2][2n-1]$."
P. 216, line 6 from bottom, *for* "$[2r+1]$" *read* "$[2r-1]$."

Dr. Hobson has sent the following correction of his paper in the present volume :—

P. 321, line 13 from bottom, *for* "contained in L_2," *read* "contained in the intervals complementary to G_2."

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON THE ROOTS OF THE EQUATION $\dfrac{1}{\Gamma(x+1)} = c$

By G. H. Hardy.

[Received 14th January, 1904.—Read 11th February, 1904.—Revised March, 1904.]

1. The present paper forms part of some investigations concerning the roots of a number of transcendental equations of particular forms, which I have undertaken in the hope of throwing some light on the exceedingly difficult and important general question of the relations subsisting between the roots of the equations comprised in the form $F(x) = \phi(x)$, where $F(x)$ is a given integral function, and $\phi(x)$ a constant polynomial, or integral function whose increase (*croissance*) is less than that of $F(x)$. I need hardly say that our present knowledge about this question is almost entirely limited to the *moduli* of the roots; what we know about the arguments is practically *nil*: and I think that the results which I have obtained may be of some interest, in spite of their very special character, as indicating to some extent the various kinds of cases which may occur.

I have considered particularly the equations

(1) $\sin x = P(x)$,

(2) $e^{-ix} = P(x)$,

(3) $e^{ax} \sin bx = P(x)$,

(a and b being real and positive).

(4) $\Pi_\rho(x) \equiv \prod_1^\infty (1+x/n^\rho) = P(x)$

—$\Pi_\rho(x)$ being one of the functions considered by Mr. E. W. Barnes in his memoir "On Integral Functions,"* and $\rho > 1$—and

$$(5) \quad \frac{1}{\Gamma(x+1)} = P(x).$$

In all of these $P(x)$ is an arbitrary polynomial. The question to which I have given especial attention is whether there is in each case any form of the polynomial for which the nature of the zeroes is abnormal. The asymptotic solutions of (1) and (2) were given in two papers in the *Messenger*.† In the case of (1) there is no abnormal case; in the case of (2) the only abnormal case is that of $P(x) \equiv 0$, the familiar Picard case of exception in which there are no roots at all. In the case of (3), which I have investigated in a recent paper in the *Quarterly Journal*,‡ there is again one abnormal case, that of $P(x) \equiv 0$, but it is abnormal in quite a different way, which essentially involves the arguments of the roots: and the same is true of (4) if $1 < \rho < 2$; but, if $2 < \rho$, there is no abnormal case. I hope on some future occasion to make a further communication concerning these equations, with especial reference to the case of $\rho = 2$. At present I shall confine myself to the equation (5). I may, however, remark that in every case the following proposition is true:—*if* $a_n(c)$ *is the* n-th *root of* $F(x) = c$, and one particular value of c is excluded from consideration, ***then the roots can be arranged in a finite number of groups, such that within each group***

$$\lim_{n=\infty} \frac{a_n(c')}{a_n(c'')} = 1.$$

It is, of course, supposed that in each group the roots are arranged according to ascending order of moduli. For the constants c, c', c'' we may substitute polynomials. That any such theorem is true *in general* I do not for one moment suppose, even if we confine ourselves to the moduli of the roots §; but it is certainly true for large classes of the most important functions. I may add that it may be shown that the exceptional case in equations (3) and (4) is exceptional in the same way with whole classes of functions.

2. I come now to the equation (5). The function $\Pi(x) \equiv \dfrac{1}{\Gamma(x+1)}$ is

* *Phil. Trans.* (A), Vol. CXCIX., p. 411.

† Vol. XXXI., p. 161, and Vol. XXXII., p. 36.

‡ Vol. XXXV., p. 261.

§ That it is true in this sense is suggested by Borel, *Fonctions entières*, p. 100.

an integral function whose apparent and real orders* are each unity. Now M. Borel has proved the two following theorems, the second being a generalization of Picard's theorem :—

(i.) If the apparent order of $F(x)$ is finite and not integral, the real order is equal to it :

(ii.) If the apparent order is an integer p, then among the functions $\phi(x)F(x)-\psi(x)$, where ϕ, ψ are integral functions whose apparent order $< p$, there is one at most whose real order $< p$.

From the second theorem it follows that the real order of all the functions $\Pi(x)-c$ is the same, with possibly one exception. Is there such an exception? The answer seems to me very interesting. It is that there is not, and yet that the case $c = 0$ is abnormal not merely as regards the distribution of the zeroes in the plane, but also as regards the increase of their moduli. In fact, the increase of the zeroes for $c \neq 0$ is like that of $n/\log n$, whereas that of the zeroes for $c = 0$ is, of course, like that of n. This shows the possibility of cases of exception of a nature too subtle to be indicated by any alteration of the real order of the function.†

The result is also interesting in connection with the apparent paradox originally noted by M. Borel, that the increases of the functions

$$(6)\quad \Pi(x),\ \sin\tfrac{1}{2}\pi x$$

are different, being those (roughly) of

$$(7)\quad e^{r\log r},\ e^{r}$$

while the increases of the moduli of their zeroes are the same. One is tempted to say that, if we substitute for $\Pi(x)$

$$(8)\quad \Pi(x)-c,$$

* See *Leçons sur les Fonctions entières*. The *real order* ρ of a function is the index of convergence of the reciprocals of the moduli of its zeroes ; the *apparent order* is the least number ρ' such that, however small be ϵ, the maximum of $|F(x)|$ on a circle of radius r is less than $\exp(r^{\rho'+\epsilon})$ for all values of r greater than a certain value. It follows from the first theorem that the Picard case can only occur if ρ is an integer ; but cases in which the behaviour of the zeroes is abnormal for a particular value of c certainly can, as is shown by the example of the function $\Pi_{\rho}(x)$.

† Two very important memoirs on integral functions have appeared in the last few years— P. Boutroux, "Sur quelques propriétés des fonctions entières (*Acta Mathematica*, Vol. XXVIII.) ; E. Lindelöf, "Mémoire sur la théorie des fonctions entières de genre fini" (*Acta Soc. Fennicæ*, Vol. XXXI.). Each of these writers has introduced greater precision into the known theorems concerning functions of non-integral order and has proved interesting results concerning functions of integral order. I refer further in the text to some of M. Boutroux's results. I am indebted to M. Boutroux for a number of suggestions concerning the results proved and referred to in this paper.

the apparent paradox disappears, as we have then for the increases of the moduli of the zeroes

$$(9) \quad n/\log n, \; n,$$

which correspond naturally enough to (7). But this, as M. Boutroux pointed out to me, is not quite a sufficient account of the matter. M. Boutroux has in fact shown that in the case of a function of integral real order there are *two* typical laws of increase of its modulus. These typical laws may be said roughly to correspond to the cases in which all the roots are real and (*a*) all positive, (*b*) equal and opposite in pairs. The two laws for the function corresponding to the law n for the zeroes would be e^r, $e^{r\log r}$, and corresponding to the law $n/\log n$ would be $e^{r\log r}$, $e^{r(\log r)^2}$. This sufficiently elucidates the behaviour of the functions (6); but seems at first sight to raise a precisely similar difficulty with reference to the function (8), since its increase is the same whether $c=0$ or not, while the increase of its zeroes differs in the two cases. This difficulty, however, disappears when we note (what will be obvious later on) that what we may call the two *principal* sets of zeroes of (8), *i.e.* those whose increase is *least*, are arranged on the "equal and opposite" type (*b*), approximately of course. M. Boutroux when writing his memoir was of course not aware of the nature of the zeroes of (8). I proceed now to the proof of the assertions which I have made about them.

2. It has been shown by Mellin* that, if $-\pi+\epsilon < am\,.\,x < \pi-\epsilon$,

$$\Pi(x) = \frac{1}{\sqrt{(2\pi)}} \exp\left\{-(x+\tfrac{1}{2})\log x + x - \sum_1^k (-)^{\nu-1} \frac{B_\nu}{2\nu(2\nu-1)} x^{1-2\nu} + I(x,\kappa)\right\}, \tag{A}$$

where
$$I(x,\kappa) = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} \frac{\pi}{\sin\pi t}\,\frac{x^t}{t}\,\zeta(t)\,dt$$
$$(-2k-1 < \kappa < -2k+1),$$

k being a positive integer, $\zeta(t)$ the Riemann ζ function, and $\log n$ having its principal value; and that $|I(x,\kappa)| < C\,|x|^{\kappa}$ where C depends only on κ and on ϵ. From this and the formula

$$\Pi(x)\,\Pi(-x-1) = -\pi^{-1}\sin\pi x$$

we easily deduce that

$$\Pi(x) = -\frac{\sin\pi x}{\sqrt{(\frac{1}{2}\pi)}} \exp\{-(x+\tfrac{1}{2})\log(-x-1) + x + 1 + \ldots\},$$

an asymptotic expression for $\Pi(x)$ valid throughout a region of the plane

* *Acta Societatis Fennicæ*, Vol. XXIX.

which includes the part hitherto excluded; and it is easy to see that the two expressions are equivalent in the domain common to their regions of validity. Moreover, these expressions hold *uniformly* for all values of the amplitude of x; *i.e.* the ratio of $\Pi(x)$ to one or other (or to either) of them differs from unity by a quantity numerically less than $C\,|\,x\,|^{\kappa}$, C being independent of the amplitude of x.

Now, if $x = re^{i\theta}$,

$$|\exp\{-(x+\tfrac{1}{2})\log x+x\}| = \exp\{-(r\log r-r)\cos\theta+r\theta\sin\theta-\tfrac{1}{2}\log r\}.$$

This tends to ∞ with r if $\tfrac{1}{2}\pi \leqslant \theta \leqslant \pi-\epsilon$ or $-\pi+\epsilon \leqslant \theta \leqslant -\tfrac{1}{2}\pi$, to 0 if $-\tfrac{1}{2}\pi < \theta < \tfrac{1}{2}\pi$. If $\pi-\epsilon < \theta < \pi$ or $-\pi < \theta < -\pi+\epsilon$, we see by the help of the other asymptotic expression that $|\,\Pi(x)\,|$ becomes on the whole exceedingly large, except in the immediate neighbourhood of its zeroes.* Thus we see that the large roots of $\Pi(x) = c$ must be sought either in the direction of the negative real axis, or in the direction of either part of the imaginary axis; and that the roots (if any) in the latter directions will lie to the right of the imaginary axis.†

The first series of roots does not particularly interest us here. It is not difficult to show that, if we draw the curves $|\,\Pi(x)\,| = |\,c\,|$, those of them which lie in the direction of the negative real axis at a great distance from the origin are small closed curves surrounding the points $-n$; and hence that the roots of $\Pi(x) = c$ tend asymptotically to the points $-n$, one and only one being associated with each point, and passing continuously into it for $c = 0$. I proceed to consider the other sets of zeroes. Suppose that

$$\Pi(x) = c = \gamma e^{i\mu}, \quad x = \xi+i\eta,$$

ξ, η being positive and η large. If we make $k = 1$ in (A), we find that

$$\log\Pi(x) = -\log\sqrt{(2\pi)}-(x+\tfrac{1}{2})\log x+x$$
$$-\frac{B_1}{2x}+\frac{1}{2\pi i}\int_{\kappa-i\infty}^{-\kappa+i\infty}\frac{\pi}{\sin\pi t}\,\frac{x^t}{t}\,\zeta(t)\,dt \quad (-3<\kappa<-1).$$

If we vary κ so that the line $(\kappa-i\infty,\ \kappa+i\infty)$ lies to the right of the point -1, as by taking $\kappa = -\tfrac{1}{2}$, we obtain

---

* As do simpler functions, such as $e^x \sin x$ when x is large and positive.

† It is essential to the truth of these statements that we should know that $|\,\Pi(x)\,|$ tends *uniformly* to ∞ with $|\,x\,|$ throughout the region $\pi-\epsilon<\theta<-\pi+\epsilon$. It is perfectly possible for the modulus of a function to tend to ∞ with $|\,x\,|$ in *every* direction, and yet for it to have an infinity of roots near ∞; but in such a case it must tend to ∞ non-uniformly. See a recent note in the *Comptes Rendus* by Prof. Mittag-Leffler.

$$\log \Pi(x) = -\log \surd(2\pi) - (x+\tfrac{1}{2})\log x + x + \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi}{\sin \pi t}\,\frac{x^t}{t}\,\zeta(t)\,dt,$$

since the residue at $t=-1$ is

$$\frac{1}{x}\zeta(-1) = -\frac{B_1}{2x}.$$

Now, if
$$\phi(x) = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi}{\sin \pi t}\,\frac{x^t}{t}\,\zeta(t)\,dt,$$

$$\frac{d\phi}{dx} = \frac{1}{2\pi i}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{\pi}{\sin \pi t}\,x^{t-1}\zeta(t)\,dt,$$

and so *
$$|\phi(x)| < C\,|x|^{-\frac{1}{2}}, \qquad \left|\frac{d\phi}{dx}\right| < C\,|x|^{-\frac{3}{2}},$$

and, if $\phi(x) = u+iv$, the moduli of $\dfrac{\partial u}{\partial \xi} = \dfrac{\partial v}{\partial \eta}$ and $\dfrac{\partial u}{\partial \eta} = -\dfrac{\partial v}{\partial \xi}$ are each numerically less than a constant multiple of $|x|^{-\frac{3}{2}}$.

Now
$$\exp\{-(x+\tfrac{1}{2})\log x + x + \phi(x)\} = c\surd(2\pi),$$
and therefore
$$-(x+\tfrac{1}{2})\log x + x + \phi(x) = \log \gamma\surd(2\pi) - i(2p\pi - \mu)$$
where p is an integer. That is

(1) $\quad -\tfrac{1}{2}(\xi+\tfrac{1}{2})\log(\xi^2+\eta^2) + \eta\tan^{-1}\dfrac{\eta}{\xi} + \xi + u = \log\gamma\surd(2\pi)$

(2) $\quad -(\xi+\tfrac{1}{2})\tan^{-1}\dfrac{\eta}{\xi} - \tfrac{1}{2}\eta\log(\xi^2+\eta^2) + \eta + v = -2p\pi + \mu,$

$\tan^{-1}\dfrac{\eta}{\xi}$ being a positive angle $< \tfrac{1}{2}\pi$ and, since η is large compared with ξ, nearly $= \tfrac{1}{2}\pi$.

Let us consider the curves (1), (2). Since η is large, and large compared with ξ, it is clear that p is large and positive and that a first approximation to the form of the curves (2) in the part of the plane under consideration is given by $\eta\log\eta = 2p\pi$. It is easy to see † that a region can be defined by inequalities of the form

$$\frac{\pi}{2} - \delta \leqslant \theta \leqslant \tfrac{1}{2}\pi, \quad r \geqslant R$$

* See Mellin, *loc. cit.* This follows from the fact that $\lim\limits_{t=-\frac{1}{2}\pm i\infty} e^{-\epsilon|t|}\zeta(t) = 0$ for any $\epsilon > 0$.

† These assertions are easily verified, and there is nothing of interest in the formal proof of them.

(δ being small and R large) within which to each value of ξ corresponds one and only one value of η for each of the curves (2), that the values of η increase with p, and that $d\eta/d\xi$ is small and negative along each of the curves. Moreover, the curve (1) consists of a single branch whose equation may be put in the form $\xi = \frac{1}{2}\pi\eta(1+\epsilon)/\log\eta$ where ϵ is very small; and along this curve $d\eta/d\xi$ is large and positive. From these facts it follows that the curve (1) cuts each of the curves (2) in one and only one point in the region in question. Each of these points is a root of $\Pi(x) = c$, and it is clear that at such a point

$$\eta = 2p\pi(1+\epsilon)/\log p, \qquad \xi = p^2\pi(1+\epsilon)/(\log p)^2.$$

Therefore *the equation* $\Pi(x) = c$ $(c \neq 0)$ *has an infinity of roots lying in the direction of the imaginary axis, and given by the formula*

$$x_p = \frac{\pi^2 p}{(\log p)^2}(1+\epsilon) + \frac{2\pi p i}{\log p}(1+\epsilon)$$

where in each bracket ϵ *is a quantity whose limit for* $p = \infty$ *is* 0. Also

$$|x_p| = r_p = 2\pi p(1+\epsilon)/\log p;$$

so that the increase of these roots is that of $p/\log p$.

It is obvious that there is a corresponding set of roots in the negative direction of the imaginary axis. The equation $\Pi(x) = c$ has therefore three sets of roots whose increases are p, $p/\log p$, $p/\log p$; and so the increase of its roots is on the whole $p/\log p$.

3. An investigation only very slightly more complicated shows that the increase of the roots of the equation

$$\Pi(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$$

is also $p/\log p$. In fact, we obtain exactly the same first approximation to them as in the simpler case. I have not been successful in an attempt to approximate to the roots with sufficient accuracy to distinguish between the nature of the roots for different values of n. In the case of the other equations referred to at the beginning of this paper a more accurate approximation is possible.

ON A CERTAIN DOUBLE INTEGRAL

By A. C. DIXON.

[Received January 19th, 1904. — Read February 11th, 1904.]

THIS paper contains a discussion of some properties of the double integral $\int_0^1\int_0^1 x^{i-1}(1-x)^{j-1}y^{l-1}(1-y)^{k-1}(1-xy)^{m-j-k}\,dy\,dx$. The most remarkable of these is that, when multiplied by a certain factor, the expression becomes a symmetric function of five independent variables. From an examination of particular cases it does not seem likely that this function can be reduced to gamma functions in general; though under some conditions it can. (*Proc. London Math. Soc.*, Vol. XXXIV., p. 401; Vol. XXXV., p. 289.)

1. Certain transformations which leave the double integral

$$\iint x^{i-1}(1-x)^{j-1}y^{l-1}(1-y)^{k-1}(1-xy)^{m-j-k}\,dx\,dy$$

unaltered in form make up a group that is of some interest. Let the transformation

$$x=\phi(X,\,Y),\qquad y=\psi(X,\,Y),$$

where X, Y are the new variables, be denoted by $(\phi(x,\,y),\ \psi(x,\,y))$. Then one member of the group is $(y,\,x)$, which simply interchanges x and y. This operation is of period 2. Others, also of period 2, are $\left(\frac{1}{x},\,\frac{1}{y}\right)$, $\left(\frac{1}{xy},\,y\right)$, $\left(1-x,\,\frac{y}{y-1}\right)$. From these may be derived $\left(y,\,\frac{1}{xy}\right)$, of period 3, and $\left(\frac{1}{1-x},\,\frac{y-1}{y}\right)$, also of period 3; and from these again $\left(\frac{1}{1-y},\,1-xy\right)$, which is of period 5.

The variables introduced in the different transformations are thirty in number, namely,

$$x,\ \frac{1}{x},\ 1-x,\ \frac{1}{1-x},\ \frac{x-1}{x},\ \frac{x}{x-1};$$

$$y,\ \frac{1}{y},\ 1-y,\ \frac{1}{1-y},\ \frac{y-1}{y},\ \frac{y}{y-1};$$

$$xy, \quad \frac{1}{xy}, \quad 1-xy, \quad \frac{1}{1-xy}, \quad \frac{xy-1}{xy}, \quad \frac{xy}{xy-1};$$

$$\frac{1-y}{1-xy}, \quad \frac{1-xy}{1-y}, \quad \frac{y-xy}{1-xy}, \quad \frac{1-xy}{y-xy}, \quad \frac{y-xy}{y-1}, \quad \frac{y-1}{y-xy};$$

$$\frac{1-x}{1-xy}, \quad \frac{1-xy}{1-x}, \quad \frac{x-xy}{1-xy}, \quad \frac{1-xy}{x-xy}, \quad \frac{x-xy}{x-1}, \quad \frac{x-1}{x-xy}.$$

Let P, Q, R, S, T denote these five rows respectively. Then the transformations of the group permute P, Q, R, S, T. For instance, (y, x) interchanges P and Q, also S and T. We may then write

$$(y, x) = (PQ)(ST),$$

and similarly

$$\left(\frac{1}{x}, \frac{1}{y}\right) = (ST), \qquad \left(\frac{1}{xy}, y\right) = (PR)(ST), \qquad \left(1-x, \frac{y}{y-1}\right) = (RS).$$

From these four all the permutations may be derived. Conversely, each permutation corresponds only to one operation of the group; for, if not, an operation of the group, say (x', y'), other than (x, y) would leave P, Q, R, S, T unchanged.

Now, since P is unchanged,

$$x' = x, \quad \frac{1}{x}, \quad 1-x, \quad \frac{1}{1-x}, \quad \frac{x}{x-1}, \quad \text{or} \quad \frac{x-1}{x}.$$

Since Q is unchanged,

$$y' = y, \quad \frac{1}{y}, \quad 1-y, \quad \frac{1}{1-y}, \quad \frac{y}{y-1}, \quad \text{or} \quad \frac{y-1}{y}.$$

Since R is unchanged, $x'y'$ is a function of xy only, which can only be if $x' = x$, $y' = y$ or $x' = x^{-1}$, $y' = y^{-1}$. The latter is out of the question, since it interchanges S and T. Hence the group of transformations is the same in structure as that of the permutations of five letters.

This may be seen in another way. Put

$$x = \frac{(a-b)(c-d)}{(c-b)(a-d)}, \qquad y = \frac{(a-e)(c-b)}{(c-e)(a-b)}.$$

Then the operations of the group are equivalent to permutations of a, b,

c, d, e. We have, in fact,

$$(ac) = \left(\frac{1}{x},\ \frac{1}{y}\right), \qquad (de) = \left(\frac{1}{y},\ \frac{1}{x}\right),$$

$$(bd) = \left(\frac{1}{x},\ xy\right), \qquad (bc) = \left(1-x,\ \frac{y}{y-1}\right),$$

$$(ac)(de) = (y,\ x), \qquad (ac)(be) = \left(\frac{1}{xy},\ y\right),$$

and P, Q, R, S, T are replaced by e, d, b, c, a respectively. Let us slightly modify the meaning of P, Q, R, S, T and take

$$P = \frac{(1-x+x^2)^3}{(x-x^2)^2},$$

Q, R, S, T the same functions of y, xy, $\dfrac{1-y}{1-xy}$, $\dfrac{1-x}{1-xy}$ that P is of x. Then P is a symmetric function of a, b, c, d; Q the same of a, b, c, e; and so on; and, if we suppose a, b, c, d, e the roots of a quintic equation, P, Q, R, S, T will be those of a resolvent of the same degree whose coefficients are all invariants: this resolvent could be derived by a Tschirnhausen transformation.

It is easily found that the leading coefficient in this resolvent is the cube of the discriminant, and that the second and third coefficients contain the second and first powers of the discriminant as factors respectively. Each coefficient is of the order 24 in those of the original quintic. The discriminant of the resolvent contains as factors the seventh power of the original discriminant and the fourth power of the skew invariant, since this latter vanishes when two pairs of roots belong to an involution in which the fifth root is self-conjugate. If ab, cd are the two pairs, we have $(abce) = (bade)$, $(acde) = (bdce)$; so that the resolvent has two pairs of equal roots.

Since the discriminant is of order 8, the skew invariant of order 18, and the discriminant of the resolvent of order 192, the last must contain another factor of order 64, which is found to be the square of an invariant of order 32 whose vanishing is the condition that four roots, such as a, b, c, d, should be homographic with b, c, a, e.*

* *Note added March 15th*, 1904.—The group of permutations of five letters is here shown to be isomorphic with a group of birational transformations of two variables. This is a case of the theorem, given by Prof. W. Burnside and Prof. E. H. Moore (see *Messenger of Mathematics*, February, 1901, pp. 148–153; and *American Journal*, Vol. XXII., pp. 279–291), that the symmetric group of degree n can be represented as a group of birational transformations in $n-3$ symbols.

2. Each transformation of the group produces a linear transformation of the letters i, j, k, l, m which occur in the double integral.

Thus i, j, k, l, m are changed

by $(y, \quad x)$ into l, k, j, i, m;

by $\left(\frac{1}{x}, \quad \frac{1}{y}\right)$ into $k-m-i+1,\ j,\ k,\ j-m-l+1,\ m$;

by $\left(\frac{1}{xy}, \quad y\right)$ into $k-m-i+1,\ m-j-k+1,\ k,\ l-i-j+1,\ m$;

by $\left(1-x, \frac{y}{y-1}\right)$ into $j,\ i,\ j-m-l+1,\ l,\ i-k-l+1$.

Constant factors of the subject of integration are here neglected.

These transformations permute the ten quantities i, j, k, l, m, $i-k-l+1$, $j-l-m+1$, $k-m-i+1$, $l-i-j+1$, $m-j-k+1$ in such a way that rows and columns are permuted simultaneously in the array:—

	a	b	c	d	e
a	$\ldots,$	$i-k-l+1,$	$m,$	$k-m-i+1,$	$l,$
b	$i-k-l+1,$	$\ldots,$	$l-i-j+1,$	$j,$	$k,$
c	$m,$	$l-i-j+1,$	$\ldots,$	$i,$	$j-l-m+1,$
d	$k-m-i+1,$	$j,$	$i,$	$\ldots,$	$m-j-k+1,$
e	$l,$	$k,$	$j-l-m+1,$	$m-j-k+1,$	$\ldots.$

The array is bordered with the letters a, b, c, d, e to show the correspondence between the two forms of the group.

2. So far the double integral has been taken as indefinite. If we now make it definite, the limits being 0, 1 for both x and y, there are ten transformations of the group that leave the limits unchanged. One of these is (y, x); another is $\left(\frac{1-x}{1-xy}, y\right)$ or $(ae)(bc)$, which substitutes j, i, m, l, k for i, j, k, l, m. These operations reverse the cyclic order of i, j, k, l, m, as also of a, b, c, e, d; hence the ten operations of the

sub-group are those which maintain or reverse the cyclic order i, j, k, l, m, and the value of

$$\int_0^1\int_0^1 x^{i-1}(1-x)^{j-1}y^{l-1}(1-y)^{k-1}(1-xy)^{m-j-k}\,dx\,dy$$

is unchanged by these ten substitutions for i, j, k, l, m.

Denote this integral by $B(i, j, k, l, m)$ on the supposition that the real parts of i, j, k, l, m are positive, so that the value is finite. It can be expressed as a single integral by means of the hypergeometric function thus

$$\frac{\Gamma k\Gamma l}{\Gamma(k+l)}\int_0^1 x^{i-1}(1-x)^{j-1}F(j+k-m,\ l,\ k+l,\ x)\,dx.$$

Since the hypergeometric function is unaffected by an interchange of its first two elements, we find that $B(i, j, k, l, m) \div \Gamma k\Gamma l$ is unchanged by the substitution for i, j, k, l, m of $i, j, l+m-j, j+k-m, m$. Since i, j, m are also unchanged, we have $B(i, j, k, l, m) \div \Gamma i\Gamma j\Gamma k\Gamma l\Gamma m$ unaffected by the same substitution, which replaces $i+j, j+k, k+l, l+m, m+i$ by $i+j, l+m, k+l, j+k, m+i$, the same quantities in another cyclic order.

Since it has been proved already that $B(i, j, k, l, m)$ is unaffected by any reversal of the cyclic order of the five letters, we find that

$$B(i, j, k, l, m) \div \Gamma i\Gamma j\Gamma k\Gamma l\Gamma m$$

is a symmetric function of $i+j, j+k, k+l, l+m, m+i$.

Writing $F(\alpha, \beta, \gamma, \delta, \epsilon, x)$ for

$$1+\frac{\alpha\beta\gamma}{\delta\epsilon}x+\frac{\alpha(\alpha+1)\beta(\beta+1)\gamma(\gamma+1)}{1.2.\delta(\delta+1)\epsilon(\epsilon+1)}x^2+\ldots,$$

we have, when the real parts of $\delta+\epsilon-\alpha-\beta-\gamma$, α, β, $\delta-\alpha$, $\epsilon-\beta$ are positive,

$$\Gamma\alpha\Gamma(\delta-\alpha)\Gamma\beta\Gamma(\epsilon-\beta)F(\alpha, \beta, \gamma, \delta, \epsilon, 1)$$
$$= \Gamma\delta\Gamma\epsilon\int_0^1\int_0^1 x^{\alpha-1}(1-x)^{\delta-\alpha-1}y^{\beta-1}(1-y)^{\epsilon-\beta-1}(1-xy)^{-\gamma}\,dx\,dy.$$

Hence $$F(\alpha, \beta, \gamma, \delta, \epsilon, 1) \div \Gamma\delta\Gamma\epsilon\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)$$

is a symmetric function of δ, ϵ, $\delta+\epsilon-\alpha-\beta$, $\delta+\epsilon-\beta-\gamma$, $\delta+\epsilon-\gamma-\alpha$.

Write p, q, r, s, t for these five variables; then when $\alpha = 0$, that

is, when $p+q+s=r+t$, the value of the symmetric function of p, q, r, s, t is $1/\Gamma p\Gamma q\Gamma s$. Another interesting result is that

$$\frac{F(a, \beta, \gamma, \delta, \epsilon, 1)}{\Gamma\delta\Gamma\epsilon\Gamma(\delta+\epsilon-a-\beta-\gamma)}$$
$$= \frac{F(\delta+\epsilon-a-\beta-\gamma, \epsilon-a, \delta-a, \delta+\epsilon-a-\beta, \delta+\epsilon-a-\gamma, 1)}{\Gamma(\delta+\epsilon-a-\beta)\Gamma(\delta+\epsilon-a-\gamma)\Gamma a},$$

from which it follows that

$$\lim_{a+\beta+\gamma=\delta+\epsilon} \frac{F(a, \beta, \gamma, \delta, \epsilon, 1)}{\Gamma(\delta+\epsilon-a-\beta-\gamma)} = \frac{\Gamma\delta\Gamma\epsilon}{\Gamma a\Gamma\beta\Gamma\gamma}.$$

4. The symmetry of the hypergeometric function $F(a, \beta\ \gamma, t)$ with respect to a, β may be shown by transformation of the definite integral as follows. We have

$$\int_0^1\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}y^{a-1}(1-y)^{\gamma-a-1}(1-xt)^{-a}\,dx\,dy$$
$$= B(\beta, \gamma-\beta)B(a, \gamma-a)F(a, \beta, \gamma, t).$$

Put $$\frac{x}{1-x} = \frac{u}{(1-u)(1-vt)}, \qquad \frac{y}{(1-y)(1-xt)} = \frac{v}{1-v}.$$

This gives $$x = \frac{u}{1-vt+uvt}, \qquad y = \frac{v(1-ut-vt+uvt)}{1-vt}$$

$$(1-x)(1-y) = (1-u)(1-v), \qquad \frac{\partial(x, y)}{\partial(u, v)} = \frac{xy}{uv},$$

and the double integral becomes

$$\int_0^1\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}v^{a-1}(1-v)^{\gamma-a-1}(1-vt)^{-\beta}\,du\,dv,$$

the same expression with a, β interchanged.

5. It would seem that there is no irreducible closed path which for all values of the indices i, j, k, l, m brings back the expression

$$x^{i-1}(1-x)^{j-1}F(j+k^{-m}, l, k+l, x)$$

to its original value. Any closed path causes a linear homogeneous transformation of F and another function F_1; if the values of x^{i-1}, $(1-x)^{j-1}$ are to be restored in general, the path must pass round each of the points 0, 1 as often positively as negatively, and the product of the multipliers in the transformation of F, F_1 must therefore be 1. If F, or any expression of the form $\lambda F+\mu F_1$, is unchanged, one of the multipliers

is 1, the other is then also 1, and the transformation is parabolic or else identical. But there are well known cases in which the group of the hypergeometric function contains no parabolic substitutions. Such cases are, for instance, those associated with the regular solids. When the first three elements are commensurable there are irreducible paths corresponding to identical substitutions, but these are not the same for all values of the elements.

6. The function $B(i, j, k, l, m)$ satisfies certain difference equations which may be found as follows :—

Multiply the identity $x+(1-x) = 1$ by

$$x^{i-1}(1-x)^{j-1}y^{l-1}(1-y)^{k-1}(1-xy)^{m-j-k}$$

and integrate.

Thus $B(i+)+B(j+, m+) = B$, where $i+$ indicates that i is to be increased by 1, the other letters being unchanged.

Other equations may be derived from this by symmetry. Write p, q, r, s, t for $i+j, j+k, k+l, l+m, m+i$, and $C(p, q, r, s, t)$ or simply C for $B(i, j, k, l, m)/\Gamma i\Gamma j\Gamma k\Gamma l\Gamma m$. Thus we have

$$\begin{aligned} C &= iC(p+, t+)+jmC(p+, q+, s+, t+) \\ &= \tfrac{1}{2}(p-q+r-s+t)\,C(p+, t+) \\ &\qquad +\tfrac{1}{4}(p+q-r+s-t)(-p+q-r+s+t)\,C(p+, q+, s+, t+). \quad (a) \end{aligned}$$

Of such equations there are thirty, since p, q, r, s, t may be interchanged in any way. If we take the one written with those derived from it by interchanging q, s, t, we can eliminate C and $C(p+, q+, s+, t+)$. Thus

$$\begin{aligned} (p-q+r-s+t)(q-s)\,C(p+, t+)+(p+q+r-s-t)(s-t)\,C(p+, q+) \\ +(p-q+r+s-t)(t-q)\,C(p+, s+) = 0, \end{aligned}$$

or, by the substitution of $p-1$ for p,

$$\begin{aligned} (p-q+r-s+t-1)(q-s)\,C(t+)+(p+q+r-s-t-1)(s-t)\,C(q+) \\ +(p-q+r+s-t-1)(t-q)\,C(s+) = 0. \end{aligned}$$

Similarly, interchanging q, r, s, we can form from (a) two more equations by means of which it is possible to eliminate C and $C(p+, t+)$: then, by diminishing p, q, r, s, t each by 1, we have

$$\begin{aligned} (q-s)(p+q-r+s-t-1)(-p+q-r+s+t-1)\,C(r-) \\ +(s-r)(p-q+r+s-t-1)(-p-q+r+s+t-1)\,C(q-) \\ +(r-q)(p+q+r-s-t-1)(-p+q+r-s+t-1)\,C(s-) = 0, \end{aligned}$$

Again, from (a), by interchanging q, t and eliminating

$$C(p+,\, q+,\, s+,\, t+),$$

we have

$$4(q-t)C = (p+q+r-s-t)(p+q-r+s-t)C(p+,\, q+)$$
$$-(p-q+r-s+t)(p-q-r+s+t)C(p+,\, t+);$$

this may be written

$$(i-j-k+m)B = (i-k+m)B(i+)-(j+k-m)B(j+),$$

in which form it seems to correspond with the reduction formula

$$(m+n)B(m,\, n+1) = nB(m,\, n).$$

To get some of these results we might also have integrated by parts.

From the equation (a) form two more, (1) by interchanging the pair pt with qs, (2) by writing $p-1$, $t-1$ for p, t. From these three eliminate $C(q+,\, s+)$ and $C(p+,\, q+,\, s+,\, t+)$. The result contains $C(p-,\, t-)$, C, and $C(p+,\, t+)$, and may therefore be written as a difference equation of the second order for B or C, with the single independent variable i, the other four j, k, l, m being treated as constant. This equation is

$$(i+m-k)(i+j-l)B(i+)+\{(i-1)(k+l-i)+jm-(i+j-l)(i+m-k)\}B$$
$$-(i-1)(k+l-i)B(i-) = 0.$$

Its general solution is discussed in Boole's *Finite Differences*, pp. 260–2 (1880).

OPEN SETS AND THE THEORY OF CONTENT

By W. H. YOUNG.

[Received January 10th, 1904.—Read January 14th, 1904.]

1. *Introductory.*

In the present paper, I begin by enunciating and proving a number of theorems which will, I think, be found to be of considerable importance in the theory of open sets, more especially in connection with what we may popularly express as "the room they fill up" (*Raumerfüllung*).

I have purposely, at the risk of some apparent repetition, considered separately the case where the points fill up sets of intervals: the gain in clearness seems to me considerable. Moreover this special case is of very great importance in itself. Thus a small portion of Theorem 4 is Arzelà's *lemma fondamentale*, from which he deduces many of his most interesting results. The proofs so far given of this lemma are all due to Arzelà; the only one I have examined with care is that in his memoir entitled "Sulle Serie di Funzioni, Parte prima" (1899—Arzelà's second proof): this proof is, as I here* show, incomplete.

Since the presentation of the present paper, I see that M. Borel has pointed out the importance in the theory of functions of a theorem including Arzelà's lemma and included in Theorem 4 of the present paper. His note is printed in the *Comptes Rendus*, December, 1903; no proof is indicated, nor is any reference given either to the work of Arzelà, or to my paper on "Closed Sets of Points defined as the Limit of Closed Sets of Points," which contains the germ of my own work on the subject.

The analogy between sets of points and sets of intervals brought out in the earlier part of the present paper leads naturally to the consideration, in the latter part, of the difficult question of the content of open sets: this question is here discussed at some length. The theory to which I have independently been led coincides in some of its main features with that developed by M. Lebesgue † in his very important

---

* See below, p. 22.

† *Annali di Matematica* (1902). [My attention was first called to this memoir after the presentation of the present paper. In consequence I have added references to the work of M. Lebesgue, wherever it seemed desirable, and I have partially adopted his nomenclature. The only other alterations made since the presentation of the paper consist in the insertion of one or two additional theorems in the section which deals with the (outer) content.—*March 16th*, 1904.]

memoir entitled "Integrale, Longueur, Aire"; and my results confirm his. This is, perhaps, not without interest, as in one or two places M. Lebesgue's treatment is rather suggestive than detailed, and the assumption that the region of space considered is finite underlies not only his definitions, but also his proofs. Whereas, moreover, the work of M. Lebesgue on this subject consists of a discussion of the properties of the contents of "measurable sets" in combination with each other, I have in what follows, regarding the matter from a somewhat more general standpoint, investigated the relations of what I call "additive sets"—a class which includes all the sets actually shown by M. Lebesgue to be measurable—to sets in general.

It has not been shown that sets which are not measurable do not exist, and it is possible that such sets do exist. This has led M. Lebesgue to adopt the terms *mesure intérieure*, and *mesure extérieure*, though he only considers those sets for which these agree. Corresponding to these terms I use the expressions "(inner)" and "(outer)" content, and in its proper place I go shortly into the details of M. Lebesgue's theory.

The definitions given of the (inner) and (outer) content of an open set are found to simplify materially the statement of a number of the properties of open sets, and are indeed suggested by them. The question then arises whether the contents so defined obey the law of addition; whether, in fact, the sum of the contents, whether (inner) or (outer), of two non-overlapping sets is equal to the (inner) or (outer) content of their sum. This is found to be the case, provided at least one of the two components is closed, or belongs to a very extended class of open sets. I have not succeeded in proving the theorem (or disproving it) in its complete generality. We here knock up against that barrier of imperfect acquaintance with open sets which is responsible for the non-determination of the question whether or no sets of points exist whose potencies lie between that of the natural numbers and that of the continuum.

If, as is possible, the addition theorem is not true for all open sets, the extended class of additive sets for which it holds possesses a peculiar interest of its own. It appears from the results of the paper that the class forms a corpus; all known operations performed on members of the corpus lead to members of the corpus. From this point of view the paper may perhaps be regarded as making a contribution of some interest to the classification of open sets, and I have availed myself of the opportunity of stating and proving several theorems which bear on this question, and which are, I believe, new.

PART I.—SETS OF INTERVALS.

2. *Finite Sets of Intervals.*

THEOREM 1.—*Given a countably infinite series $D_1, D_2, \ldots$ of sets of intervals, each of which contains only a finite number of intervals such that each interval of D_{n+1} is contained in an interval of D_n (with possibly one or both end points common), there is at least one point common to an interval from each set; and the common points form a closed set.*—For, since the number of intervals in D_n is finite, the internal and end points form a closed set, and, by hypothesis, the closed set of points D_{n+1} is a component of the closed set D_n; hence, by Cantor's Theorem of Deduction,* the first part of the conclusion follows; the second statement is also the direct consequence of a well-known extension of that theorem.

THEOREM 2.—*If to the hypothesis of Theorem 1 we add that the content*† *of each D_n is greater than some positive quantity $\geqslant g$, the common points form a closed set of points D' of content $\geqslant g$, so that they have the potency c.* For, if possible, let the content be less than g, and let the difference be greater than e. Then we can enclose all the points in a finite set of intervals of content less than $(g-e)$. Out of the set D_n let us cut those parts which are common to D_n and the intervals just constructed: there remain over a finite number of intervals of content greater than e. The intervals so constructed for successive values of n satisfy the requirements of Theorem 1; so that there is at least one point common to them, and therefore to the original sets D_n, contrary to the assumption that all the common points had been cut out. The assumption was then inadmissible that the common points could be enclosed in a finite set of intervals of content less than g. Q. E. D.

3. *Infinite Sets of Intervals.*

If we remove the restriction that the number of intervals in D_n is finite, these conclusions are inadmissible, since the points of D_n do not then form a closed set.‡ The following simple example proves this.

Example 1.—Let D_{m+1} consist of all the abutting intervals between the points whose numbers in the binary scale are 1^n $(n \geqslant m)$ (Fig. 1).

---

* See Part II., Theorem 1.

† That is, the content of the equivalent set of non-overlapping intervals. *Proc. London Math. Soc.*, Vol. xxxv., p. 386, §4.

‡ Cf. *Quarterly Journal of Pure and Applied Mathematics*, No. 138 (1903), p. 110.

Here the only limiting point of intervals one from each set is the point unity, and is external to the intervals of every set.

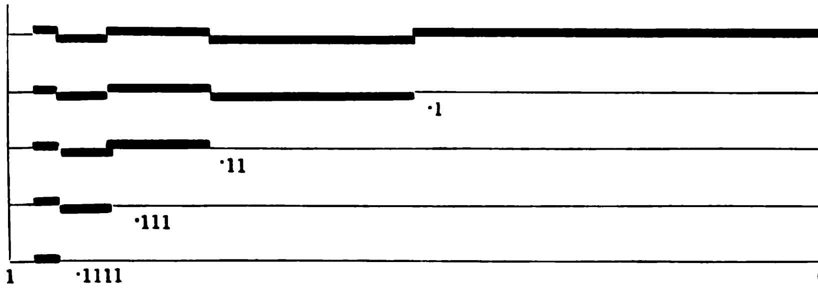


FIG. 1.

In this example the content of the sets decreases without limit; the theorem proved by me in a paper entitled "On Closed Sets of Points defined as the Limit of a Sequence of Closed Sets of Points"* shows that when the sets of intervals have no point common this must always be the case. As the application of Theorem 1 *supra* considerably simplifies the proof originally given of that theorem, I might have been tempted to repeat the theorem here. As a matter of fact, however, subsequent investigations† showed that the enunciation of that theorem might be further extended, and the result stated with greater precision, without complicating the proof. The following theorem is therefore substituted for it, and includes it as a special case of its first part (*a*) :—

THEOREM 3.—*Given a countably infinite series $D_1, D_2, \ldots$ of sets of intervals such that* (1) *each interval of D_{n+1} is contained in an interval of D_n for every value of n, and* (2) *the content I of each set D_n is greater than some positive quantity g, then* (*a*) *there is a set of points such that each is internal to an interval of D_n for every value of n, and* (*b*) *it contains closed components of content $> g-e$, where e is as small as we please; so that the potency*‡ *of these points is c.*

Let the non-overlapping intervals which have the same internal points as D_n be arranged in countable order, and denoted by $D_{n,r}$, for successive integral values of r.

Let us determine a finite number of the intervals $D_{1,r}$ such that the content of the remaining is less than $\frac{e}{2^{1+1}}$; and from each end of each

* *Proc. London Math. Soc.*, Vol. XXXV., p. 282.

† *Ibid.*, p. 284.

‡ Some other mathematicians have used the terms (1) "cardinal number," (2) "power," for this concept.

of these intervals (in finite number) $D_{1,r}$ let us cut off a fraction $\frac{1}{2^{1+3}}, \frac{e}{I_1}$ of its length. The sum of these pieces is less than $\frac{e}{2^{1+1}}$, and therefore the finite number of curtailed intervals, which we denote by D_1', has content $> g-\frac{1}{2}e$.

The parts of intervals $D_{2,r}$ which lie inside D_1' evidently have content $> I_2-\frac{1}{2}e$: choosing out a finite number of these so that the content of the remainder may be less than $\frac{e}{2^{2+1}}$, and curtailing them at each end by a fraction $\frac{1}{2^{2+3}}, \frac{e}{I_2}$ of their length, we get a finite set of intervals D_2' of content $> g-\frac{e}{2}-\frac{e}{2^2}$ lying inside the intervals D_1'.

Proceeding thus with each successive set of intervals $D_{n,r}$, we obtain from it a finite set D_n' of content $> g-\frac{e}{2}-\frac{e}{2^2}-\ldots-\frac{e}{2^n}$, *a fortiori* $> g-e$, lying inside the finite set D_{n-1}', for every value of n. Applying Theorem 2 to these sets, we deduce that they have in common a closed set of points of content $> g-e$. By construction these points are internal to the original intervals; which proves the theorem.

THEOREM 4.—*Given an infinite number of sets of intervals, in a finite segment* (A, B) *of length* L, *such that the content of each set of intervals is greater than some positive quantity* g, *then a set of points of potency* c *exists, which is internal to an infinite series of these sets of intervals, and contains closed components of content* $> g-e$, *where* e *is as small as we please.*

For consider the non-overlapping intervals having the same internal points as any one of the sets D_1. Their content $> g$, and therefore we can choose out a finite number of them whose content is greater than g. Suppose this done for all the sets: then in each set we have only a finite number of non-overlapping intervals.

Let the integer m be determined so that

$$mg \leqslant L < (m+1)g. \qquad (1)$$

Let us consider a group of n of the sets, where n is a sufficiently large integer, later to be more particularly specified.

The parts of (A, B), if any, which are covered by these n sets doubly form a finite number of intervals, possibly overlapping, whose content

we denote by $I_{1,n}$. The parts which are simply covered, therefore, form a finite set of non-overlapping intervals of content $> n(g-I_{1,n})$, whence

$$n(g-I_{1,n}) < (m+1)g;$$

therefore $$I_{1,n} > \{1-(m+1)/n\}g. \tag{2}$$

Let us choose an integer n' so that $(m+1)/n' < \frac{1}{2}e$, that is,

$$n' > 2(m+1)/e; \tag{3}$$

then $$I_{1,n'} > (1-\tfrac{1}{2}e)g. \tag{4}$$

Grouping the given sets together in distinct groups of n', and taking the corresponding sets of double non-overlapping intervals, we have conditions exactly similar to those with which we started, only that, instead of g, we have $(1-\frac{1}{2}e)g$.

To these new sets we apply the same reasoning as before, taking, however, $\frac{1}{2}e$ instead of e, and substituting for n an integer n'' such that $n'' > 4(m+1)/e$ and grouping the sets of double intervals in groups of n''. The content of the double parts corresponding to any such group being denoted by $I_{2,n''}$, it follows that

$$I_{2,n''} > (1-\tfrac{1}{2}e)(1-\tfrac{1}{4}e)g > (1-\tfrac{1}{2}e-\tfrac{1}{4}e)g > (1-e)g. \tag{5}$$

There will, therefore, certainly be such parts for every one of the groups, and they will, by the construction, be at least quadruply covered by the original sets.

In this way we can always proceed a stage further: the sets of intervals which we construct at each successive stage always have content $> (1-e)g$. Returning to the equation (2), we see that, since the whole set of intervals* in (A, B) which are covered at least doubly by the given sets has a content I'_1 greater than or equal to $I_{1,n}$ for all values of n, $I'_1 \geqslant g$. Similarly, denoting by I'_2 the content of the set of intervals in (A, B) which are covered at least quadruply by the given intervals, $I'_2 \geqslant g$; and, generally, $I'_n \geqslant g$, where I'_n is the content of the set of intervals which are covered by at least 2^n of the given sets.

Now, since the intervals corresponding to the content I'_n certainly lie inside those of content I'_{n-1}, we can apply to this series Theorem 3, since the content of each is certainly greater than $g-e$, which proves the theorem.

* There might, of course, be points of (A, B) external to these intervals which belong to a finite or infinite number of given sets, but they do not affect the argument.

This theorem includes Arzelà's *lemma fondamentale*, the enunciation of which is as follows:—

Sia y_0 un punto limite per un gruppo qualsiasi di numeri (y); e indichi $G_0 = (y_0, y_2, \ldots)$ una successione, comunque scelta, di numeri (y) tendenti al limite y_0. Assumendo le variabili come coordinate ortogonali di un punto nel piano, si consideri il gruppo delle rette $y = y_1$, $y = y_2, \ldots$; nell' intervallo $a \ldots b$ sopra ciascuna si segnino dei tratticelli distinti l'uno dall' altro, in numero finito che può variare da retta a retta e anche crescere indefinitamente via via che y_s si approssima a y_0. La somma dei tratticelli $\delta_{1,s}, \delta_{2,s}, \ldots, \delta_{n,s}$ segnati sulla $y = y_s$ sia d_s. *Se per ogni valore* $s = 1, 2, \ldots$ *si ha sempre* $d > g$; g *numero determinato positivo, necessariamente esiste tra* a, e, b *almeno un punto* x_0 *tale che la retta* $x = x_0$ *incontra un numero infinito di tratti* δ.

In other words, assuming that the sets of intervals in the enunciation of Theorem 4 are finite (a restriction which is subsequently removed), Arzelà asserts that there is at least one point x_0 common to intervals of every set, *i.e.*, either an internal or end point of such intervals.

Arzelà's first proof, which dates from the year 1885,* and occupies four pages royal octavo, involves the consideration, *seriatim*, of a number of different possibilities. The complicated character of this first proof, and perhaps also the fundamental nature of the result, induced him to attempt to give an alternative proof of a simpler character in 1899.† The line of argument commences much in the same way as that employed by myself. By taking the sets of intervals in groups of $m+2$ he obtains the equation (2) for the particular value $n = m+2$, viz.,

$$I_{1,\,m+2} > g/(m+2).$$

Since, as he says, the process can now be repeated indefinitely, Arzelà infers the existence of a sequence of sets of intervals, each set contained in the preceding set. He then asserts that a sequence of single intervals exists, one from each set, and each interval contained in the preceding interval of the sequence. As the sets of intervals are no longer finite, such an assumption would need proof even if it appeared that their contents had a positive lower limit. There is, however, nothing in Arzelà's argument to show that the contents do not diminish without limit, and the example on p. 19 of the present paper shows that, should this happen, the required conclusion would be illegitimate.

---

* *Loc. cit.*

† "Sulle Serie di Funzioni," Parte prima, *R. Acc. d. Sc. di Bologna*, 1899.

PART II.—SETS OF POINTS.

5.

I now proceed to show how to replace the intervals of the preceding part of the paper by closed sets of points of positive content. Our sets of intervals, when infinite in number, will then become open sets of points. The theorems I am about to obtain will therefore contain the earlier ones as special cases.

That we can do this is very instructive, and suggests at once the possibility of extending the theory of content to open sets. From our present point of view, we may say that we have indeed already ascribed content to an open set of points, viz., the points of an infinite set of intervals.*

The definition already adopted of the content of a set of intervals is the most natural one, and is indeed the only one of any conceivable use; it would certainly not be reasonable to substitute for it the content of the set of points got by closing it, which may be the whole continuum even when the content of the intervals is as small as we please. There would appear therefore to be no sufficient reason for defining the content of an open set to be that of the set got by closing it. Moreover, in the important special case in which the open set is expressible as the limit of the sum of n closed sets when n is infinite, we are led to define its content as the limit of the content of the sum.

Whether such a definition is logically valid, and whether it agrees with our previous notions of the properties of content as gained from the study of closed sets, requires of course discussion, as also the further question whether it is possible to extend the definition so as to embrace other kinds of open sets. I shall return to the subject in the third part of the paper. The theorems about to be proved will then be required.

6.

LEMMA 1.—*If G_1 and G_2 be two closed sets of points having no point common, the set consisting of G_1 and G_2 together is a closed set of content equal to the sum of their contents.*

This follows at once from the fact that the points of G_2 must, in this case, be internal to a finite number of the black intervals† of G_1.

* *Loc. cit.*

† That is, the intervals free of points of G_1, except that their end points belong to G_1.

LEMMA 2.—*If a closed set of content I contain a closed component of content J, it contains a closed component of content $I-J-\epsilon$ (where ϵ is as small as we please), having no point common with the former component.*

By the preceding lemma no closed component could have content greater than $I-J$.

Let e be any assumed small quantity, and let us shut up all the points of the given closed component in a finite number of intervals of content lying between J and $J+e$. The points of the given set which are not *internal* to these intervals form, as is easily seen, a closed set; if the content of this latter set were less than $I-J-e$, we could enclose all its points in a finite number of intervals of content less than $I-J-e$, which together with the intervals first described would form a set of a finite number of intervals of content less than I, enclosing all the points of a closed set of content I; which is impossible. Thus the content of the closed component in question is not less than $I-J-e$; which proves the lemma.

LEMMA 3.—*If G_1 and G_2 be two closed sets of points of content I_1 and I_2, (a) the set consisting of all the points common to G_1 and G_2 is a closed set, say G' of content I', and (b) the set consisting of all points belonging to one or both of G_1 and G_2 is a closed set, say G of content I. Further,* (*c*) $I_1+I_2 = I+I'$.

For (*a*), if P be a limiting point of G', it is a limiting point both of G_1 and G_2, and therefore a point of both, that is a point of G'; so that G' is closed.

(*b*) If P be a limiting point of G, it must be a limiting point of one at least of G_1 and G_2, and is therefore a point of that one, and therefore a point of G; so that G is closed.

(*c*) By Lemma 2, G consists of the closed set G_1 and a complementary component containing closed sets of content as near $I-I_1$ as we please, but not any whose content exceeds $I-I_1$. Since this complementary component is also the complementary component of G' with respect to G_2, by the same lemma, it contains closed sets of content as near as we please to I_2-I', but none whose content exceeds I_2-I'. Hence $I-I_1 = I_2-I'$, which is equivalent to the statement to be proved.

In the proofs of the above lemmas I have for convenience employed the term "interval"; a moment's consideration, however, shows us that nowhere has the assumption been made that the sets of points are linear. In other words, the lemmas are true for closed sets of points in space of any number of dimensions. It may be remarked at once that in space of

more than one dimension there is no gain in simplicity in considering sets of regions: in such a space the conception of a set of points replaces with advantage that of a set of regions.

7.

I now proceed to give for sets of points the theorems analogous to those proved in Part I. for sets of intervals.

THEOREM 1'.—*Given a countably infinite series of closed sets of points, G_1, G_2, ..., such that each point of G_{n+1} is also a point of G_n, there is at least one point common to all the sets, and the common points form a closed set.*

THEOREM 2'.—*If to the hypothesis of Theorem 1' we add that the content of each G_n is greater than some positive quantity g, the common points form a closed set G' of content $\geqslant g$; so that they have the potency c.*

As already remarked, Theorem 1' is a known theorem. The proof of Theorem 2' is as follows:—

If possible, let the content be I', where I' is less than g. Denote by I_1, I_2, ... the contents of G_1, G_2, ... By Lemma 2 we can find a closed component of G_1, all of whose points are distinct from those of G', and whose content is $I_1 - I' - e$, where e is a positive quantity, smaller than some assigned quantity. This set has in common with G_2 a closed set, whose content, by Lemma 3, is equal to $I_1 - I' - e + I_2 - K$, where K is the content of the closed set constituted by G_2 and the closed component of G_1 above found, and is certainly less than I_1. The content of this component of G_2 is therefore greater than $I_2 - I' - e$; *a fortiori*, greater than $g - I' - e$.

In other words, we have found a component of G_2 which is closed and has no points in common with G', whose content is greater than $g - I' - e$.

We can therefore repeat the argument, and obtain in each succeeding set such a closed component, each component lying inside the one previously obtained. It follows then, by Theorem 1, that there are points other than G' common to all the given sets, contrary to the hypothesis. Therefore, &c. Q. E. D.

8. *Open Sets.*

THEOREM 3'.—*Given a countably infinite series G_1, G_2, ... of sets of points such that the upper limit I_n of the content of closed components in G_n is greater than a positive quantity g, the same for all values of n,*

each set G_n being contained in the foregoing G_{n-1}, then a set of points exists of potency c, common to all the sets, and this set contains closed components of content greater than $g-e$, where e is as small as we please.

By the definition of I_n, we can find a closed component G_1' of G_1 such that, its content being denoted by I_1', $I_1-\frac{1}{2}e < I_1' \leqslant I_1$; and, for all values of n greater than 1, we can, in like manner, find a closed component G_n'' of G_n such that, its content being denoted by I_n'',

$$I_n - \frac{1}{2^n} e < I_n'' \leqslant I_n.$$

Those points of G_2'' which belong to G_1' form a closed set, whose content is greater than $I_2-\frac{1}{2}e-\frac{1}{4}e$ [since the set consisting of all the points belonging to one or both of the sets G_1' and G_2'' is a component of G_1, so that its content is not greater than I_1, by Lemma 3; therefore, the content of the set common to G_1' and G_2'' is greater than

$$I_1-\tfrac{1}{2}e+I_2-\tfrac{1}{4}e-I_1 \quad \text{or} \quad I_2-\tfrac{1}{2}e-\tfrac{1}{4}e].$$

Let us denote this closed component of G_2 by G_2'. *Then G_2' is contained in G_1', and has content greater than $g-e$.*

Similarly we can determine a closed component G_3' of G_3'' and G_2', of content greater than $g-e$; generally we determine successively closed components of each G_n'' and G_{n-1}', of content greater than $g-e$.

Applying Theorem 2′ to these sets G_n', the result follows.

THEOREM 4′.—*Given an infinite number of sets of points $G_1, G_2, \dots$, components of a closed set of finite content* L, such that the upper limit of the contents of the closed components of G_n is greater than some positive quantity g, the same for all values of n, then an infinite series of these sets exists, having in common a set of points of potency c, the content of whose closed components has an upper limit $\geqslant g$.*

Let us choose out a closed component of each set of content greater than g, and let these be denoted by $G_1', G_2', \dots$. Let e be any small positive quantity, and let the integer m be determined such that

$$mg \leqslant L < (m+1)g. \tag{1}$$

* It will be seen that it is sufficient if L is the upper limit of the content of closed sets in the whole set, which does not need to be closed; this is brought out in the re-statement of this theorem as Theorem 7.

Let us consider a group of n of the closed sets G', where n is a sufficiently large integer, subsequently to be further specified.

The points common to any particular pair of the sets of the group form a closed set of points (Lemma 3); therefore, since the sum of any finite number of closed sets is a closed set, the points common to at least two of the sets of the group form a closed set: let us denote it by $G_{1,n}$, and its content by $I_{1,n}$.

The points of $G_{1,n}$ which belong to any particular set of the group G' form a closed component of $G_{1,n}$, whose content is therefore less than or equal to $I_{1,n}$; by Lemma 2, therefore, there is a closed set of content greater than $g-I_{1,n}$, consisting entirely of points belonging to no set of the group, except G'. Corresponding to each of the n sets G' we can find such a closed component, and they will have no common points; so that they form a closed set of content greater than $n(g-I_{1,n})$, by Lemma 3. Hence, by (1), $n(g-I_{1,n}) < (m+1)g$; and therefore

$$I_{1,n} > \{1-(m+1)/n\}\, g. \tag{2}$$

Thus the set $G_{1,n}$ certainly exists, and has the potency c, for all values of n greater than $m+1$.

Let us determine an integer n' such that $(m+1)/n' < \tfrac{1}{2}e$, that is, $n' > 2(m+1)/e$. Then

$$I_{1,n'} > (1-\tfrac{1}{2}e)\, g. \tag{3}$$

Grouping our sets G' together in distinct groups of n' sets, and taking the sets of points belonging to at least two sets of each in turn of these groups, say G_1'', G_2'', ..., we have the same conditions as before, only the content of each closed set is now greater than $(1-\tfrac{1}{2}e)\, g$, instead of g.

Repeating on these sets the process just gone through, we obtain sets of quadruple points of the original sets whose content $I_{2,n}$ satisfies the inequality

$$I_{2,n} > \{1-(m+1)/n\}(1-\tfrac{1}{2}e)\, g. \tag{4}$$

Thus sets $G_{2,n}$, consisting of points common to at least 2^2 of the given sets, certainly exist, and have the potency c, for all values of n greater than $m+1$.

As before (using $\tfrac{1}{2}e$ instead of e), we can then determine n'' so that

$$I_{2,n''} > (1-\tfrac{1}{4}e)(1-\tfrac{1}{2}e)\, g > (1-\tfrac{1}{2}e-\tfrac{1}{4}e)\, g > (1-e)\, g. \tag{5}$$

This process can be continued *ad infinitum*, and at each stage we see that *there are sets $G_{r,n}$ (consisting of points common to at least 2^r of the given sets), of potency c, and of content greater than $(1-e)\, g$, where e is as small as we please.*

Now the set, in general open, consisting of *all* the points belonging to at least 2^r of the given sets is certainly contained in the set consisting of all the points common to at least 2^{r-1} of the given sets, and, by the above, these sets satisfy the other condition of Theorem 3, ($g-e$ being substituted for g). Hence, by Theorem 3, the result follows.

PART III.—ON THE GENERAL THEORY OF CONTENT.

9.

We have seen that, in the case of an open set, the upper limit of the content of closed components plays a most important *rôle*. In the lemmas and theorems relating to open sets, enunciated and proved, this concept has to them precisely the relation that content itself has to closed sets. With Lebesgue, I shall call it *the inner measure of the content* or briefly *(inner) content of the open set.*

DEFINITION.—*The (inner) content of a set is defined to be the upper limit of the content of its closed components.*

The introduction of this term simplifies the statements of the lemmas and theorems of Part II.: thus Lemmas 1 and 2 can be replaced by the following simple proposition:—

THEOREM 5.—*If a closed set G be the sum of two non-overlapping sets, one at least of which is closed, the content of G is the sum of the (inner) contents of the components.*

Theorems 2′ and 3′ are replaced by the following:—

THEOREM 6.—*Given a countably infinite series of sets of points, whose (inner) contents have a positive lower limit g, such that each set is contained in the preceding set, then a set of points of potency c exists, common to all the sets, and the (inner) content of this set is g.*

Theorem 4′ is replaced by the following:—

THEOREM 7.—*Given an infinite number of sets of points, components of a set of finite (inner) content, the (inner) contents of these sets having a positive lower limit g, then an infinite number of these sets exists, having in common a set of (inner) content $\geqslant g$.*

10.

The (inner) content, so defined, is certainly a magnitude, and, in the case of a closed set, the (inner) content is the content itself. The question arises whether the (inner) content possesses all the properties

which we are accustomed to associate with the term "content" as long as this term was confined to closed sets. First, we ask, *is the (inner) content of the sum of two non-overlapping sets always equal to the sum of their (inner) contents ?*

All that has been proved in the preceding sections is that this is the case provided the sum of the two sets as well as one of the components is closed. We can, however, at once extend the result to the case when the sum is open. In other words—*Even if the sum of two non-overlapping sets be open, its (inner) content is the sum of their (inner) contents, provided one at least of the components is closed.**

For, if the content of the closed component be a, and the (inner) content of the sum $a+b$, we can, by the definition, find a closed component of content $a+b-e$, where e is as small as we please. The part common to these two closed components must have content $\geqslant a-e$, and $\leqslant a$ [since, otherwise, the remaining component of the first closed component would have (inner) content $> e$, and we could therefore find in it a closed component having no point common with that of content $a+b-e$, and these two together would form a closed component of the whole set of content $> a+b$].

In the closed component of content $a+b-e$ there must then, by Theorem 5, be another distinct component of (inner) content $\geqslant b-e$ and $\leqslant b$. This being true for all values of e, it follows that the (inner) content of the original open component is not less than b. But it cannot be greater than b, since otherwise we could find a closed component which with the first given component would form a closed set of content greater than $a+b$. Thus the second component has (inner) content b; which proves the theorem.

Summing up the result so far, we have the following theorem :—

The (inner) content of the sum of two non-overlapping sets, one of which is closed, is the sum of their (inner) contents.

11.

Two cases remain :—The sum of two non-overlapping open sets is (1) closed ; or (2) open.

If we assume that what we may, for shortness, call the (inner) addition theorem holds in Case (1), it is easy to deduce that it holds in Case (2).

For, if I be the (inner) content of the sum, and a and b of the com-

* This theorem takes us at once beyond the range of Lebesgue's investigations.

ponents, we can find a closed component of the sum, of content $I-e$, and this cuts out of the components two (open) sets, whose contents, by the same argument as before, lie between a and $a-e$, and b and $b-e$, respectively. The sum of these two (open) sets, being closed, has, under the supposition that the (inner) addition theorem holds in Case (1), content lying between $a+b-2e$ and $a+b$, but must be equal to $I-e$. Since this is true for all values of e, the result follows.

12.

We are now left with the discussion of Case (1). By means of the theorems of the present paper, we can reduce the problem of determining whether in this case the (inner) addition theorem holds to the following: —*Can the sum of two open sets, each of (inner) content zero, be a closed set of positive content?*

To show this we proceed as follows:—

Suppose, if possible, we have a closed set of content $a+b+c$, and it can be divided into two open components, whose (inner) contents are b and c respectively. In these open components there exist closed components of content $b-e$ and $c-e$ respectively, where e is as small as we please; the content of their sum is then $b+c-2e$. The remaining points of the whole set form a set, in general open, whose (inner) content, by Lemma 2, is $a+2e$, and which is the sum of two non-overlapping sets, the (inner) content of each of which is, by what has been proved, not greater than e. Hence, by the usual argument, we can find a *closed* component of the whole set of content $a+e$, which is the sum of two non-overlapping sets, the sum of whose (inner) contents is not greater than e. With respect to these sets we can now repeat the argument, using $\frac{1}{2}e$ instead of e, and so on. Ultimately, by Lemmas 2′ and 3′, we shall determine a closed set whose content is a, divided into two non-overlapping components, both of whose (inner) contents are zero.

With our present imperfect knowledge of open sets, it seems to me impossible to assert definitely that such a case could not arise. I can only say that I do not know of any such case.

In the next section it is shown that the (inner) addition theorem holds when one of the components is any set whatever and the other component is any one of a large class of open sets.

13.

We begin with a few preliminary definitions.

DEFINITION 1.—*If G_1, G_2, ... be a series of sets of points such that, for all values of n, G_n is contained in G_{n+1}, and G be the set such that*

every set G_n is contained in G, while every point of G belongs to some definite G_n, G is said to be the (generalised) outer limiting set of the series.

DEFINITION 1′.—If G_1, G_2, ... be all *closed* sets, G is said to be an *ordinary outer* limiting set.

In the case of such an ordinary outer limiting set,* the upper limit of the content of its closed components is the limit of the content of G_n. Thus, with our present definition, *the (inner) content of an ordinary outer limiting set is the limit of the content of G_n.*

We shall see that the word "(inner)" is here superfluous.

DEFINITION 2.—*If, instead of being contained in G_{n+1}, G_n contains G_{n+1}, the set consisting of the points common to all the sets G_n is called a "generalised inner limiting set."*

DEFINITION 2′.—*If each set G_n consists of all the points of a set of open or closed intervals, G is called an "ordinary inner limiting set."*

I may here call attention to the fact that in my former investigations I used the term "inner limiting set" only for the case where G_n is a set of *open* intervals. It is easily proved that, if the intervals be taken closed, at most a countably infinite set of new points are introduced.†

By Theorem 3′ or 6 *the (inner) content of a generalised inner limiting set is the limit of the (inner) content of G_n*; and, since the content of a set of intervals is the same whether the intervals be closed or not, an ordinary limiting set evidently has the same content whether the intervals be closed or not. Here again the term "(inner)" will be seen to be superfluous.

The process of forming a generalised inner limiting set from the defining series I have elsewhere called *deduction.*

THEOREM 8.—*If the (inner) addition theorem for the (inner) contents holds when one of the two components is a set of a certain type, it is also true when one of the components is the outer limit of sets of that type.*

In other words, if, for every value of n, the addition theorem holds for G_n and *any* other set O_n, it holds for G and any other set O. For let the sum of G and O be H, and let O_n be the set which added to G_n

---

* *Quart. Jour. of Pure and Applied Math.*, No. 138 (1903), p. 191.

† This was proved in my first paper on the subject, "Zur Lehre der nicht abgeschlossenen Punktmengen," *Ber. d. K. Sachs. Ges. d. Wiss. zu Leipzig*, August 1, 1903, p. 290.

makes up H. Then, using the letters indiscriminately for the sets and their (inner) contents, $G_n + O_n = H$.

Now, as G_n increases towards G, O_n diminishes towards O, each O_n lying in the preceding O_{n-1}. Therefore, by Theorem 8′, the (inner) content O is itself the limit of the (inner) content O_n. Also $\text{Lt}\, G_n + \text{Lt}\, O_n = H$; therefore $\text{Lt}\, G_n + O = H$.

Now the (inner) content of G is evidently not less than the limit of G_n. Therefore $G+O \geqslant H$, the letters denoting (inner) contents. But, evidently, $G+O \leqslant H$; therefore $G+O = H$, the letters denoting either sets or their contents.

COR. 1.—*The (inner) addition theorem holds when one of the components is an ordinary outer limiting set, by the conclusion of Art.* 10.

COR. 2.—*An outer limiting set which is the limit of a sequence of sets each of which has the property that the (inner) addition theorem holds for it and any other set whatever has for (inner) content the limit of the (inner) contents of the sets of the sequence.*

COR. 3.—The theorem

$$\text{Lt (inner) content} = \text{(inner) content of Lt}$$

is true for a sequence of expanding open sets when the expansion is due *to the increase of a component for which the (inner) addition theorem holds.*

THEOREM 9.—*If the (inner) addition theorem is true when one of the two components is a set of a certain type, it is also true when one of the components is an inner limiting set deduced from an infinite series of sets of this type.*

In other words, if, for all values of n, $G_n + O = H_n$ (the letters being used indiscriminately for a set and its content), and each G_n is contained in the preceding set G_{n-1}, then $G+O = H$. In fact, H is itself an inner limiting set, and therefore its (inner) content is

$$\text{Lt}\, H_n = \text{Lt}\, G_n + O = G + O.$$

COR.—*The (inner) addition theorem holds if one of the components is an ordinary inner limiting set.*

Proof.—An infinite set of closed intervals is a special case of an ordinary outer limiting set, and therefore, by Cor. 1 to Theorem 8, the (inner) addition theorem applies when one of the components consists of the points belonging to such a set of intervals. Hence, applying our present theorem, it holds for the deduced set of a sequence of such sets of intervals. Q. E. D.

Applying the results of this section, we see that, if we keep applying in any order Theorems 8 and 9 to any series of ordinary outer or inner limiting sets, the sets so obtained must always have the property in question.

We have thus already obtained a large class of open sets possessing the property in question; we can, however, extend this class still further.

THEOREM 10.—*If each of two sets which do not overlap belong to this class, their sum also possesses the property in question.*

Let G_1 and G_2 be two such sets, and H any other set whatever. Also let G be the sum of the two sets. Let the (inner) contents of G_1, G_2, and G be denoted by I_1, I_2, and I, and that of H by J. Then, since G_1 belongs to the class, we have at once $I_1+I_2 = I$. For the same reason I_1+J is the (inner) content of (G_1+H). Hence also, since G_2 belongs to the class, the (inner) content of $(G_1+H)+G_2$ is I_1+J+I_2, *i.e.*, it is $I+J$. In other words, the (inner) content of $G+H$ is $I+J$. Therefore, &c.,

Q. E. D.

THEOREM 11.—*If each of two sets one of which is a component of the other belong to the class, so does their difference.*

Use the same notation as in the preceding theorem, G denoting the larger of the two sets, and G_1 say, the component belonging to the class. As before, $I = I_1+I_2$. We have to prove that G_2 belongs to the class. Suppose this is not the case; then the (inner) content of G_2+H must be greater than I_2+J, say I_2+J+k. But, by hypothesis, G_1 belongs to the class; hence the (inner) content of $G_1+(G_2+H)$ is $I_2+J+k+I_1$, *i.e.* it is $I+k+J$. But G_1+G_2 is G, and G belongs to the class; therefore the (inner) content of G_1+G_2+H is $I+J$; therefore k must be zero. Therefore, &c. Q. E. D.

THEOREM 12.—*If a set belonging to this class be divided into two components the sum of whose inner contents is equal to that of the original set, each of the components belongs to the class.*

Let G be the set, G_1 and G_2 the components, H any other set whatever. Denote the corresponding (inner) contents by I, I_1, I_2 and J. Suppose, if possible, that the (inner) content of G_1+H be not I_1+J; then it must be greater than I_1+J. Add the set G_2 to the set G_1+H. Then the (inner) content of the set of $(G_1+H)+G_2$ would be greater than I_1+I_2+J. But, by hypothesis, $I_1+I_2 = I$; therefore the (inner) content of the set G_1+H+G_2 would be greater than $I+J$; that is, the (inner) content

of the set $G+H$ would be greater than $I+J$. But G belongs to the class in question; therefore the (inner) content of $G+H$ is equal to $I+J$. Therefore, &c. Q. E. D.

THEOREM 13.—*If G_1 and G_2 be two sets of this class, of (inner) content I_1 and I_2, (a) the set consisting of all the points common to G_1 and G_2 is a set of this class, say G', of (inner) content I'; and (b) the set consisting of all the points belonging to one or both of G_1 and G_2 is a set of this class, say G, of (inner) content I; further, (c) $I_1+I_2 = I+I'$.*

For suppose the (inner) contents of the parts of G_1 and G_2 which are not common to be I_1-x and I_2-y respectively. Then, since the (inner) addition theorem holds for G_2, $I_2+(I_1-x) = I$. Similarly, since the (inner) addition theorem holds for G_1, $I_1+(I_2-y) = I$; whence

$$x = y = I_1+I_2-I.$$

Also $I'+(I_1-x) \leqslant I_1$; therefore $I' \leqslant x$, that is,

$$I' \leqslant I_1+I_2-I. \tag{1}$$

Again, take in each component a closed set of content greater than I_1-e, I_2-e respectively. Then the common part of these closed sets lies in G', and has therefore content $\leqslant I'$. The sets of points belonging to one or both of these closed sets lies in G, and has therefore content $\leqslant I$. Then, by Lemma 3, $(I_1-e)+(I_2-e) < I+I'$, however small e may be, that is,

$$I' \geqslant I_1+I_2-I. \tag{2}$$

Comparing (1) and (2), we have

$$I_1+I_2 = I+I'. \qquad \text{Q. E. D.}$$

Again, *the (inner) contents of the parts of G_1 and G_2 which are not common are I_1-I' and I_2-I'.* In fact, from the result just obtained, we have $x = I'$. It at once follows, by Theorem 12, *that the sets G, G', G_1-G', G_2-G', all belong to the class in question.* Q. E. D.

The theorems which we have obtained enable us, starting from closed sets, to build up a very extended class of open sets, possessing the property that the (inner) addition theorem holds for any one of them in combination with any set whatever. The great generality of the class obtained suggests the possibility that the (inner) addition theorem holds for all sets without exception. We must be careful, however, not to jump to this conclusion. We have, at most, shown that all known open sets belong to the class in question. All the known operations employed on members

of the class lead to members of the same class: in modern phraseology, they form a *corpus*. If we could assert that there were no other open sets than those formed from closed sets by these processes, we should have settled, once for all, the difficult question of the classification of open sets.

14.

In connection with the class of operations made use of in the last section, the following theorems, which bear also on the classification of open sets, will be of interest, and are needed in what follows.

THEOREM 14.—*An inner limiting set of a sequence of inner limiting sets is an ordinary inner limiting set.*

THEOREM 15.—*An outer limiting set of ordinary outer limiting sets is an ordinary outer limiting set.*

Proof.—Let the sets defining G_n be $G_{n,1}$, $G_{n,2}$, ..., for all values of n. For shortness, let me use the symbol $<$ to mean "is contained in," and $>$ to mean "contains."

Then, when G is a generalised inner limiting set, $G_1 > G_2$. Hence, if $G_{1,r} < G_{2,r}$, we can remedy this by taking, instead of $G_{2,r}$, the common part of $G_{1,r}$ and $G_{2,r}$, which is also a closed set and contains G_2. Doing this for all values of r, $G_{2,r} > G_{2,r+1}$, and $G_{1,r} > G_{2,r}$.

Doing this in succession for the sets defining G_3, G_4, ..., we get the following table:—

$$\begin{array}{ccccccccccc}
G_{1,1} & > & G_{1,2} & > & G_{1,3} & > & G_{1,4} & > & \ldots\ldots & > & G_1 \\
\vee & & \vee & & \vee & & \vee & & & & \vee \\
G_{2,1} & > & G_{2,2} & > & G_{2,3} & > & G_{2,4} & > & \ldots\ldots & > & G_2 \\
\vee & & \vee & & \vee & & \vee & & & & \vee \\
G_{3,1} & > & G_{3,2} & > & G_{3,3} & > & G_{3,4} & > & \ldots\ldots & > & G_3 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \vee \\
 & & & & & & & & & & \vdots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & G.
\end{array}$$

This being so, consider the sequence of closed sets $G_{1,1}$, $G_{2,2}$, $G_{3,3}$, ..., and let their inner limiting set be denoted by G'.

If P be a point of G', it belongs to $G_{m,m}$, for all values of m, and therefore to $G_{m,n}$, for all values of $n > m$, and therefore to G_m, for all values of m; that is, P is a point of G.

If, on the other hand, P is a point of G, we can assign an integer m such that P is a point of G_m, and therefore of $G_{m,m}$, for all values of m;

that is, P is a point of G'. *Thus G is identical with G', and is, as was asserted, an ordinary inner limiting set.*

Next, if G be a (generalised) outer limiting set of outer limiting sets, $G_1 < G_2$. If $G_{1,r} > G_{2,r}$, we can remedy this by taking, instead of $G_{2,r}$, the set consisting of all points belonging to one or both of $G_{1,r}$ and $G_{2,r}$, which is also a closed set and contained in G_2. Doing this for all values of r, $G_{2,r} < G_{2,r+1}$ and $G_{1,r} < G_{2,r}$.

Doing this in succession for all the sets defining $G_3, G_4, \ldots$, we get the following table:—

$$\begin{array}{cccccc}
G_{1,1} < & G_{1,2} < & G_{1,3} < & G_{1,4} < & \ldots\ldots < & G_1 \\
\wedge & \wedge & \wedge & \wedge & & \wedge \\
G_{2,1} < & G_{2,2} < & G_{2,3} < & G_{2,4} < & \ldots\ldots < & G_2 \\
\wedge & \wedge & \wedge & \wedge & & \wedge \\
G_{3,1} < & G_{3,2} < & G_{3,3} < & G_{3,4} < & \ldots\ldots\ldots & G_3 \\
\wedge & \wedge & \wedge & \wedge & & \wedge \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \wedge \\
 & & & & & \vdots \\
\ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots & G.
\end{array}$$

This being so, consider the sequence of closed sets $G_{11}, G_{22}, G_{33}, \ldots$, and let their outer limiting set be denoted by G'. If P be any point of G', we can assign an integer m such that P is a point of $G_{n,n}$, and therefore of G_n, for all values of $n > m$; that is, P is a point of G.

If, on the other hand, P be any point of G, we can assign an integer m such that P is a point of G_m. Then, since G_m is an outer limiting set, we can assign an integer r such that P is a point of $G_{m,r}$. If now $m \geqslant r$, P is a point of $G_{m,m}$; but, if $m < r$, P is a point of $G_{r,r}$: in either case, P is a point of G'.

Thus G is identical with G', and is, as was asserted, an ordinary outer limiting set.

These theorems belong to a class of theorems of the same kind, bearing on the question of the classification of open sets. I content myself here with giving the following additional theorems.

THEOREM 16.—*The difference of two closed sets is both an ordinary outer and an ordinary inner limiting set.*

First, to prove it is an ordinary outer limiting set. Enclose the smaller closed set in a finite number of open intervals each of length less than e. The points of the larger closed set left over form a closed

set. This closed set, as e decreases without limit, generates the difference of the two given closed sets. Q. E. D.

Next to prove that it is an ordinary inner limiting set. Enclose the larger closed set in intervals each of length less than e. These cover up a finite number of non-overlapping segments. Let d be any one of these segments: then the points of the smaller closed set which lie in d form a closed set, *inside* the black intervals of which lie all the points of the set in question which lie inside d. Taking all such black intervals in all the segments d, we have a set of intervals containing the whole set in question. As we diminish e, we get a series of sets of intervals each lying inside the preceding, and each containing the set in question. The inner limiting set of this series will therefore certainly contain the set in question; but, since each such set of intervals lies inside the corresponding finite number of segments, this inner limiting set is a component of the larger closed set, and contains no point of the smaller closed set; so that the set in question contains this inner limiting set. Thus the set in question is none other than this ordinary inner limiting set. Q. E. D.

THEOREM 17.—*If we subtract a closed set from either an ordinary outer or an ordinary inner limiting set, we still get an ordinary outer or an ordinary inner limiting set.*

In the former case the theorem is a direct consequence of Theorems 15 and 16. In the latter case the difference of the two sets is the ordinary inner limiting set of the parts of the defining intervals of the ordinary inner limiting set that are internal to the black intervals of the closed set.

THEOREM 18.—*If we subtract an ordinary outer limiting set from an ordinary inner limiting set containing it, the difference is an ordinary inner limiting set; and, if we subtract an ordinary inner limiting set from an ordinary outer limiting set containing it, the difference is an ordinary outer limiting set.*

The first part of the theorem is proved in precisely the same way as the second part of the preceding theorem, only that, instead of a single closed set, we have a sequence of closed sets each containing the preceding, and therefore a sequence of sets of black intervals each containing the succeeding.

To prove the second part we proceed as follows:—

Let $D_1, D_2, \ldots$ denote the successive sets of intervals defining the inner limiting set D, and let $P_1, P_2, \ldots$ denote the closed sets of which

$D_1, D_2, \ldots$ are the black intervals; also let $G_1, G_2, \ldots$ denote the closed sets defining the outer limiting set G. The points common to G_n and P_n form a closed set, say K_n, contained in G and having no point common with D; further, given any point of G not belonging to D, we can assign an integer m such that, for all integers n greater than m, that point is a point of P_n (since it is not a point of D), and an integer m' such that, for all integers n greater than m', it is a point of G_n (since it is a point of G); therefore, if m'' denote the larger of m and m', the point is a point of K_n, for all integers n greater than m''. Thus the outer limiting set of the series of closed sets K_n, each one of which evidently contains the succeeding, is the difference $G-D$. Q. E. D.

15.

In Art. 12 I showed that, in the discussion of the question whether, or no, the inner addition theorem holds always, we might confine our attention to sets of zero (inner) content. We may remark that *the general problem of classifying open sets may be reduced to the corresponding problem for sets of zero (inner) content.*

In fact, if we take any open set of (inner) content a, two cases at most can present themselves: either it contains a closed set of content a or it contains closed sets of content as near a as we please. In the former case the given set is the sum of a closed set of content a and an open set of (inner) content zero; in the latter case we may first subtract a closed set of content $a-e$, and so obtain an open set of content e; in this latter set we may subtract a closed set of content e', where e' is as small as we please; and so on. We thus get, by successive subtraction of closed sets, a series of open sets, each lying inside the preceding and having zero for the lower limit of their contents; their deduced set is therefore either altogether absent or has content zero. In the former case the given open set is an ordinary outer limiting set; in the latter case it is the sum of an ordinary outer limiting set and a set of zero (inner) content. In other words, we have the following theorem:—

THEOREM 19.—*Every set of (inner) content a is either a closed set or an ordinary outer limiting set, or is equal to the sum of one or other of these and of a set of zero (inner) content.** As the properties of an ordinary outer limiting set may be regarded as known, this theorem confirms the statement made above as to the classification of open sets.

---

* Cp. Lebesgue, *loc. cit.*, Art. 7.

16.

The definition adopted makes the (inner) content of an open set depend on that universally adopted for a closed set; moreover, we get as the (inner) content the content [for we shall see that we can here suppress the term (inner)] of a certain ordinary outer limiting set contained in it. If we attempt to give a definition of content equally applicable to all sets of points, we are met at once by difficulties which might seem to be insuperable.

The ordinary definition of the content of a closed set is as follows:— Describe little intervals of constant length e round the points of the set: these fill up a finite set of intervals the content of which is, in the limit, when e is indefinitely diminished, the content of the closed set.

If this definition be applied to an open set, it, of course, gives us the same content as that of the set got by closing it, and thus fails to distinguish between the set and its component.

In the definition given of the content of a closed set it is, however, unnecessary to take the intervals all of the same length: not only so; it is not necessary to specify that they have a positive lower limit In fact, if round every point of a closed set we describe a little interval, say $< e$, according to any law, it follows by the extension of the Heine-Borel theorem, since the set is closed, that it will be internal to a finite number of these intervals. The equivalent non-overlapping set will also consist of a finite number of intervals only, and its content, when e is indefinitely diminished, will give us the same quantity as before.

If we try to apply this modified form of the definition of the content of a closed set to open sets in general, we are met by a similar difficulty to that which occurred before. Whereas in the case of a closed set no other points are left in ultimately, when e is indefinitely diminished, this is not true of open† sets, unless they belong to the class of what we have called "ordinary inner limiting sets." Thus, if it be legitimate to ascribe content to an ordinary inner limiting set and to define it in this manner, the process in question, when applied to an open set in general, would gives us the content of an ordinary inner limiting set of which it is a component. With Lebesgue, I shall call the content defined in this manner the outer measure of the content, or, briefly, the "(outer) content."

DEFINITION.—*Round every point of the set G describe a little interval;*

* For simplicity of explanation I confine myself to linear sets of points; it is not difficult to make the necessary modifications of language in the general case.

† For a discussion of the points which must come in, see "On Sequences of Sets of Intervals containing a given Set of Points," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, Part 4, p. 262.

find the content of the set of intervals so formed; this content has a lower limit for the various possible modes of construction; this lower limit is called the "(outer) content of the sets of points."

17. *Measurable Sets.*

For closed sets we know that (inner) and (outer) content are merely different aspects of the same thing, the content of the closed set. Lebesgue uses the term *measurable set* for a set for which the (inner) and (outer) contents coincide; for such a set we may, without scruple, use the term "content."

It is evident that any definition of the content which agrees in the least with our fundamental ideas must make the content of a set greater than, or at least equal to, that of any of its components; so that, if the (outer) and (inner) contents ever do not coincide, the former gives us an upper limit and the latter a lower limit for the content. Thus, in the case of measurable sets no other definition of the content is possible.

Lebesgue proves the following properties of measurable sets in a *finite* segment of the straight line :—(1) The set consisting of all the points belonging to one or more of a finite or countably infinite number of measurable sets is itself measurable; (2) the set consisting of all the points common to a finite or countably infinite number of measurable sets is itself measurable; (3) the contents of measurable sets in combination with one another obey the law of addition; (4) the content of an inner or outer limiting set of measurable sets is the limit of the content of the defining sets; (5) the class of measurable sets has in any finite segment the potency of all possible sets and includes all ordinary sets.

I do not propose to assume any of these results, firstly, because the theorems I require, in so far as they could be deduced from theorems of Lebesgue's, are capable of a direct proof of a simple character; but, secondly, because I have not found it necessary to assume that the region of operation is finite, an assumption without which Lebesgue's proofs* would not be valid; so that the doubt arises whether his results can be assumed to hold when the region of operation is the whole of space or a more than finite portion of it.

From the point of view of an exhaustive classification of open sets, these results of Lebesgue's are not sufficient, unless it can be shown that none but measurable sets exist. This point is still open to question. If there are other sets, then, as will be shown, all the ordinary sets

* See, for example, Lebesgue, p. 239, line 3.

enumerated and indicated in Lebesgue's paper are included in a class which is included in the class of measurable sets, but may consist of only a part of it: this class has itself the potency of all sets in any segment finite or infinite, and, from the point of view of content, possesses most important characteristics; this is none other than the class of sets which in combination with *any other set whatever* are such that the sum of the (inner) contents is the (inner) content of the sum, and the sum of the (outer) contents is the (outer) content of the sum. It will be noticed that all that Lebesgue has proved for measurable sets is that this is true of measurable sets in combination with other measurable sets. I shall, for definiteness, allude to the class of sets for which the (inner) addition theorem holds as the (inner) additive class, and that for which the (outer) addition theorem holds as the (outer) additive class; the class above referred to will then belong to both these classes, and I call it *the additive class.*

Theorem 8 of the first part of this paper shows that for an ordinary inner limiting set the (outer) content coincides with the (inner) content; it shows, moreover, that, in the case of an ordinary inner limiting set, however we construct the intervals round the points of that set, the content of those intervals always approaches the same limit when the intervals are decreased without limit, viz., the content of the ordinary inner limiting set, provided ultimately no points are left in except those of the given inner limiting set.

In the case of a set which is not an inner limiting set we cannot so construct the intervals that no other points are left in, and there might seem to be a certain degree of arbitrariness in the selection of those points which are to be admitted.

According to the law of construction adopted, we may, as the length of the separate intervals is indefinitely decreased, approach the actual lower limit, that is the (outer) content, or some other quantity lying between this and the content of the set got by closing the given set.

If I be the (inner) content of a set, it is evident that the set cannot be enclosed in a set of intervals of content less than I; thus the defining property of measurable sets may be expressed by saying that *a set of (inner) content I is measurable if, and only if, it can be enclosed in a set of intervals of content $I+e$, where e is as small as we please.* This property is, as we saw, possessed *par excellence* by ordinary inner limiting sets. It is remarkable that it is also possessed by ordinary outer limiting sets, though, except in particular cases, an ordinary outer limiting set cannot be defined as the inner limiting set of a sequence of sets of intervals.

To prove the property in question, we begin by proving it for the special case when the ordinary outer limiting set is the difference of two closed sets.

Let the contents of the two sets be I_1 and I_2; so that the (inner) content of their difference is I_1-I_2.

First, let us enclose the larger set in a finite number r of intervals, whose sum is I_1+e. Let any one of these intervals be denoted by d.

The points of G_2 which lie in d form a closed set: let it be denoted by G_2', and its content by I_2'; so that $\Sigma G_2' = G_2$ and $\Sigma I_2' = I_2$. The black intervals of G_2' inside d have content $d-I_2'$, and *inside* these lie all those points of G_1-G_2 which lie in d. Thus all the points of G_1-G_2 lie inside all these intervals in the r intervals d, whose sum is $\Sigma d-\Sigma I_2' = I_1+e-I_2$, which proves the theorem in this case.

To deduce the theorem in the general case we proceed as follows:— Suppose the set to be the limit of G_n, when n is infinite. Let the content of G_n be I_n. Shut up G_1 in a finite number of intervals of sum $I_1+\frac{e}{2}$; G_1-G_2 in a set whose content is $I_2-I_1-\frac{e}{2^2}$; and so on. Evidently, in this way, we get an infinite set of intervals, in general overlapping, containing all the points of the set G, whose content is therefore certainly not greater than $\mathrm{Lt}\, I_n+e$, that is $I+e$; so that G is measurable. Thus we have the theorem:

An ordinary outer or inner limiting set is measurable, that is, if its content be I, it can be shut up in an infinite set of intervals whose content lies between I and $I+e$, and it contains closed components of content lying between $I-e$ and I, where e is as small as we please.

We might consider in detail all the sets obtained from open sets by means of the processes of Art. 15, and prove that they all possess this property. The following theorem, however, proves not only this, but that all sets belonging to what I have called the inner additive class possess this property.

THEOREM 20.—*If a set is such that when added to any other set whatever which has no points in common with it the sum of the (inner) contents is the (inner) content of the sum, the set in question is measurable.*

Let I_1 be the (inner) content of the set, and I_2 be that of the set of points required to close it, and I that of the whole set so obtained; then, by hypothesis, $I = I_1+I_2$. As usual, let the sets whose contents are I, I_1, and I_2 respectively be denoted by G, G_1, and G_2.

Take a closed component G_2' of content $> I_2-\frac{1}{2}e$ in G_2. The set G_1 lies, of course, in the black intervals of this set. Next shut up the set G in a finite number of intervals $d_1, d_2, \ldots, d_n$, of content $< I+\frac{1}{2}e$.

In any one of these intervals d_r, the points of G_2' form a closed set, of content I_r' say, where $\sum_1^n I_r' > I_2-\frac{1}{2}e$.

The points of G_1 which lie in d_r lie in the black intervals of this closed component of G_2', that is, in intervals whose sum is d_r-I_r'. Thus *all* the points of G_1 are enclosed in a set of intervals whose sum is

$$\sum_1^n \{d_r-I_r'\} < I+\tfrac{1}{2}e-I_2+\tfrac{1}{2}e < I_1+e.$$

This, therefore, proves the theorem.

It is easy to see that, if a set does not belong to the (inner) additive class, we can no longer assert that it possesses the property in question. Take, for example, a closed set of content a, and suppose it, if possible, divided into two components which do not belong to the (inner) additive class, so that the sum of their (inner) contents is less than a. Then, if both these components have the property in question, we could enclose the closed set in an infinite set of intervals whose sum is less than a, and therefore in a finite number of these intervals; which is impossible. Thus at least one of the components cannot have the property in question.

We have not, however, proved that, if there are sets which do not belong to the (inner) additive class, they may not be further sub-divided into those which are and those which are not measurable.

18. *The (Outer) Content.*

The properties which we have found for the (inner) content have their exact counterparts for the (outer) content; so that we cannot say that either concept seems more fundamental than the other.

A set of (outer) content J is evidently measurable if, and only if, it contains closed components of content $J-e$, where e is as small as we please.

That this is the case when the set belongs to what I called the (outer) additive class is shown as follows; the theorem is the counterpart to Theorem 20.

THEOREM 21.—*If a set be such that, when added to any set whatever*

having no point common with it, the sum of the (outer) contents is the (outer) content of the sum, the set in question is measurable.

As before, let G_1 be the set, G_2 the set required to close it, and G the sum of G_1 and G_2, and let the corresponding (outer) contents be J_1, J_2, and J.

Let us enclose G in a finite number of intervals of content lying between J and $J+e$, and G_2 in a set of intervals of content lying between J_2 and J_2+e.

The points of the former intervals which are not internal to the latter intervals form a closed set of content lying between $J-J_2-e$ and $J-J_2$; that is, between J_1-e and J_1, by the hypothesis. The points of this closed set which also belong to the closed set G form a closed component of G, which, since it has no point common with G_2, is also a closed component of G_1. Let its content be denoted by K; then we can enclose it in a finite number of intervals of content less than $K+e$, and these, together with the intervals constructed round G_2 contain all the points of G; hence $K+J_2+2e \geqslant J_1+J_2$, that is $K \geqslant J_1-2e$, which proves the theorem.

COR.—The sets of the additive class are all measurable.

It is easily seen that Theorem 5 holds if for (inner) we substitute (outer). Corresponding to Theorem 6 we have the following:—

THEOREM 6′.—*The (outer) content of a generalised outer limiting set is the limit of the (outer) content of the defining set G_n.*

Let J_n be the (outer) content of G_n and J of the outer limiting set G, and let us denote the limit of J_n when n is indefinitely increased by j. It is evident that, as each G_n is contained in the following G_{n+1}, the quantities J_n never decrease, and j is their upper limit.

Let us commence at such a set G_1 that, e being any small positive quantity, $j-e \leqslant J_n \leqslant j$, for all values of n, and let $e_1+e_2+\ldots < e$. Enclose G_n in a set of intervals of content less than J_n+e_n, for all values of n.

Then the parts common to the $(n-1)$-th and n-th sets of intervals contain G_{n-1}, and must therefore have content $\geqslant J_{n-1}$. Thus, if we take all the intervals together which we have constructed, we have a set of overlapping intervals containing every point of G, and their content is less than or equal to $(J_1+e_1)+(J_2-J_1+e_2)+\ldots+(J_n-J_{n-1}+e_n)+\ldots$, that is, less than $j+e$. Thus $J<j+e$. But J cannot be less than j; for otherwise we could enclose G in a set of intervals of content less than j, which is evidently impossible. Thus $J=j$. Q. E. D.

COR.—From Theorems 6 and 6′ the theorem follows that *an outer or inner limiting set of measurable sets is measurable and has for content the limit of the contents of the defining sets.*

Corresponding to Theorem 7 we have the following:—

THEOREM 7′.—*Given an infinite number of sets of points, components of a set of finite (outer) content L, the (outer) contents of these sets having a positive upper limit g, then an infinite number of these sets exists, which can all be enclosed simultaneously in a set of intervals of content $< g+e$, where e is as small as we please.*

If more than a finite number of the sets have zero (outer) content, the theorem is obviously true; we assume therefore that this is not the case; then there is certainly at least one proper upper limit $g' \leqslant g$ such that, for all values of e, there are a more than finite number of the sets whose (outer) contents lie between $g'-e$ and g', both inclusive.

This being so, let us replace the sets by ordinary inner limiting sets containing them, having the same (outer) content and contained in an outer limiting set of content L,* and let $G_1, G_2, G_3, \ldots$ be a countable set of these ordinary inner limiting sets such that, if the content of G_n be denoted by I_n, $g' \geqslant I_n > g' - \frac{e^n}{2^{n+1}}$.

Then, since an ordinary inner limiting set has the same (inner) and (outer) content, we can, since they are all contained in a set of content L, and have content $> g' - \frac{e}{2^2}$, apply to these sets the result of Theorem 4′, that is, there must be a countable number of them, say, in order, $G'_1, G'_2, G'_3, \ldots$, having in common a set of (inner) content $\geqslant g' - \frac{e}{2^2}$, and therefore containing an ordinary outer limiting set of content $\geqslant g' - \frac{e}{2^2}$. Let us denote this latter by C_1.

Similarly, there must be a countable number of the sets $G'_1, G'_2, \ldots$, whose contents are greater than $g' - \frac{e}{2^3}$, and among these we can find a countably infinite set $G''_2, G''_3, \ldots$, having in common a set of (inner) content $\geqslant g' - \frac{e}{2^3}$, and therefore containing an ordinary outer limiting

* It is easy to see how to do this; we can enclose each of the sets in a set of intervals of content within e of its content, and the whole set in a set of intervals of content lying between L and $L+e$; if we now omit any parts of the former intervals external to the latter intervals, and let e describe a sequence having zero as limit, we get the sets above referred to.

set of content $\geqslant g' - \frac{e}{2^3}$. Let us call this C_2. In this way we obtain a series of the sets $G'_1, G''_2, G'''_3, \ldots$, and a corresponding series of ordinary outer limiting sets $C_1, C_2, C_3, \ldots$, such that C_1 is contained in all the sets $G'_1, G''_2, \ldots, C_2$ in all but the first, C_3 in all but the two first, and so on.

By Theorem 15 the outer limiting set of $C_1, C_2, \ldots$ is an ordinary outer limiting set—let us call it C—and its content is the limit of the content of C_n, that is g'.

Now, since G'_1 and C_1 are both additive sets, their difference has content $\leqslant \frac{e}{2^2}$. Similarly, the difference between G''_2 and C_2 has content $\leqslant \frac{e}{2^3}$, and so on. Thus, if we enclose C in a set of intervals of content $< g' + \frac{1}{4}e$, we shall be able to enclose the remaining points of G'_1 in a set of intervals of content $< \frac{e}{2^2} + \frac{e}{2^3}$, and the remaining points of G''_2 in a set of intervals of content $< \frac{e}{2^3} + \frac{e}{2^4}$, and so on. In this way we enclose simultaneously $G'_1, G''_2, G'''_3, \ldots$ in a set of intervals of content $< g' + e$. These intervals, of course, contain the original sets from which we obtained $G'_1, G''_2, G'''_3, \ldots$; so that this proves the theorem.

19. *The (Outer) Additive Class.*

It is not difficult to show that all closed sets belong to the outer additive class. That the (outer) content of the sum G of two non-overlapping sets G_1 and G_2 is the sum of their (outer) contents, provided both G and G_1 are closed, has already been pointed out as the correlative to Theorem 5; that this is still the case if G is open can be shown as follows.

Let G' be an ordinary inner limiting set containing G and having as content the (outer) content of G, that is I. G' contains G_1 (the closed set), and the other component (which contains G_2), is, by Theorem 17, an ordinary inner limiting set, and has therefore, by what has been proved for the (inner) content, content $I - I_1$; therefore $I_2 \leqslant I - I_1$; but, since G can certainly be enclosed in a set of intervals of content as near as we please to $I_1 + I_2$, we cannot have $I_1 + I_2 < I$; therefore $I_1 + I_2 = I$.

Thus we have the theorem:—

The (outer) content of the sum of two sets which do not overlap is the sum of their (outer) contents, provided one of the component sets is closed.

It does not follow that, if the (outer) addition theorem holds when the

sum is closed, it holds generally. Instead of this, however, if we could assume that it holds when the sum consists of all the points of an interval, we could, as in § 11, show that the theorem would be true generally.

The sum of the (outer) contents of two non-overlapping sets is evidently not less than the (outer) content of the sum; thus the question corresponding to that asked on p. 30 is the following :—

Can a segment of length a be divided into two sets of points the sum of whose (outer) contents is greater than a?

By applying Theorem 6′, we can, precisely as in the corresponding discussion of the (inner) additive class, prove the following :—

THEOREM 8′.—*The (outer) addition theorem holds for an inner limiting set of sets of the (outer) additive class.*

COR. 1.—*The (outer) additive class includes all ordinary inner limiting sets.*

THEOREM 9′.—*The (outer) additive class includes all the outer limiting sets of sets of that class.*

COR.—*This class includes all ordinary outer limiting sets.*

The proof given of Theorem 10 serves, with the mere alteration of the word "(inner)" into "(outer)" to prove the corresponding theorem, viz. :—

THEOREM 10′.—*If each of two sets which do not overlap belong to the (outer) additive class, their sum also belongs to that class.*

Similarly, with the same alteration, and writing "less than" for "greater than" and $-k$ for k, the next proof can be applied, and we get the following :—

THEOREM 11′.—*If each of two sets one of which is a component of the other belong to the (outer) additive class, so does their difference.*

Similarly,

THEOREM 12′.—*If a set belonging to the (outer) additive class be divided into two components the sum of whose (outer) contents is equal to that of the original set, each of the components belongs to that class.*

The proof of the next theorem requires a few more alterations, and is therefore given here at length.

THEOREM 13′.—*If G_1 and G_2 be two sets of the (outer) additive class of (outer) content I_1 and I_2, (a) the set consisting of all the points common to G_1 and G_2 is a set of this class, say G', of outer content I', and (b) the set*

consisting of all the points belonging to one or both of G_1 and G_2 is a set of the class, say G of (outer) content I; further (c) $I_1+I_2=I+I'$.

For suppose the (outer) contents of the parts of G_1 and G_2 which are not common to be I_1-x and I_2-y respectively. Then, since the (outer) addition theorem holds for G_2, $I_2+(I_1-x)=I$; similarly, since the (outer) addition theorem holds for G_1, $I_1+(I_2-y)=I$; whence

$$x=y=I_1+I_2-I.$$

Also $I'+(I_1-x)\geqslant I_1$; therefore $I'\geqslant x$, that is,

$$I'\geqslant I_1+I_2-I. \tag{1}$$

Again, take inner limiting sets of content I_1 and I_2 respectively containing G_1 and G_2. The common part of these contains G', and has therefore content $\geqslant I'$. The set of points belonging to one or both contains G and has therefore content $\geqslant I$. Therefore, by Theorem 13,

$$I_1+I_2\geqslant I+I'. \tag{2}$$

Comparing (1) and (2), we have

$$I_1+I_2=I+I'.$$

Q. E. D.

Again, the (outer) contents of the parts of G_1 and G_2 which are not common are I_1-I' and I_2-I', since, by the above, $x=y=I'$.

It follows, by Theorem 12′, that the sets G, G', G_1-G', G_2-G' all belong to the class in question. Q. E. D.

20. *The Additive Class.*

The theorems proved enable us without further proof to sum up the chief properties of the additive class.

DEFINITION.—*The additive class consists of all sets which have the property that, if one of them be added to any other set, having no point common with it, the sum of the contents, whether (inner) or (outer), is the corresponding content of the sum.*

(1) The additive class consists entirely of measurable sets, that is, the (inner) and (outer) contents are the same; so that we may properly speak of the content of any additive set.

(2) The additive class includes all closed sets, and ordinary inner and outer limiting sets.

(3) The additive class includes all inner and outer limiting sets of additive sets.

(4) The additive class includes the sum and difference of any two additive sets.

(5) If G_1 and G_2 be two sets of the additive class, their common component G' and the set G, consisting of all the points belonging to one or both of them, both belong to the additive class, and the sum of the contents of the two former sets is the same as the sum of the contents of the two latter sets.

(6) The additive class includes all sets of (outer) content zero or (inner) content infinity, and has therefore in any portion of the straight line the potency of all possible sets.

This last property requires proof.

If E be a set of infinite (inner) content, it is evident that the outer content will also be infinite, and that the sum of E and any other set will contain closed components of content as large as we please, and cannot be enclosed in a set of intervals of finite content; thus E belongs to the additive class. Next, let E be a set of (outer) content zero; then the (inner) content of E must also be zero; so that E is measurable.* Let G be any set of (inner) content a and (outer) content b, having no point common with E. Then $G+E$ can be enclosed in a set of intervals of content as near as we please to b, but not in a set of content less than b; thus b is the (outer) content of the sum. Again, $E+G$ contains closed sets of content as near as we please to a. Suppose it contains a closed set K of content a' greater than a. Let E' be an ordinary inner limiting set containing E and having zero content. Then, since K and E' are both additive sets, their common part K' is additive and has content zero. Therefore $(K-K')$ is additive and has content a'. But $(K-K')$ is a component of G, and G contains no components of content higher than a; so that this is impossible; therefore $E+G$ does not contain any components of content higher than a; so that a is the content of $E+G$. Thus E is additive. Q. E. D.

Now, if F be a perfect set of content zero, any component E of F has (outer) content zero, and belongs therefore to the additive class; but the potency of the components of F is evidently the same as that of all possible sets. This proves the whole of (6).

It is unnecessary to say more to show the importance of this class of sets; it includes all the familiar sets and has all the advantages of Lebesgue's class of measurable sets, while, if there be other than

---

* Cp. Lebesgue, § 6.

measurable sets, it possesses distinct advantages over the class of measurable sets *in toto*. The fundamental property of additive sets embodied in the definition enables us to extend the theory of content to all sets of the additive class without any scruple. The extent to which that theory can be still further extended, on the one hand to the (inner), and on the other to the (outer), additive class, and a step further to all measurable sets, has been now fully discussed. The only point which remains uncertain is whether or no sets other than these exist.

20.

It will be noticed that the additive class includes all countable sets, and that, with the definition of the content of an additive set which I have adopted, we have the theorem that *the (inner) content of every countable set is zero.*

Again, *the content of the set of irrational numbers in any segment of a straight line is that of the segment itself.*

By making use of the theorems of the present paper, we prove not only this theorem, but the more general one for space of any number of dimensions. For the sake of variety, and also because it throws fresh light on the subject, I give an independent proof of the theorem for one dimension.

Take the following construction :—

Divide the segment (0, 1) into m parts, where m is any odd number except unity. Blacken the central part.

Divide each of the $(m-1)$ remaining parts into m^2 parts, and blacken each central part.

Then divide each of the $(m-1)(m^2-1)$ remaining parts into m^3 parts, and blacken each central one; and so on.

The set consisting of the end-points and external points of the set of intervals constructed thus is easily seen to be a perfect set, nowhere dense, whose content is the same as the corresponding H. J. S. Smith's set of the second kind, viz.,

$$1-\frac{1}{m}-\frac{1}{m^2}\left(1-\frac{1}{m}\right)-\frac{1}{m^3}\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m^2}\right)-\dots,$$

which lies between 1 and $1-1/(m-1)$.*

Thus, by suitably choosing m, we can get a perfect set, nowhere

* For $m = 3$, the content, expressed in the ternary scale, is 1 02 212 2000 01001 01 ... .

dense, in the segment (0, 1), whose content is as near as we please to unity. The points of this perfect set are not all irrational, but I will now show how to obtain from it a similar set in which every point is irrational.

A theorem of Scheeffer* asserts that, *given two sets, one closed and nowhere dense, and the other countable, and any two quantities a and b, we can find a quantity c, $a < c < b$, such that, if one of the sets of points be pushed a distance c along the straight line, all the points of the countable set lie inside the black intervals of the closed set.*

Choose as the countable set all the rational numbers between 0 and 1, and as the closed set the perfect set just constructed, so that its content is greater than $1-\frac{1}{2}e$, where e is as small as we please. Then we can find a positive quantity $c < \frac{1}{2}e$, such that, shifting the perfect set to the left a distance c, all its points which remain in the segment (0, 1) become irrational. Since these points form a perfect set nowhere dense of content greater than $1-e$, *we have in this way constructed a perfect set of irrational numbers in the segment* (0, 1) *of content as near as we please to unity.* Q. E. F.

* *Acta Math.*, 5.

ON UPPER AND LOWER INTEGRATION

By W. H. YOUNG.

[Received and Read January 14th, 1904.]

1. The problem of finding the value of an upper or lower integral has rarely been considered except in connection with the light it throws on the simpler one of ordinary integration. It is, however, only a restricted class of functions that admit of integration in Riemann's sense, namely, continuous functions and those pointwise discontinuous functions the content* of whose points of continuity is that of the continuum. For all other functions the problem of upper and lower integration is of primary importance. As far as I know no calculus has as yet been discovered for the evaluation of such generalised integrals.† If it be urged that such functions rarely occur, the answer is immediate: Such functions present themselves in the most unlooked-for places. Thus, take the function which has the value 1 at every point of a closed set whose content is not zero and zero at every other point of the space considered. This function is a pointwise discontinuous function—in point of fact, semi-continuous—belonging to the non-integrable class. If the space considered be of n dimensions, the problem of finding the content of the set is equivalent to that of finding the upper n-ple integral of the semi-continuous function in question.

In the present paper I propose to show how to reduce the general problem of upper and lower integration to that of ordinary integration. If we assume a complete knowledge of the content of sets of points connected with the function, we have, I prove, to carry out merely a single integration. In the general case, I show how to replace an upper n-ple integral by an n-fold upper integral; in other words, a repetition

* In the sense explained in my paper on "Open Sets and the Theory of Content." Cf. also Lebesgue.

† This paper was written simultaneously with the preceding memoir, at a time when the writer was unacquainted with the work of M. Lebesgue. The result of Theorem 2 is in perfect accord with Lebesgue's expression for his integral as the common limit of two difference summations (*Annali di Matematica*, 1902, p. 253); in fact, it is easily shown that, in the case of an (upper) lower semi-continuous function, the Lebesgue integral coincides with the upper (lower) integral. It may be further remarked that, in the general case, the Lebesgue integral may itself be expressed in precisely my form.

In accordance with the alterations made in the preceding memoir (cp. footnote, p. 16), I have made a few verbal alterations in the present paper; I have also elaborated the proof of the final theorem, which, in its original form, was too condensed.—*April 2nd*, 1904.

n times of the process of finding an upper single integral, and similarly for lower integrals. Each stage of the process may then, if we please, be effected by an ordinary integration. In this way we avoid the introduction of the content of any but linear sets. A particular case of this result is that the content of a closed set of points in $n+1$ dimensions can be expressed as an n-fold upper integral, and thus found by means of n single integrations.

The expression for the upper n-ple integral in a region Ω as an ordinary single integral is $K\Omega + \int_K^{K'} Idk$, where I is the content of those points of the region for which the maximum of the function is greater than or equal to k, and the limits are suitably chosen. As a corollary from this and the corresponding theorem for a lower integral, we at once deduce the known necessary and sufficient conditions that an n-ple ordinary integral of a function should exist. Moreover, the expression gives us the content of any closed set in space of n dimensions in the form $\int I dz$, where I is the content of points of an S_{n-1} for which the section of the set by straight lines perpendicular to the S_{n-1} has content $\geqslant z$.

The argument turns on the introduction of auxiliary semi-continuous functions, and the splitting up of the range of the independent variables into suitably chosen sets of points, instead of into ordinary regions, as in the case of integration properly so called.

The success of the introduction of the auxiliary semi-continuous functions is due to the following property of such functions, which I enunciate and prove in the course of the paper:—

The upper n-ple integral in any region in space of n dimensions of an upper semi-continuous function of n variables is always equal to the corresponding n-fold upper integral; that is, it can be found by the process of upper single integration, repeated n times.

With regard to the splitting up of the region considered into sets of points, it appears at once that the process adopted is applicable, with suitable modifications, when the range of the independent variables, instead of being a closed region, is any closed set of points. The formulæ obtained, indeed, furnish us with a new definition of upper and lower integration with respect to any closed set of points, and hence also of ordinary integration with respect to such a set, when this is possible; it gives us, moreover, the necessary and sufficient conditions for the possibility.

2. In the present section I give, for the sake of clearness, a few preliminary definitions and explanations. I first remark that I shall, for convenience, suppose that the regions in which the functions we are

dealing with are defined are finite and simply connected; also that the functions themselves are everywhere finite, and have finite upper and lower limits.

Consider any function. Take any point of the region for which it is defined; describe round this point an n-dimensional sphere having the point as centre; the upper limit of the values of the function in this sphere will itself tend towards a definite lower limit as the radius of the sphere is diminished: we call this the *maximum* of the function at the point. The *minimum* at the point is similarly defined. The excess of the maximum at P over the minimum at P is called the *oscillation* of the function at P. I shall, as usual, call a region *open* if, with any point of the region as centre, we can describe an n-dimensional sphere lying entirely within the region.

Taking any function whatever of n variables, defined for a region of space of n dimensions, divide the region up into any finite number of partial regions, and multiply the content of each such part by the upper limit of the values of the function in that part, and sum for all the parts: the limit of this sum, when the content of each part is indefinitely diminished, and the number of parts accordingly increased, is called the *upper n-ple integral* of the function. The *lower n-ple integral* is similarly defined.

Now suppose, for simplicity of wording, that $n = 3$; and, to further simplify the ideas, let the region considered be a rectangular parallelepiped, having edges a, b, and c along the three axes. Find the upper integral of the function with respect to z, regarding x and y as constant, the limits being 0 and c. Find the upper integral with respect to y of the function of x and y so obtained, regarding x as constant, the limits being 0 and b. Finally find the upper integral with respect to x of the function of x so obtained between the limits 0 and a. This final upper integral I shall call the *three-fold upper integral* of the original function, taken over, or with respect to, the parallelepiped. It is clear how we may generalise this conception, and give the corresponding definition for the *n-fold upper integral* of a function of n variables with respect to any closed n-dimensional region.

The theory of ordinary multiple integration begins by showing that, when an ordinary n-ple integral exists, it is always equal to the n-fold ordinary integral; so that, moreover, the order of integration is immaterial. The corresponding theorem in our case does not hold. It is, however, at once obvious that the upper n-ple integral is greater than or equal to the n-fold upper integral. I shall prove in Art. 8 that the equivalence does exist in the case of upper semi-continuous functions;

a corresponding theorem holds, of course, for lower semi-continuous functions.

The consideration of semi-continuous functions is due to Baire*; they are defined as follows:—

DEFINITION.—*A function is said to be an upper semi-continuous function if its value at every point is the upper limit of the values assumed by the function in the neighbourhood of the point when this neighbourhood is indefinitely diminished.*

A corresponding definition holds for a lower semi-continuous function: we have only to replace the word "upper" by "lower" in the above definition. Baire has shown not only that these functions are pointwise discontinuous, but that they possess the following characteristic property:—

THEOREM.—*The points at which an upper semi-continuous function has a value greater than or equal to k form, for each value of k, a closed set. Similarly the points at which a lower semi-continuous function has a value less than or equal to k form a closed set.*

It is scarcely necessary to add that an upper semi-continuous function actually assumes its maximum in any interval or region, and a lower semi-continuous function its minimum. We note also that a function not otherwise semi-continuous may be so at a particular point.

The Associated Semi-continuous Functions of a Discontinuous Function.

Take any function whatever. At every point of the region for which it is defined the function possesses a maximum, in the sense explained above. This system of values determines, therefore, a new function. I shall call it the *upper limiting function of the given function*, or simply the *associated upper limiting function*. Taking the minimum instead of the maximum, we have in the same way the definition of the *associated lower limiting function*. Finally, taking the excess of the maximum over the minimum—that is, the oscillation—we have a third function, which I shall call the *associated oscillation function*.

It is proved by Baire that the first and third of these functions are upper semi-continuous functions, while the second is a lower semi-continuous function. Baire also proves that a function and its associated upper limiting function have, in any open region, the same maximum. The fact that this is not in general true of a closed region is the explanation of the slightly complicated character of the proof of the theorem of the next article.

* Baire, *Ann. di Mat.* (3), Vol. III. (1899).

3. We can now enunciate the following theorem :—

THEOREM 1.—*The upper n-ple integral of a discontinuous function of any number of variables is unaltered if we replace the discontinuous function by its associated upper limiting function. The lower n-ple integral is in like manner unaltered if we replace the discontinuous function by its associated lower limiting function.*

I give the detailed proof for two dimensions: the proof is, however, of a perfectly general character, and requires at most a few trifling verbal alterations to make it valid for space of any number of dimensions. Assume any small positive quantity e, and let us determine a corresponding e', such that, if the region of integration be divided into a finite number of small regions d, each of linear dimensions less than e', the following two properties hold :—(1) $\Sigma \bar{F} d$ is greater than $\bar{\int} F\,dw$ by less than e, (2) $\Sigma \bar{f} d$ is greater than $\bar{\int} f\,dw$ by less than e: here f is the function, F the associated upper limiting function, $\bar{F}$ and $\bar{f}$ the upper limits of F and f respectively in the region d, dw is the element of area, and $\bar{\int}$ denotes upper integral. This is evidently possible.

Suppose $ABCD$ to be one of the regions d (Fig. 1). Then, $\bar{f}$ being the upper limit of f in $ABCD$, $\bar{f}$ is $\geqslant$ the upper limit of f in the open region $ABCD$, that is, $\geqslant$ the upper limit of F in the same open region, and therefore $\geqslant \bar{F}'$, where $\bar{F}'$ is the upper limit of F in a closed region $A'B'C'D'$, lying inside $ABCD$, but nearly coinciding with it.

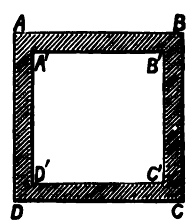


FIG. 1.

Suppose the boundary of $A'B'C'D'$ drawn so that the part of $ABCD$ outside $A'B'C'D'$ (shaded in the figure) has content less than k, where, if M be the upper limit of F in the whole region of integration and m the number of small regions d, $mMk < e$. Then, however we subdivide the shaded region and multiply each part d' by the upper limit of F inside it and sum, the sum will be less than e/m. Thus $\Sigma \bar{F} d$ over the now subdivided regions (*i.e.*, the original m regions divided into shaded and unshaded parts in the manner indicated) lies between $\Sigma \bar{F}'(A'B'C'D')$ and $e + \Sigma \bar{F}'(A'B'C'D')$, But the dimensions of the new regions are still less than e'; therefore $\Sigma \bar{F} d$ over the new regions lies between $\bar{\int} F\,dw$ and $e + \bar{\int} F\,dw$, and may be denoted by $e_1 + \bar{\int} F\,dw$, where e_1 lies between 0 and e. Thus

$$\Sigma \bar{F}'(A'B'C'D') \leqslant e_1 + \bar{\int} F\,dw \leqslant e + \Sigma \bar{F}'(A'B'C'D'). \qquad (1)$$

Now, since, as was pointed out, $\bar{f} \geqslant \bar{F}'$,

$$\Sigma \bar{f}(ABCD) \geqslant \Sigma \bar{F}'(ABCD) > \Sigma \bar{F}'(A'B'C'D'). \qquad (2)$$

Comparing (1) and (2),

$$e_1 + \bar{\int} F dw < e + \Sigma \bar{f}(ABCD);$$

that is, since that latter summation is greater than $\bar{\int} f dw$ by less than e, and e can be made as small as we please,

$$\bar{\int} F dw \leqslant \bar{\int} f dw.$$

But, since F is never less than f at any point x, $\bar{\int} F dw \geqslant \bar{\int} f dw$; therefore $\bar{\int} F dw = \bar{\int} f dw$. Q. E. D.

4. It is clear from the preceding article that we may in discussing upper and lower integrals confine our attention to semi-continuous functions. As, moreover, an upper semi-continuous function becomes a lower semi-continuous function if its sign be changed, and *vice versa*, we may confine our attention to upper semi-continuous functions. All the results I am about to obtain will hold *mutatis mutandis* for lower semi-continuous functions.

THEOREM 2.—*If I be the content of those points for which an upper semi-continuous function has a value greater than or equal to k, then I is an integrable (monotone) function of k, and the upper integral of the function (upper n-ple integral, if there be n independent variables), in any finite region S, is equal to* $SK + \int_K^{K'} I dk$, where K' is any finite quantity greater than or equal to the greatest value assumed by the function in the region considered, and K is any finite quantity less than or equal to the lower limit of the values assumed by the function.

For, as k increases from K up to K', I never increases; it is therefore a monotone function. Since K and K' are finite, the points at which the function I makes a jump greater than or equal to e must, for each value of e, be finite in number, thus forming a closed set of zero content; hence I is an integrable function of k.

For simplicity I only give the proof of the second part of the theorem for one-dimensional space; the necessary verbal alterations for space of n dimensions can easily be made.

Since we know beforehand that an upper integral of the given function, say f, exists, we can determine a small positive quantity e such that, if the whole segment in which we are operating (region of integration) be divided into small segments, finite in number and each less than e, and if we multiply the length of each of these by the upper limit of the

values of f in it, and sum for all the small regions, the result of the summation differs from the upper integral in question by less than some assigned small quantity e'. For brevity I shall write e_n for $e/2^n$; so that

$$e = e_1+e_2+\ldots.$$

Let us divide the segment (K', K) of the K-axis into $n+1$ equal parts, where n is any chosen integer, and denote the points of division by $K_1, K_2, \ldots, K_n$. Then the points x of the segment (A, B) at which $f(x) \geqslant K_r$ form a closed set, say G_r of content I_r, contained in the closed set G_{r+1} of all the points of (A, B) at which $f(x) \geqslant K_{r+1}$, which is itself contained in the closed set G_{r+2} of all the points at which $f(x) \geqslant K_{r+2}$, and so on.

We can therefore enclose all the points of G_1 in a finite number of intervals, in general overlapping, each less than e', and the content of them lying between I_1 and I_1+e_1.

The remaining segment or segments of (A, B) have content lying between $L-I_1-e_1$ and $L-I_1$; and the points of any one of the sets G_r which lie in them or on their boundaries form a closed set of content lying between $I_r-I_1-e_1$ and I_r-I_1.

In this segment, or these segments, we can now, in like manner, enclose all the points of G_2 which lie in them or on their boundaries, in a finite number of intervals, each less than e', so that their content lies between $I_2-I_1-e_1$ and $I_2-I_1-e_2$.

The segment or segments of (A, B) now left over have content lying between $L-I_1-e_1-I_2+I_1-e_2$ and $L-I_1-I_2+I_1$, that is, between $L-I_2-e_1-e_2$ and $L-I_2$; and inside these segments the points of G_r form a closed set of content lying between $I_r-I_2-e_1-e_2$ and I_r-I_2.

Proceeding thus, we must, after at most n stages, have shut up all the points of (A, B) in a finite number of intervals, of course overlapping, each of length less than e'.

If we take only such parts of these intervals as do not overlap, and multiply the length of each part by the corresponding upper limit of f, we see that we get something less than

$$(I_1+e_1)\,K'+(I_2-I_1+e_2)\,K_1+(I_3-I_2+e_3)\,K_2+\ldots$$
$$\ldots+(I_n-I_{n-1})\,K_{n-1}+(S-I_n)\,K_n\,;$$

that is, less than $SK+(I_1+I_2+\ldots+I_n+S)\,k'+eK'$, where k' is the n-th part of the interval (K, K').

Since this is greater than the upper integral in question, but differs from it by less than e', the result at once follows, when we make n increase without limit.

COR.—*If I be the content of those points of a region for which any function is defined which are such that the maximum of the function there has a value $\geqslant k$, then the upper integral of the function in the region is equal to $SK+\int_k^{k'} I\,dk$, and a similar statement holds for the lower integral.*

5. From the theorem of the previous section we may, if we please, deduce at once the known necessary and sufficient conditions that a given function of n variables should possess an n-ple integral for a given region. From § 3 it is evident, in fact, that the excess of the upper n-ple integral over the lower n-ple integral is the upper integral* of the associated oscillation function, and, since in this case zero is lower than or equal to the lower limit, it follows, by the theorem of the preceding section, that this upper integral is equal to the integral $\int D\,dk$, taken between proper limits, where D is the content of the set of those points at which the oscillation is greater than or equal to k. Since D is essentially positive, this integral can vanish when, and only when, D vanishes for every value of k. In other words, the necessary and sufficient conditions are that the points at which the oscillation of the function is greater than or equal to k form a closed set of zero content.

We may, moreover, deduce from the theorem of the last section other consequences. It is evident that the expression $KS+\int_K^{K'} I\,dk$ in the corollary at the end of the last section has a definite meaning and value, equally whether the function in question be defined for an ordinary region or for a closed set of points. We are therefore led to define it in this case as *the upper integral of the function with respect to a closed set*; similarly we can define *the lower integral of the function with respect to a closed set*, and it is plain that we at once have the condition that these two should be equal. In other words, we have, as we may say, the condition that an ordinary integral with respect to a set of points exists in the following form:—

THEOREM 3.—*The necessary and sufficient conditions that a function defined for a closed set of points possesses an integral with respect to that set are as follows:—The points at which the oscillation of the function is greater than or equal to k form a closed set of zero content.*

Or, which is the same thing:—

The points of continuity of the function with respect to the fundamental closed set form a set of content equal to that of the fundamental set.

Here the word "oscillation" is used in an obvious extended sense. I shall here content myself with these indications in connection with this

* Here we tacitly assume Lemma 2, p. 64, below.

matter. It is not difficult to reconstruct the whole theory from the beginning in such a way that the closed set of points plays the *rôle* previously assumed by the region, as the range of variation of the independent variable.

6. THEOREM 4.—*If $X(x)$ denote the content of the section of a closed set by the ordinate through the point x, then $X(x)$ is a semi-continuous function of x, and the upper integral $\bar{\int} X(x)\,dx$ of $X(x)$ with respect to x is the content of the closed set.*

More generally, taking space of n dimensions, let $X(x)$ denote the content of the hyperplane section of a closed set by the S_{n-1} through the point x of the x-axis perpendicular to that axis; then the same is true.

It is easy to obtain other generalisations of the above theorem, but I shall not in the present paper occupy myself further with the matter.

Proof.—The first part of the theorem follows from Theorem 4′ of my paper on "Open Sets and the Theory of Content," the enunciation of which is as follows:—

THEOREM 4′.—*Given an infinite number of sets of points, components of a set of finite (inner) content L, G_1, G_2, ..., such that the upper limit of the contents of the closed components of G_n—that is, the "(inner) content" of G_n—is greater than some positive quantity g, the same for all values of n; then an infinite series of these sets exists, having in common a set of points of potency c and (inner) content greater than or equal to g.*

In our case the set G is closed. Take then any set of lines, say L, parallel to the axis of y, having a single limiting line, say p. Choose them in such a way that, as they approach p, the contents of the corresponding ordinate sections (which are, of course, also closed) of the given set G have a definite limit, say I. Then we have to prove that the section of the set G by p has content greater than or equal to I.

We can evidently commence L at such a line that all the corresponding ordinate contents lie between $I-e$ and $I+e$, where e is as small as we please. Project all these ordinate sections on to the line p. Then, by Theorem 4′, there is a set, say C, of (inner) content greater than or equal to $I-e$, contained in an infinite number of these projections. That is, taking any point of C, and drawing its ordinate, this ordinate meets an infinite number of the lines L in points of G; the limiting point of these points lies, by our choice of L, on p, and is therefore the point of C which we took. Since G, and therefore the ordinate section of G by p, is closed, this point belongs to the ordinate section by p. Thus this ordinate section contains closed components of content as near as we please to I; so that its own content is not less than I. Q. E. D.

Now consider the second part of Theorem 4. For simplicity let us take a plane set G, and let the region of integration be taken to be a square of side L.

Let e be any chosen small positive quantity. Then, by the definition of content, I denoting the content of G, we can determine a small positive quantity e'', such that, if the square be divided up into small rectangles of linear dimensions less than e'', the content of those small rectangles which contain points of G lies between I and $I+e$.

Also, since we know that there is an upper integral $\bar{\int} X(x)\,dx$, we can certainly find a small positive quantity e' less than e'', such that, if the segment (A, B) of the x-axis in which we are operating be divided in any way into a finite number of parts, then, provided the length of each part be less than e', the expression $\Sigma \overline{X}(x)\,dx$ is greater than $\bar{\int} X(x)\,dx$ by less than e.

Now $X(x)$ lies between 0 and L: let us then choose any integer n, such that $ne > L$, and consider the closed sets $G_1, G_2, \ldots,$ where G_r consists of all the points x at which $X(x) \geqslant \frac{n-r}{n} L$. Thus G_n is itself the section of G by the x-axis, and each G_r contains the preceding G_{r-1}, while the remaining points, viz., those of $G_r - G_{r-1}$, are such that at each

$$\frac{n-r}{n} \leqslant X(x) < \frac{n-r+1}{n} L.$$

First, let us determine e_1', so that, if the points of G_1 be enclosed in intervals each of length less than e_1', the content of these intervals lies between I_1 and I_1+e. Then let us enclose the ordinate section of G by the ordinate through x, in a finite number of small vertical intervals, each of length less than e'.

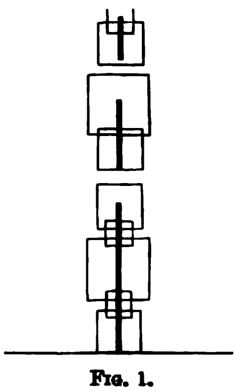


FIG. 1.

In the remaining parts of the ordinate, marked black in the figure, there are no points of G inside or on the boundaries. Let P be any point of such a black part; then, since G is closed, P is not a limiting point of G, so that we can describe a small square, with P as centre, whose sides are parallel to the coordinate axes, and of length less than e_1', so that there is no point of G inside this square or on its periphery. This being done for all points P in all the black parts, we can, by the Heine-Borel theorem, determine a finite number of the squares

such that each point of the black parts is *internal* to one at least of the squares. There being now only a finite number of squares to consider, there is, of course, a definite smallest of them. Let the length of its side be $e_1(x)$, and let us continue its vertical sides up and down to bound off a strip parallel to the y-axis, of breadth $e_1(x) < e_1'$, having the ordinate through x as central line.

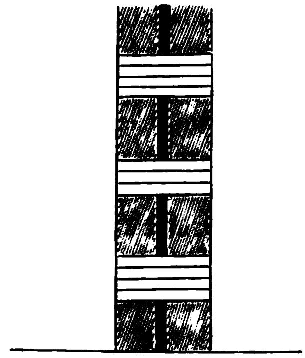


FIG. 2.

If we draw parallels to the x-axis through the ends of the vertical intervals originally drawn round the points of the ordinate section x, this strip is divided up into rectangles, of which those which contain the black parts of the ordinate x, shaded in the figure, contain no points of G inside or on the periphery, while the linear dimensions of the others are less than e', and their content lies between $X(x)\,e_1(x)$ and $Le_1(x)$.

Now, n was chosen so large that $X(x)$ differs from L by less than e, and therefore from the upper limit of the values of $X(x)$ in $e_1(x)$ by less than e. Thus the content of the plain rectangles lies between $[\overline{X}(x)-e]\,e_1(x)$ and $[\overline{X}(x)+e]\,e_1(x)$.

Now let us construct such strips and rectangles for every point of G_1. By the extension of the Heine-Borel theorem, there are then a finite number of these strips, each of breadth less than e_1', which enclose every point of the closed set G_1. Each strip is divided, as before, into plain and shaded rectangles.

Proceeding in like manner to enclose those points of G_2 in the remaining portions of the square (including those which lie on the boundaries), and then the points of G_3 not already enclosed, and so on, we eventually get the whole square divided up into plain and shaded rectangles; in the shaded rectangles there are no points of G, and the content of the plain rectangles lies between $\Sigma\,[\overline{X}(x)-e]\,d$ and $\Sigma\,[X(x)+e]\,d$, where d is the breadth of any vertical strip and $\overline{X}(x)$ the upper limit of the contents of the ordinate sections in d. Also, since the linear dimension of each rectangle are, by construction, less than e', and therefore less than e'', their content lies between I and $I+e$, and may be denoted by

$$I+\theta e \quad (0 < \theta \leqslant 1).$$

Again, $\Sigma\,\overline{X}(x)\,d$ differs from $\overline{\int} X(x)\,dx$ by less than e, and may be

denoted by $\overline{\int} X(x)\,dx + \theta' e$. Thus

$$\overline{\int} X(x)\,dx + \theta' e - Le \leqslant I + \theta e \leqslant \overline{\int} X(x)\,dx + \theta'' e + Le.$$

Since e can be chosen as small as we please, it follows that

$$I = \overline{\int} X(x)\,dx.$$

Q. E. D.

COR.—If the sections of a plane closed set by straight lines be translated parallel to themselves according to any law by which the plane set remains closed, the content of the set will not be altered.

7. THEOREM 5.—From Theorems 2 and 4 *we at once obtain the following expression, giving the content of a closed plane set as a single definite integral*, viz., $\int I\,dk$, taken between proper limits, where I stands for the content of those points of the x-axis at which the ordinate sections of the given plane set are greater than or equal to k.

It is clear that this result may be extended in various ways to sets in space of higher dimensions. It should be noticed that these formulæ are exactly analogous to the ordinary formulæ in the integral calculus for areas, volumes, &c., and include them as particular cases.

From this formula a number of particular consequences are at once deduced. For example, *the known necessary and sufficient condition that a plane closed set should have zero content is that points x whose ordinate sections of the plane set have content greater than or equal to k should form a set of zero content.*

8. It remains to give the following theorem, which includes that of § 6 as a particular case:—

THEOREM 6.—*The r-fold upper integral of any upper semi-continuous function of n variables ($r > n$), taken, that is, with respect to r of the variables seriatim, is an upper semi-continuous function of the remaining ($n-r$) variables; and the n-fold upper integral is equal to the upper n-ple integral.*

We require the following lemmas:—

LEMMA 1.—*If $x = a$ be a point of uniform convergence of a series of upper (lower) semi-continuous functions,* it is a point of upper (lower) semi-continuity of their sum.*

I give the proof for an upper semi-continuous function.

* It is clear from the proof that it is only necessary that the functions should be semi-continuous at the point in question.

Note that the lemma is equally true, and the proof will not require modification, if the uniform convergence be simple only.

Let $F(x) = S_n(x) + R_n(x)$, where n has been chosen such that an interval, say, $(a-h, a+h)$, can be found, for the whole of which $R_n(x)$ is less than an assigned small quantity e: this can always be done, since the convergence at the point a is (simply) uniform. Inside this interval we can, since $S_n(x)$ is a semi-continuous function at the point a, find another interval, $(a-h', a+h')$, say, for which the maximum value of $S_n(x)$ differs from $S_n(a)$ by less than e. In this interval the maximum value of $F(x)$ cannot exceed the maximum value of $S_n(x)$ by more than e; and the value at a of $F(x)$ cannot differ from $S_n(a)$ by more than e. Hence the maximum value of $F(x)$ in this interval cannot exceed $F(a)$ by more than $3e$. In other words, given a small positive quantity $3e$, we can find an interval, surrounding the point a, such that the maximum value of $F(x)$ in this interval differs from $F(a)$ by less than $3e$; which proves the lemma.

COR.—If $F(x)$ is the limit of a sequence of upper (lower) semi-continuous functions which converges (simply) uniformly, it is an upper (lower) semi-continuous function.

It is evident that the proof is unaltered.

LEMMA 2.—*The upper integral of the sum of two upper semi-continuous functions is equal to the sum of their upper integrals.*

Similarly for the lower integrals of lower semi-continuous functions. I have proved this theorem elsewhere.*

LEMMA 3.—*The upper (lower) integral of an upper (lower) semi-continuous function which is the limit of a uniformly convergent sequence of upper (lower) semi-continuous functions is the limit of the sequence of upper (lower) integrals of these functions.*

Proof.—As usual, take the case of upper integrals and functions only. We have

$$F(x) = \text{Lt}\, S_n(x) = S_n(x) + R_n(x),$$

where throughout the whole interval under consideration $R_n(x)$ is numerically less than an assigned positive quantity e as small as we please. The theorem now at once follows from the definition of upper

* "On an Extension of the Heine-Borel Theorem," *Messenger of Mathematics*, New Series, No. 393.

integration; for the difference between the upper integrals of $F(x)$ and $S_n(x)$ can evidently not exceed e multiplied by the length of the interval.

I now proceed to the proof of Theorem 6, confining myself, for the sake of simplicity, to the case when n is 2.

First, then, to prove that the upper integral of any upper semi-continuous function of two variables x and y, taken with respect to one of them, say y, is an upper semi-continuous function of the other x.

Let I_k be the content of the closed set of points for which the function, $f(x, y)$, say, is greater than or equal to k; and let $I_k(x)$ be the content of the closed section of this set by the ordinate x; so that, by Theorem 4,

$$I_k = \bar{\int} I_k(x)\,dx.$$

Then $\bar{\int} f(x, y)\,dw$ over the whole region S

$$= KS + \int_K^{K'} I_k\,dk = KS + D, \text{ say.}$$

Also $\bar{\int} f(x, y)\,dy$ along the ordinate x of length $\bar{y}$, say,

$$= K\bar{y} + \int_K^{K'} I_k(x)\,dk = K\bar{y} + D', \text{ say.}$$

Since the sum of two semi-continuous functions is itself semi-continuous, and $K\bar{y}$ is continuous, it is only necessary to prove that the integral D' is a semi-continuous function of x.

Now, $I_k(x)$ is an upper semi-continuous function of x for all values of k; the same is therefore true of the finite summation $\sum_{r=1}^{r=n} I_{k_r}(x)\,k'$, where k' denotes, as before, one n-th part of the interval (K, K'), and the values k_r $(r = 1, 2, \ldots, n)$ are any values in the corresponding n equal divisions of (K, K').

Let M be any finite quantity greater than the maximum of $I_k(x)$ for all values of k and of x. Then, since $I_k(x)$, for constant x, is a never increasing monotone function of k, the least value of the summation will be obtained by taking the right-hand value of k in each interval k', and the greatest value by taking the left-hand value; these two summations differ by less than Mk'.

Let us assign any small quantity e; then we can determine n_1, so that, for all values of $n > n_1$, $Mk' < e$. The integral $\int_K^{K'} I_k(x)\,dk$ or D', for constant x, lies between these two summations, and differs therefore from the summation $\sum_{r=1}^{r=n} I_{k_r}(x)\,k'$ by less than e, where e was assigned independent of x.

Thus the upper semi-continuous functions represented by these summations, for successive values of n, converge uniformly towards the integral D' as limit. It follows, by Lemma 1, that the latter integral D' is a semi-continuous function of x. This proves the first part of the theorem.

To prove the second part we note that, by Lemma 2,

$$\sum_{r=1}^{r=n} I_{k_r} k' = \sum_{r=1}^{r=n} k' \overline{\int} I_{k_r}(x)\, dx = \overline{\int} \sum_{r=1}^{r=n} [I_{k_r}(x)\, k']\, dx\, ;$$

that is,
$$D - e' = \overline{\int} (D' + e'')\, dx,$$

where e' is as small as we please, and $e'' \leqslant e$. Therefore

$$D = \overline{\int} D' dx\, ;$$

also
$$S = \int \overline{y}\, dx,$$

whence
$$KS + D = \overline{\int} (K\overline{y} + D')\, dx,$$

which proves the theorem.

THE TILE THEOREM

By W. H. YOUNG.

[Received April 12th, 1904.—Read April 14th, 1904.]

IN a note entitled "On an Extension of the Heine-Borel Theorem,"* I defined a "tile" as consisting of a definite interval or region with a definite corresponding point, called the "point of attachment."

A set of tiles, as they occur in application, usually has the property that, if d_P be a tile of the set having the point of attachment P and covering the region d, every tile having P as point of attachment and covering a part of d belongs to the set: when this is the case, I say that the tiles may be "chipped" as much as we please.

The points on the boundary of d are not to be considered as "covered" by the tile d_P.

In the note quoted, I gave a theorem, analogous to the Heine-Borel theorem for intervals, respecting a set of tiles whose points of attachment consist of all the points of a segment or region; I showed also how this theorem may be applied to give a simple proof of the theorem that the upper integral of the sum of two upper semi-continuous functions is the sum of their upper integrals.

In the following note I give a more general form of the tile theorem. This theorem is independent in conception and in proof of the Heine-Borel theorem; it was suggested to me by the necessity of such a theorem to conveniently prove certain general theorems in the theory of integration.

THEOREM.—*Given a set of tiles, each of which may be chipped as much as is convenient, whose points of attachment fill up a measurable set S; then we can determine a finite or countably infinite number of the tiles having the following properties:*—(1) *The linear dimensions of each d_{P_i} are less than e;* (2) *each point of S is covered by one or more of the tiles d_{P_i};* (3) *the point of attachment P_i of the tile d_{P_i} is not covered by any other of the tiles;* (4) *the sum of the tiles differs from the content of*

* *Messenger of Mathematics*, New Series, No. 393.

S by less than e′; here e and e' are any assigned small positive quantities.

Proof.—By the definition of the content of a measurable set,* we can assign a set of intervals of content lying between S and $S+\frac{1}{2}e'$, and having all the points of S as internal points. Since the given tiles may be chipped, we can chip them so that (1) is satisfied, while they cover no point external to this set of intervals. The content of the intervals covered by the tiles will then lie between S and $S+\frac{1}{2}e'$.

Now let us replace the whole set of tiles by a countable set from among them covering the whole set S.† Let these in order be

$$d', d'', d''', \ldots . \tag{a}$$

We may suppose that no one of these tiles is completely covered by any other, since any such tile could have been omitted in choosing the set (*a*). Similarly, if at any stage of the process to be adopted one tile comes to be completely covered by another, I shall suppose that we then omit it. We take the first tile d' in the countable order (*a*) and call it d_{P_1}.

Since it is not completely covered by any tile, it divides the remaining tiles into two sets, those on the right and those on the left of itself; I shall consider first the former, and chip them so that none of them extend as far as the point of attachment P_1 on the left.

Then there may, or may not, be certain of these tiles which overlap with the first tile d_{P_1}; if there are any, let the first in the order (*a*) which does so be called d_{P_2}, and let the remaining tiles be chipped so as not to extend as far as P_2 on the left. Also, if P_2 do not lie outside d_{P_1}, or if the interval covered by both the first two tiles d_{P_1} and d_{P_2} be not less than $2^{-4}e'$, we chip d_{P_1} until both of these are the case, without, of course, chipping so much that no interval is covered by both the first two tiles.

We now take as d_{P_3} the first in the order (*a*) which overlaps with d_{P_2}, and proceed as before, using only $2^{-5}e'$ instead of $2^{-4}e'$, and so on. We thus get a finite or countably infinite set of the tiles each overlapping with the next, the sum of the overlapping parts being less than $2^{-3}e'$. Similarly we treat the tiles to the left of d_{P_1}. In all we have

---

* Lebesgue, *Ann. di Mat.*, 1902. Cf. W. H. Young, *supra*, p. 40.

† For a proof of the possibility of this, see "Overlapping Intervals," *Proc. London Math. Soc.*, Vol. xxxv.

here a finite or countably infinite set of the tiles, overlapping all along, the sum of the overlapping parts being less than $\frac{1}{4}e'$.

These do not necessarily cover over every point of S; if not, however, let $d^{(r)}$ be the first of the tiles, in the order (a), which is separated by an interval or point from the tiles already considered. Then we perform the same process as before, starting from $d^{(r)}$ as we did from d', using, however, the quantity $2^{-(r+3)}e'$ instead of $2^{-4}e'$.

This process is to be continued till we have gone over all the tiles (a). During our process we never uncover a point of S, so that (2) is satisfied; but we ensure that no point of attachment is covered by any tile but the one corresponding to it (3); finally, the sum of the overlapping parts is less than $2^{-2}e'+2^{-(r+1)}e'+\ldots$, that is, certainly less than $\frac{1}{2}e'$. Now the content of the intervals covered by the tiles lies between S and $S+\frac{1}{2}e'$; therefore the sum of the tiles lies between S and $S+e'$, (4). This proves the theorem.

ON THE UNIQUE EXPRESSION OF A QUANTIC OF ANY ORDER IN ANY NUMBER OF VARIABLES, WITH AN APPLICATION TO BINARY PERPETUANTS

By P. W. WOOD.

[Received February 25th, 1904.—Read March 10th, 1904.]

THE present paper contains an extension of the results of a previous paper "On the Unique Expression of Binary and Ternary Forms."* It will be seen that the method employed admits of much wider application than is here made, and that there is some freedom of choice in the form of unique expression; in any case this choice must depend on the nature of the problem to which the result is to be applied.

No new results of any importance are given, but there is a new proof that the minimum weight of an irreducible perpetuant of degree δ is $2^{\delta-1}-1$. Some results concerning the grade of perpetuants are also deduced.

1. It is known that any quantic of order n in the δ variables $x_1, x_2, \ldots, x_\delta$ can be expressed linearly in terms of $\binom{n+\delta-1}{\delta-1}$ linearly independent forms of the same order n in the variables $x_1, x_2, \ldots, x_\delta$.

Consider the relation $S \equiv \sum_r y_r^{n-m_r} p_{r_x}^{m_r} = 0$

of total order n in the δ variables $x_1, x_2, \ldots, x_\delta$. The positive integers m_r, not necessarily all different, satisfy the equation

$$\sum_r \binom{m_r+\delta-1}{\delta-1} = \binom{n+\delta-1}{\delta-1};$$

each y_r is a linear function of $x_1, x_2, \ldots, x_\delta$, and $p_{r_x}^{m_r}$ is a quantic of order m_r in those variables. If we can show that, when we have chosen the linear forms y_r, we cannot find quantics $p_{r_x}^{m_r}$, other than identical zeros, satisfying the relation $S \equiv 0$, then it follows that any quantic F_x^n of order n in $x_1, x_2, \ldots, x_\delta$ is *uniquely* expressible in the form

$$F_x^n = S;$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1904.

for the $\binom{n+\delta-1}{\delta-1}$ terms of S are linearly independent, since otherwise we should have a relation of the same form as $S\equiv 0$; and, moreover, if the expression were not unique, there would arise, on equating two such expressions, a relation of the form $S\equiv 0$, which, by hypothesis, is not possible, unless each quantic involved is identically zero; and, in this case, the two expressions would be identical.

2. The first step in thus determining a form of unique expression is to find relations satisfying the two following conditions :—

(1) When the linear forms have been chosen, each quantic involved must be identically zero.

(2) The total number of constants involved in the quantics is $\binom{n+\delta-1}{\delta-1}$.

The relation of any order is completely determined, when we know the linear forms involved and the power to which each linear form is raised. In the first instance a special choice of the linear forms will be made: let $[rstu\ldots w]\equiv x_1+(r)x_2+(rs)x_3+\ldots+(rstu\ldots w)x_\delta$, where each of $r, s, t, \ldots, w$ is any one of the numbers $1, 2, 3, \ldots, \lambda$, and $(rstu\ldots v)$ is a constant. The only restriction to be placed on these constant coefficients is that no two of them are equal, if they differ *only* in the last number: thus $(rstuv\ldots\lambda\mu)$ and $(rstuv\ldots\lambda\mu')$ are unequal, but we may have $(rstu\ldots\lambda'\mu)=(rstu\ldots\lambda\mu)$ or $(rstuv)=(r'stuv)$, &c. The number of such linear forms $[rstuv\ldots w]$ is $\lambda^{\delta-1}$, and, by virtue of this restriction on the coefficients, these $\lambda^{\delta-1}$ linear forms are all distinct. Any linear forms occurring in a relation will be taken as one of the $\lambda^{\delta-1}$ forms thus determined.

If $O_s\equiv (s)\dfrac{\partial}{\partial x_1}-\dfrac{\partial}{\partial x_2}$, then O_s annihilates each of the $\lambda^{\delta-2}$ forms $[s\ldots]$, s being any one of $1, 2, 3, \ldots, \lambda$; in this way we obtain λ linear differential operators $O_1, O_2, \ldots, O_s, \ldots, O_\lambda$.

3. If n is the total order of the relation, then n may be taken to be one of $m\lambda, m\lambda+1, \ldots, m\lambda+s, \ldots, m\lambda+\lambda-1$; m and λ being any positive integers. We shall now find the orders of the quantics involved in each of the λ relations, corresponding to the λ different forms of n, to ensure the number of constants therein involved being $\binom{n+\delta-1}{\delta-1}$, for each of the λ different forms of n.

In the relation of total order $\{(m+1)\lambda-s\}$, suppose there are $B_s^{(t)}$

quantics of order $(m-t+1)$: then we must find integers B_s, positive or zero, satisfying the equations

$$\binom{(m+1)\lambda-s+\delta-1}{\delta-1}=\sum_{t=1} B_s^{(t)}\binom{m+\delta-t}{\delta-1}$$

$$(s=1,\ 2,\ 3,\ \ldots,\ \lambda) \quad (1)$$

for all values of m.

Assume that

$$\sum_{t=1}(B_1^{(t)}x^{\lambda t-1}+B_2^{(t)}x^{\lambda t-2}+\ldots+B_s^{(t)}x^{\lambda t-s}+\ldots+B_\lambda^{(t)}x^{\lambda t-\lambda})\equiv\phi(x).$$

Now $$(1-x^\lambda)^{-\delta}=\sum_t\binom{m+\delta-t}{\delta-1}x^{\lambda(m-t+1)},$$

and therefore

$$\sum_t B_s^{(t)}\binom{m+\delta-t}{\delta-1}=\text{coefficient of } x^{\lambda(m+1)-s} \text{ in } \phi(x)(1-x^\lambda)^{-\delta},$$

also

$$\binom{(m+1)\lambda-s+\delta-1)}{\delta-1}=\text{coefficient of } x^{\lambda(m+1)-s} \text{ in } (1-x)^{-\delta};$$

therefore, since the equation (1) holds for all values of m and for the values $1,\ 2,\ \ldots,\ \lambda$ of s, we must have

$$\phi(x)(1-x^\lambda)^{-\delta}\equiv(1-x)^{-\delta},$$

and so $$\phi(x)\equiv(1+x+x^2+\ldots+x^{\lambda-1})^\delta,$$

and therefore $B_s^{(t)}=$ coefficient of $x^{\lambda t-s}$ in $(1+x+x^2+\ldots+x^{\lambda-1})^\delta$, for $s=1,\ 2,\ 3,\ \ldots,\ \lambda$.

It is obvious that the quantities B thus given are positive integers or zero, and that

$$B_s^{(t)}=0,$$

if $\lambda t-s>(\lambda-1)\delta$, *i.e.*, if $t>\dfrac{(\lambda-1)\delta+s}{\lambda}$.

4. The method of proving the impossibility of the relations will be inductive, and the relations will be so chosen that the effect of operating on the relation of total order $\{(m+1)\lambda-s\}$ with O_s is to give a relation of the same constitution as the relation of total order $\{(m+1)\lambda-s-1\}$, for the values $1,\ 2,\ 3,\ \ldots,\ \lambda$ of s.

We have

$$\text{(i.)}\quad O_r\{[s\ldots]^{(m+1)(\lambda-1)+\theta-s}\,p_x^{m-\theta+1}\}\equiv[s\ldots]^{(m+1)(\lambda-1)+\theta-s-1}\,p_x^{m-\theta+1},$$

if $r\neq s$,

$p_x'^{m-\theta+1}$ being a quantic in the same variables, which is not identically zero;

(ii.) $O_s\{[s\ldots]^{(m+1)(\lambda-1)+\theta-s}p_x^{m-\theta+1}\} \equiv [s\ldots]^{(m+1)(\lambda-1)+\theta-s}O_s\{p^{m-\theta+1}\}.$

Suppose there are, in the relation of total order $(m\lambda+\lambda-1)$, for the values $1, 2, 3, \ldots, \lambda$ of s, $A_s^{(t)}$ forms $[s\ldots]$ occurring to a power

$$\{(m+1)(\lambda-1)+t-1\},$$

the orders of the corresponding quantics in that relation being $(m-t+1)$; then, since the relations are such that the effect of operating with O_s on the relation of order $\{(m+1)\lambda-s\}$ is to give a relation of the same constitution as the relation of total order $\{(m+1)\lambda-(s+1)\}$, for the values $1, 2, 3, \ldots, \lambda$ of s, the quantities A must satisfy the following equations for all values of m:—

$$\binom{(m+1)\lambda-1+\delta-1}{\delta-1} = \sum_{t=1}(A_1^{(t)}+A_2^{(t)}+\ldots+A_s^{(t)}+\ldots+A_\lambda^{(t)})\binom{m+\delta-t}{\delta-1},$$

$$\binom{(m+1)\lambda-2+\delta-1}{\delta-1} = \sum_{t=1}(A_1^{(t-1)}+A_2^{(t)}+\ldots+A_s^{(t)}+\ldots+A_\lambda^{(t)})\binom{m+\delta-t}{\delta-1},$$

... ...

$$\binom{(m+1)\lambda-s+\delta-1}{\delta-1}$$
$$= \sum_{t=1}(A_1^{(t-1)}+A_2^{(t-1)}+\ldots+A_{s-1}^{(t-1)}+A_s^{(t)}+\ldots+A_\lambda^{(t)})\binom{m+\delta-t}{\delta-1},$$

$$\binom{(m+1)\lambda-(s+1)+\delta-1}{\delta-1}$$
$$= \sum_{t=1}(A_1^{(t-1)}+A_2^{(t-1)}+\ldots+A_s^{(t-1)}+A_{s+1}^{(t)}+\ldots+A_\lambda^{(t)})\binom{m+\delta-t}{\delta-1},$$

... ...

$$\binom{m\lambda+\delta-1}{\delta-1}$$
$$= \sum_{t=1}(A_1^{(t-1)}+A_2^{(t-1)}+\ldots+A_s^{(t-1)}+\ldots+A_{\lambda-1}^{(t-1)}+A_\lambda^{(t)})\binom{m+\delta-t}{\delta-1}$$

$$\binom{m\lambda-1+\delta-1}{\delta-1}$$
$$= \sum_{t=1}(A_1^{(t-1)}+A_2^{(t-1)}+\ldots+A_s^{(t-1)}+\ldots+A_\lambda^{(t-1)})\binom{m+\delta-t}{\delta-1}.$$

Here we take $A_s^{(0)}$ as zero, for $s = 1, 2, 3, \ldots, \lambda$.

The assumption that relations of this nature can, as far as the orders of the quantics are concerned, actually be found is, remembering the results of § 3, justified, if we can show that each A is a positive integer or zero and that

$$A_1^{(t-1)}+A_2^{(t-1)}+\ldots+A_{s-1}^{(t-1)}+A_s^{(t)}+\ldots+A_\lambda^{(t)} = B_s^{(t)},$$

for all values of t concerned and for the values $1, 2, 3, \ldots, \lambda$ of s.

Subtract each of the above equations from the preceding; then, for all values of m and for the values $1, 2, 3, \ldots,$ of s, we have

$$\binom{(m+1)\lambda-s+\delta-2}{\delta-2} = \sum_{t=1} (A_s^{(t)}-A_s^{(t-1)}) \binom{m+\delta-t}{\delta-1}$$

$$= \sum_{t=1} A_s^{(t)} \binom{m+\delta-t-1}{\delta-1}.$$

Hence $A_s^{(t)}$ is the value taken by $B_s^{(t)}$ when in the latter δ is replaced by $(\delta-1)$, and therefore $A_s^{(t)} =$ coefficient of $x^{\lambda t-s}$ in

$$(1+x+x^2+\ldots+x^{\lambda-1})^{\delta-1},$$

for $s = 1, 2, 3, \ldots, \lambda$.

Hence each A is a positive integer or zero, and

$$A_s^{(t)} = 0,$$

if $\lambda t-s > (\lambda-1)(\delta-1)$; *i.e.*, if $t > \lambda^{-1}\{(\lambda-1)(\delta-1)+s\}$.

The value of $A_s^{(t)}$ may be found in the form of a series, for

$$A_s^{(t)} = \text{coefficient of } x^{\lambda t-s} \text{ in } (1+x+x^2+\ldots+x^{\lambda-1})^{\delta-1},$$

i.e., in
$$(1-x^\lambda)^{\delta-1}(1-x)^{-(\delta-1)}$$

or in
$$\sum_{\rho=0}^{\rho=\delta-1} (-)^\rho \binom{\delta-1}{\rho} x^{\lambda\rho} . \sum_\rho x^{\lambda(t-\rho)-s} \binom{\lambda(t-\rho)-s+\delta-2}{\delta-2}.$$

Hence
$$A_s^{(t)} = \sum_{\rho=0} (-)^\rho \binom{\delta-1}{\rho} \binom{\lambda(t-\rho)-s+\delta-2}{\delta-2},$$

and, similarly,

$$B_s^{(t)} = \sum_{\rho=0} (-)^\rho \binom{\delta}{\rho} \binom{\lambda(t-\rho)-s+\delta-1}{\delta-1},$$

writing δ for $\delta-1$ in the value of $A_s^{(t)}$.

(In the special case of $\lambda = 2$, the A's and B's are binomial coefficients.)

It should be noticed that all the preceding results are valid when m is zero. Finally,

$$A_1^{(t-1)}+\ldots+A_{s-1}^{(t-1)}+A_s^{(t)}+\ldots+A_\lambda^{(t)}$$

$$= \text{sum of coefficients of } \left\{\begin{matrix} x^{\lambda t-\lambda}, & x^{\lambda t-\lambda+1}, & \ldots, & x^{\lambda t-s} \\ x^{\lambda(t-1)-1}, & x^{\lambda(t-1)-2}, & \ldots, & x^{\lambda(t-1)-(s-1)} \end{matrix}\right\}$$

$$\text{in } (1+x+x^2+\ldots+x^{\lambda-1})^{\delta-1}$$

$$= \text{coefficient of } x^{\lambda t-s} \text{ in } (1+x+x^2+\ldots+x^{\lambda-1})^{\delta-1}(1+x+x^2+\ldots+x^{\lambda-1})$$

$$= B_s^{(t)},$$

and so, as far as the orders of the quantics are concerned, it is possible to find relations of this nature. We have

$$\sum_t B_s^{(t)} = \sum_{s=1}^{s=\lambda} \sum_{t=1} A_s^{(t)} = \text{sum of coefficients of } (1+x+x^2+\ldots+x^{\lambda-1})^{\delta-1};$$

therefore $\quad \sum_t B_s^{(t)} = \lambda^{\delta-1}$, for the values 1, 2, 3, ..., λ of s,

and therefore $\sum_t A_s^{(t)} = \lambda^{\delta-2}$, for the values 1, 2, 3, ..., λ of s.

Hence in each relation (except for certain small values of m, when quantics of negative order disappear) each of the $\lambda^{\delta-2}$ forms $[s\ldots]$, s being any one of 1, 2, 3, ..., λ, occurs exactly once.

It remains to discover the power to which each of the $\lambda^{\delta-1}$ linear forms $[\ldots]$ occurs in each relation.

5. We shall require the result:—

If any linear differential operator

$$O \equiv a_1\frac{\partial}{\partial y_1}+a_2\frac{\partial}{\partial y_2}+\ldots+a_r\frac{\partial}{\partial y_r}$$

annihilates the product (PQ) of two quantics P and Q in the variables $y_1, y_2, \ldots, y_\delta$, then O annihilates each of P and Q.

For, by the theory of partial differential equations, PQ must be a quantic in the variables $a_1y_2-a_2y_1$, $a_1y_3-a_3y_1$, ..., $a_1y_r-a_ry_1$, y_{r+1}, ..., y_δ, and, since each of its factors P, Q is a quantic, P and Q must be quantics in the same variables, and so each of P and Q is also annihilated by O.

6. We shall now show how, when we know the constitutions of the λ relations for $(\delta-1)$ variables, we can find the constitutions of the λ relations for δ variables. Assuming that the relation in δ variables of order

$\{(m+1)\lambda-(s+1)\}$ can be satisfied only by each quantic being identically zero, our purpose is to show that the relation of total order $\{(m+1)\lambda-s\}$ is likewise impossible: if we can do this, we need only prove the impossibility of the relation of total order unity.

Consider first the case of two variables: the relations will be

$$\sum_{t=1}^{t=\lambda} [t]^{(\lambda-1)(m+1)} p_{t_x}^m \equiv 0$$

(of total order $m\lambda+\lambda-1$),

$$[1]^{(\lambda-1)(m+1)} p_{1_x}^{m-1} + \sum_{t=2}^{t=\lambda} [t]^{(\lambda-1)(m+1)-1} p_{t_x}^m \equiv 0$$

(of total order $m\lambda+\lambda-2$),

... ...

$$\sum_{t=1}^{t=s-1} [t]^{(\lambda-1)(m+1)-s+2} p_{t_x}^{m-1} + \sum_{t=s}^{t=\lambda} [t]^{(\lambda-1)(m+1)-s+1} p_{t_x}^m \equiv 0$$

(of total order $m\lambda+\lambda-s$),

$$\sum_{t=1}^{t=s} [t]^{(\lambda-1)(m+1)-s+1} p_{t_x}^{m-1} + \sum_{t=s+1}^{t=\lambda} [t]^{(\lambda-1)(m+1)-s} p_{t_x}^m \equiv 0$$

[of total order $m\lambda+\lambda-(s+1)$],

... ...

$$\sum_{t=1}^{t=\lambda-1} [t]^{(\lambda-1)m+1} p_{t_x}^{m-1} + [\lambda]^{(\lambda-1)m} p_{\lambda_x}^m \equiv 0$$

(of total order $m\lambda$).

For, in the relation of total order $\{(m+1)\lambda-s\}$, the number of constants involved in the binary quantics is

$$m(s-1)+(m+1)(\lambda-s+1) = (m+1)\lambda-s+1;$$

and, if we operate on that relation with O_s, the resulting relation is of the same constitution as the relation of total order $\{(m+1)\lambda-(s+1)\}$; and therefore, by hypothesis, each quantic in this resulting relation is identically zero; hence, by § 5, in the original relation each quantic p_{t_x} is identically zero, if $t \neq s$, and therefore the quantic $p_{s_x}^m$ is also identically zero.

Hence, if the relation of total order n is impossible, so also is the relation of total order $(n+1)$, for all values of n, and the relation of total order unity is

$$[x_1+(\lambda-1)x_2]P+[x_1+(\lambda)x_2]Q \equiv 0,$$

which requires $P = Q = 0$, since $(\lambda) \neq (\lambda-1)$, by § 2.

Therefore each of the relations is impossible for all values of n.

The present result for binary forms is a special case of the more general theorem* :—

If a_x, b_x, c_x, ... be a system of r distinct linear forms, and α, β, γ, ... be r positive integers satisfying the relation

$$\alpha+\beta+\gamma+\ldots = n-r+1,$$

then it is impossible to find binary forms A, B, C, ..., of orders α, β, γ, ... respectively, such that

$$a_x^{n-\alpha}A+b_x^{n-\beta}B+c_x^{n-\gamma}C+\ldots = 0.$$

7. In the general case of δ variables, operate on the relation of total order $\{(m+1)\lambda-s\}$ with O_s: then, by virtue of the choice made for the orders of the quantics, the resulting relation must, by hypothesis, have each of the quantics therein involved identically zero.

If $$O_s\{[t\ldots]^{(\lambda-1)(m+1)+\theta-s}p_x^{m-\theta+1}\} \equiv 0 \quad (s \neq t),$$

then, by § 5, $$p_x^{m-\theta+1} \equiv 0,$$

and, if $$O_s\{[s\ldots]^{(\lambda-1)(m+1)+\theta-s}p_x^{m-\theta+1}\} \equiv 0,$$

then $p_x^{m-\theta+1}$ must either be identically zero, or be a quantic in the $(\delta-1)$ variables

$$y_s[\equiv x_1+(s)x_2],\ x_3,\ x_4,\ \ldots,\ x_\delta.$$

So, if we assume the impossibility of the relation of total order $\{(m+1)\lambda-s-1\}$, the relation of total order $\{(m+1)\lambda-s\}$ is either also impossible or is a relation in the $(\delta-1)$ variables y_s, x_3, x_4, ..., x_δ, since only those terms involving the linear forms $[s\ldots]$ can contain quantics different from identical zeros. In the latter case the relation of total order $\{(m+1)\lambda-s\}$ becomes

$$\sum_{t=1}\Big\{\sum_{A_s^{(t)}}[s\ldots]^{(\lambda-1)(m+1)+t-s}p_x^{m-t+1}\Big\} \equiv 0 \quad [\text{in } (\delta-1) \text{ variables}],$$

where $\sum_{A_s^{(t)}}$ implies the summation of $A_s^{(t)}$ different quantics p_x and linear forms $[s\ldots]$.

Now, by § 4, since $A_s^{(t)}$ differs from $B_s^{(t)}$ only in having $(\delta-1)$ in place of δ, as far as the number of forms $[s\ldots]$ and their indices are concerned, this relation *may* be the standard form of relation of total order $\{(m+1)\lambda-s\}$ in $(\delta-1)$ variables, so that, by hypothesis, every quantic

* J. H. Grace, *Algebra of Invariants*, by Grace and Young, Appendix III., p. 375.

is identically zero. In other words the $A_s^{(t)}$ forms $[s\ldots]$ with index $\{(\lambda-1)(m+1)+t-s\}$ may be divided up into sets, such that there are $A_{ss'}^{(t)}$ linear forms $[ss'\ldots]$ with that index, and so on, and the resulting relation will be of the same character as that we have been considering, with $(\delta-1)$ in place of δ.

(For such relations the operators by means of which each is derived from that of total order higher by unity will be like

$$O_{ss'} \equiv (ss')\frac{\partial}{\partial y_s}-\frac{\partial}{\partial x_3}, \quad \text{for} \quad s'=1, 2, 3, \ldots, \lambda.)$$

Thus in general the forms $[s\ldots]$ must be so arranged in the relation of total order $\{(m+1)\lambda-s\}$ that the corresponding terms must by themselves, when we replace the variables x_1, x_2 by the single variable $[x_1+(s)x_2]$, constitute the corresponding relation in $(\delta-1)$ variables, wherein, by hypothesis, each quantic must be identically zero; and this is true for each of the λ relations.

Now all the λ relations are, for any number of variables, completely determined when we know the power to which each linear form is raised in the relation of total order $\{(m+1)\lambda-1\}$; for, if the linear forms $[1uvw\ldots]$, $[2uvw\ldots]$, ..., $[suvw\ldots]$, ..., $[\lambda uvw\ldots]$ have indices ϕ_1, ϕ_2, ..., ϕ_s, ..., ϕ_λ respectively in the relation of total order $\{(m+1)\lambda-1\}$, then in the relation of total order $\{(m+1)\lambda-s\}$, they have indices $\phi_1-(s-2)$, $\phi_2-(s-2)$, ..., $\phi_{s-1}-(s-2)$, $\phi_s-(s-1)$, ..., $\phi_\lambda-(s-1)$ respectively.

It will therefore be sufficient to show how the relation of total order $\{(m+1)\lambda-1\}$ in δ variables may be determined from the relation of total order $\{(m+1)\lambda-1\}$ in $(\delta-1)$ variables.

The linear forms in $(\delta-1)$ variables $x_1, x_2, \ldots, x_{\delta-1}$ are given by symbols like $[uvw\ldots]$, where $[\ldots]$ contains $(\delta-2)$ numbers, each of which is one of $1, 2, 3, \ldots, \lambda$. Take the standard form of relation of order $\{(m+1)\lambda-s\}$ in $(\delta-1)$ variables, and replace each linear form $[uvw\ldots]$ by $[suvw\ldots]$; generalize the quantics in that relation by introducing another variable x_δ: then the resulting expression will consist of those $\lambda^{\delta-2}$ terms of the relation of total order $\{(m+1)\lambda-s\}$ in δ variables, which involve linear forms $[s\ldots]$; this is obvious from what precedes.

Let θ_u be the index of the linear form $[uvw\ldots]$ in the relation of total order $\{(m+1)\lambda-1\}$ in $(\delta-1)$ variables; then its index in the relation of total order $\{(m+1)\lambda-s\}$ in $(\delta-1)$ variables will be $\theta_u-(s-2)$ if $u<s$, and will be $\theta_u-(s-1)$ if $u \geqslant s$.

Hence, in the relation of order $\{(m+1)\lambda-s\}$ in δ variables, the index of $[suvw\ldots]$ is $\theta_u-(s-2)$ if $u<s$, and is $\theta_u-(s-1)$ if $u \geqslant s$; therefore, in the relation of total order $\{(m+1)\lambda-1\}$ in δ variables, the index of $[suvw\ldots]$ is $\theta_u-(s-2)+(s-1)=\theta_u+1$ if $u<s$, and is

$$\theta_u-(s-1)+(s-1)=\theta_u \quad \text{if } u \geqslant s;$$

and this holds for each of the λ relations. Hence the following simple law:—

Take the relation of order $\{(m+1)\lambda-1\}$ in $(\delta-1)$ variables $x_1, x_2, \ldots, x_{\delta-1}$; replace each form $[uvw\ldots]$ successively by $[suvw\ldots]$, where s takes the values $1, 2, 3, \ldots, \lambda$, and, if $s>u$, increase the index of the linear form by unity; generalize each quantic of the relation by introducing another variable x_δ. Then the sum of all the $\lambda^{\delta-1}$ terms thus obtained when equated to zero is the required form of relation of total order $\{(m+1)\lambda-1\}$ in the δ variables $x_1, x_2, \ldots, x_\delta$.

By repeated application of this law we can build up our relation of total order $\{(m+1)\lambda-1\}$ in δ variables from the relation of the same order in 2 variables, and, since in the latter the index of every linear form is $(\lambda-1)(m+1)$, we have finally:—

In the relation of total order $\{(m+1)\lambda-1\}$ in δ variables, the index of any linear form $[rstuv\ldots w]$ is $\{(\lambda-1)(m+1)+\theta\}$, where θ is the number of places in the sequence of the $(\delta-1)$ numbers $r, s, t, u, v, \ldots, w$ (each number being one of $1, 2, 3, \ldots, \lambda$), where a number immediately precedes a smaller number.

This completely determines the constitution of the relation of total order $\{\lambda(m+1)-1\}$ in δ variables, and so, by a preceding result, the constitution of all the relations in δ variables.

In the first relation there are $A_s^{(t)}$ linear forms $[su\ldots w]$ with an index $\{(m+1)(\lambda-1)+t-1\}$, and we can prove otherwise that there are $A_s^{(t)}$ ways of choosing $(\delta-1)$ numbers from $1, 2, 3, \ldots, \lambda$, such that s is the first and that $(t-1)$ of the numbers immediately precede smaller numbers; for the result is certainly true when δ is 1 or 2, and, assuming it true for $(\delta-1)$ numbers, it is true for δ numbers by virtue of the relation of § 4:—

$$A_1^{(t-1)}+A_2^{(t-1)}+\ldots+A_{s-1}^{(t-1)}+A_s^{(t)}+\ldots+A_\lambda^{(t)}=B_s^{(t)},$$

for, if u is any one of $1, 2, 3, \ldots, s-1$, then there are to be only $(t-2)$ numbers in the sequence $uvw\ldots$ immediately preceding smaller numbers, and $B_s^{(t)}$ differs from $A_s^{(t)}$ only in having δ in place of $(\delta-1)$.

8. With this choice of linear forms it follows that the relation of total order $(n+1)$ is impossible, if that of total order n is impossible. We proceed to show that the relation of total order unity requires each of the quantics (constants) involved to be zero. The presence of a quantic of negative order implies that that quantic has been annihilated by successive differentiations in the process of deriving the relation of order n from that of order $(n+1)$.

Putting $m=0$ in the relation of total order $(m\lambda+1)$, and taking quantics of negative orders as absolute zeros, the relation of total order unity is
$$\sum_{A^{(1)}_{\lambda-1}} [\lambda-1\ldots]P+\sum_{A^{(1)}_{\lambda}} [\lambda\ldots]Q\equiv 0,$$
where $A^{(1)}_{\lambda-1}=\delta-1$; $A^{(1)}_{\lambda}=1$, by evaluation.

Each of the $(\delta-1)$ forms $[\lambda-1\ldots]$ and the single form $[\lambda\ldots]$ has in the relation of total order $(m\lambda+\lambda-1)$ an index $(m+1)(\lambda-1)$; hence no $[\ldots]$ may here include a number preceding a smaller number. So the δ linear forms here involved must be $[\lambda\lambda\ldots\lambda]$ and $[\lambda-1\ldots\lambda-1.\lambda.\lambda\ldots\lambda]$, where r of the numbers $(\lambda-1)$ are followed by $(\delta-r-1)$ of the numbers λ, r being one of $1, 2, 3, \ldots, \delta-1$.

Let
$$[\lambda\lambda\ldots\lambda]\equiv x_1+b_2x_2+b_3x_3+\ldots+b_\delta x_\delta;$$
$$[\lambda-1\ldots\lambda-1]\equiv x_1+a^{(1)}_2x_2+a^{(1)}_3x_3+\ldots+a^{(1)}_{\delta-1}x_{\delta-1}+a^{(1)}_\delta x_\delta;$$
$$[\lambda-1\ldots\lambda-1.\lambda]\equiv x_1+a^{(1)}_2x_2+a^{(1)}_3x_3+\ldots+a^{(1)}_{\delta-1}x_{\delta-1}+a^{(2)}_\delta x_\delta;$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$[\lambda-1\ldots\lambda-1.\lambda\ldots\lambda]\equiv x_1+a^{(1)}_2x_2+\ldots+a^{(1)}_{\delta-r}x_{\delta-r}+a^{(2)}_{\delta-r+1}x_{\delta-r+1}+\ldots$$
$$\ldots+a^{(r+1)}_\delta x_\delta$$
(wherein the last r numbers of $[\ldots]$ are λ's),
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$[\lambda-1.\lambda\ldots\lambda]\equiv x_1+a^{(1)}_2x_2+a^{(2)}_3x_3+\ldots+a^{(\delta-r-1)}_{\delta-r}x_{\delta-r}+\ldots$$
$$\ldots+a^{(\delta-1)}_\delta x_\delta.$$
Here the coefficients are unrestricted, except that, by § 2,
$$a^{(1)}_2\neq b_2 \quad\text{and}\quad a^{(2)}_r\neq a^{(1)}_r,\quad\text{for}\quad r=3, 4, \ldots, \delta-1, \delta.$$
We have to show that we cannot find constants $P_0, P_1, P_2, \ldots, P_{\delta-2}, Q$, different from zero, satisfying
$$Q(x_1+b_2x_2+\ldots+b_\delta x_\delta)$$
$$+\sum_{r=0}^{r=\delta-2}P_r(x_1+a^{(1)}_2x_2+\ldots+a^{(1)}_{\delta-r}x_{\delta-r}+a^{(2)}_{\delta-r+1}x_{\delta-r+1}+\ldots+a^{(r+1)}_\delta x_\delta)\equiv 0.$$

AUTHORS of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. *d.*	*s.* *d.*	*s.* *d.*
4 pages	Free	3 6	5 0	6 0
8 „	„	4 6	6 6	8 0
12 „	„	7 0	10 6	13 0
16 „	„	8 0	12 0	15 0
32 „	„	14 0	21 0	26 0

It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. FRANCIS HODGSON, 89 Farringdon Street, E.C., at the following rates:—

Vol. I.	Nos. 1–8	... Jan. 1865	to Nov. 1866	...	10*s.*
„ II.	„ 9–20	... Nov. 1866	„ 1869	...	16*s.*
„ III.	„ 21–40	... „ 1869	„ 1871	...	20*s.*
„ IV.	„ 41–66	... „ 1871	„ 1873	...	31*s.* 6*d.*
„ V.	„ 67–75	... „ 1873	„ 1874	...	15*s.*
„ VI.	„ 76–88	... „ 1874	„ 1875	...	21*s.*
„ VII.	„ 89–103	... „ 1875	„ 1876	...	21*s.*
„ VIII.	„ 104–123	... „ 1876	„ 1877	...	21*s.*
„ IX.	„ 124–140	... „ 1877	„ 1878	...	21*s.*
„ X.	„ 141–155	... „ 1878	„ 1879	...	18*s.*
„ XI.	„ 156–165	... „ 1879	„ 1880	...	12*s.* 6*d.*
„ XII.	„ 166–179	... „ 1880	„ 1881	...	16*s.*
„ XIII.	„ 180–194	... „ 1881	„ 1882	...	18*s.*
„ XIV.	„ 195–214	... „ 1882	„ 1883	...	20*s.*
„ XV.	„ 215–232	... „ 1883	„ 1884	...	20*s.*
„ XVI.	„ 233–252	... „ 1884	„ 1885	...	20*s.*
„ XVII.	„ 253–279	... „ 1885	„ 1886	...	25*s.*
„ XVIII.	„ 280–304	... „ 1886	„ 1887	...	25*s.*
„ XIX.	„ 305–342	... „ 1887	„ 1888	...	30*s.*
„ XX.	„ 343–369	... „ 1888	„ 1889	...	25*s.*
„ XXI.	„ 370–398	... „ 1889	„ 1890	...	25*s.*
„ XXII.	„ 399–429	... „ 1890	„ 1891	...	27*s.* 6*d.*
„ XXIII.	„ 430–449	... „ 1891	„ 1892	..	20*s.*
„ XXIV.	„ 450–474	... „ 1892	„ 1893	...	22*s.*
„ XXV.	„ 475–499	.. „ 1893	„ 1894	...	22*s.*
„ XXVI.	„ 500–534	... „ 1894	„ 1895	...	30*s.*
„ XXVII.	„ 535–574	... „ 1895	„ 1896	...	35*s.*
„ XXVIII.	„ 575–611	... „ 1896	„ 1897	...	32*s.* 6*d.*
„ XXIX.	„ 612–657	... „ 1897	„ 1898	Pt. I.	20*s.*
				Pt. II.	20*s.*
„ XXX.	„ 658–678	... „ 1898	to Mar. 1899	...	20*s.*
„ XXXI.	„ 679–709	... April 1899	„ Dec. 1899	...	20*s.*
„ XXXII.	„ 710–737	... Jan. 1900	„ June 1900	...	20*s.*
„ XXXIII.	„ 738–763	.. Nov. 1900	„ Feb. 1901	...	20*s.*
„ XXXIV.	„ 764–789	... Mar. 1901	„ April 1902	...	20*s.*
„ XXXV.	„ 790–819	... May 1902	„ Jan. 1903	...	20*s.*

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s.* each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series will be 25*s.* each Volume.

The price of cases for binding the Volumes of the Second Series will be 1*s.* 3*d.*

// PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 2.—PART 2.

CONTENTS.

Issued August 13, 1904.

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. LOVE. W. BURNSIDE.

34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

The Croft, Bromley Road, Catford, S.E.—W. BURNSIDE.

Equate to zero the coefficients of $x_1, x_2, \ldots, x_\delta$ successively; then

$$Q + \sum_{r=0}^{r=\delta-2} P_r = 0, \tag{i.}$$

$$b_2 Q + a_2^{(1)} \sum_{r=0}^{r=\delta-2} P_r = 0, \tag{ii.}$$

$$b_3 Q + a_3^{(1)} \sum_{r=0}^{r=\delta-3} P_r + a_3^{(2)} P_{\delta-2} = 0, \tag{iii.}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$b_t Q + a_t^{(1)} \sum_{r=0}^{r=\delta-t} P_r + a_t^{(2)} P_{\delta-t+1} + \ldots + a_t^{(t-1)} P_{\delta-2} = 0, \tag{$\dot{t}$}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$b_\delta Q + a_\delta^{(1)} P_0 + a_\delta^{(2)} P_1 + \ldots + a_\delta^{(\delta-1)} P_{\delta-2} = 0. \tag{$\dot{\delta}$}$$

From (i.) and (ii.), since $a_2^{(1)} \neq b_2$,

$$Q = 0, \qquad \sum_{r=0}^{r=\delta-2} P_r = 0;$$

then, from (ii.) and (iii.), since $a_3^{(2)} \neq a_3^{(1)}$,

$$P_{\delta-2} = 0, \qquad \sum_{r=0}^{r=\delta-3} P_r = 0;$$

then, from (iii.) and (iv.), since $a_4^{(2)} \neq a_4^{(1)}$,

$$P_{\delta-3} = 0, \qquad \sum_{r=0}^{r=\delta-4} P_r = 0;$$

and so on, taking the equations in pairs.

From the first t equations we deduce

$$Q = P_{\delta-2} = \ldots = P_{\delta-t+1} = \sum_{r=0}^{r=\delta-t} P_r = 0,$$

and, taking all the equations, we have

$$Q = P_0 = P_1 = P_2 = \ldots = P_{\delta-2} = 0.$$

Hence the preceding relations are impossible for all values of m.

It follows, from § 1, that

Any quantic F_x^n of order n in the variables $x_1, x_2, \ldots, x_\delta$, where n is one of $m\lambda, m\lambda+1, \ldots, m\lambda+\lambda-1$, m and λ being any integers, is uniquely expressible in the following form:—if $n = m\lambda+\lambda-s$,

s being any one of 1, 2, 3, ..., λ,

$$F_x^m \equiv \Sigma\,[a, \beta, \gamma, \ldots, \epsilon]^{(\lambda-1)(m+1)-s+1+\theta+\xi}\, p_x^{m-\theta-\xi},$$

where ξ is zero if $a \geqslant s$, and is unity if $a < s$, θ being the number of places in the sequence $a\beta\gamma\ldots\epsilon$, where a number precedes immediately a smaller number. The linear forms $[a\beta\gamma\ldots\epsilon]$ are defined as in § 2.

Application to Binary Perpetuants.

9. Consider the expression

$$M \equiv \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_\delta x_\delta; \tag{1}$$

and suppose the coefficients μ satisfy the relation

$$\mu_{r^{(\epsilon)}} - \mu_{r^{(\epsilon-1)}} + \ldots + (-)^{\epsilon-1}\mu_{r^{(1)}} = 0,$$

where $r^{(\epsilon)}, r^{(\epsilon-1)}, \ldots, r^{(2)}, r^{(1)}$ are any ϵ of the suffixes $\delta, \delta-1, \ldots, 3, 2, 1$, such that $r^{(\epsilon)} > r^{(\epsilon-1)} > r^{(\epsilon-2)} > \ldots r^{(2)} > r^{(1)}$. In this case M is expressible in the form

$$M \equiv \nu_1 y_1 + \nu_2 y_2 + \ldots + \nu_{\delta-1} y_{\delta-1},$$

where the ν's are linear functions of the μ's, and each y is either one of the x's or is a sum of consecutive x's, *e.g.*,

$$y \equiv x_s + x_{s+1} + \ldots + x_{t-1} + x_t.$$

The values of the y's and the ν's may, in fact, be written down by inspection: they are:—

$$\left.\begin{cases} y_{r^{(\theta)}} = x_{r^{(\theta)}} + x_{r^{(\theta)}+1} + \ldots + x_{r^{(\theta+1)}-1} + x_{r^{(\theta+1)}} & (\theta = 1, 2, 3, \ldots, \epsilon-1) \\ \left.\begin{array}{l} y_\kappa = x_\kappa, \ \text{ if } \ \kappa < r^{(\epsilon)} \\ y_{\kappa-1} = x_\kappa, \ \text{ if } \ \kappa > r^{(\epsilon)} \end{array}\right\} & (\kappa \neq r^{(\theta)}, \text{ for } \theta = 1, 2, 3, \ldots, \epsilon) \end{cases}\right\}, \tag{2}$$

and

$$\left\{\begin{array}{ll} \nu_\kappa = \mu_\kappa, \ \text{ if } \ 1 \leqslant \kappa < r^{(1)} & \\ \nu_{\kappa-1} = \mu_\kappa, \ \text{ if } \ r^{(\epsilon)} < \kappa \leqslant \delta & \\ \left.\begin{array}{l} \nu_{r^{(\theta)}} = \mu_{r^{(\theta)}} - \mu_{r^{(\theta-1)}} + \ldots + (-)^{\theta-1}\mu_{r^{(1)}} \\ \nu_{r^{(\theta)}+\xi} = \mu_{r^{(\theta)}+\xi} - \nu_{r^{(\theta)}}, \quad \text{if } \ \xi < r^{(\theta+1)} - r^{(\theta)} \\ \phantom{\nu_{r^{(\theta)}+\xi}} = \mu_{r^{(\theta)}+\xi} - \mu_{r^{(\theta)}} + \mu_{r^{(\theta-1)}} + \ldots + (-)^{\theta}\mu_{r^{(1)}} \end{array}\right\} & (\theta = 1, 2, 3, \ldots, \epsilon-1) \end{array}\right\}.$$

It will be seen that $\nu_{r^{(\epsilon)}}$ is zero by virtue of relation (1). The relation (1) is completely determined by the suffixes involved, and, since the number of relations each involving s suffixes is $\binom{\delta}{s}$, the total number of such relations is

$$\binom{\delta}{1}+\binom{\delta}{2}+\ldots+\binom{\delta}{s}+\ldots+\binom{\delta}{\delta} = 2^{\delta}-1.$$

10. Let $x_s = \dfrac{(a_s a_{s+1})}{a_{s_z} a_{s+1_z}} \Pi$, for $s = 1, 2, 3, \ldots, \delta$, where

$$\Pi \equiv a_{1_z}, a_{2_z}, \ldots, a_{\delta+1},$$

$a_1, a_2, \ldots, a_{\delta+1}$ being the symbols of binary quantics of infinite order in the variables z_1, z_2; then

$$x_s+x_{s+1}+\ldots+x_{t-1}+x_t = \frac{(a_s a_{t+1})}{a_{s_z} a_{t+1_z}} \Pi.$$

The y forms in (2) therefore involve the symbolical determinants

$$(a_{r^{(\theta)}} a_{r^{(\theta+1)}+1}) \qquad (\theta = 1, 2, 3, \ldots, \epsilon-1),$$

$$(a_\kappa a_{\kappa+1}), \text{ if } \kappa \neq r^{(\theta)}, \text{ for } \theta = 1, 2, 3, \ldots, \epsilon;$$

and so any expression such as M, whose coefficients μ satisfy the relation (1), may be divided into two parts involving the suffixes

$$\left.\begin{array}{l} 1, 2, 3, \ldots, r^{(1)},\ r^{(2)}+1, \ldots, r^{(3)},\ r^{(4)}+1, \ldots, r^{(5)},\ r^{(2\theta)}+1, \ldots, r^{(2\theta+1)}, \text{ \&c.} \\ \text{and} \\ r^{(1)}+1, \ldots, r^{(2)},\ r^{(3)}+1, \ldots, r^{(4)},\ r^{(5)}+1, \ldots, r^{(6)},\ r^{(2\theta+1)}+1, \ldots, r^{(2\theta+2)}, \text{ \&c.} \\ \hfill \text{respectively} \end{array}\right\}, \tag{3}$$

where the dotted lines ... are to be replaced by consecutive numbers, so that there is a break in the sequence whenever we arrive at a suffix number $r^{(\theta)}$ involved in the relation (1), and no number occurs in the two sets.

If we substitute the preceding values for $x_1, x_2, \ldots, x_\delta$, then M^{w} is a sum of terms each of which is a product of perpetuants of forms of the two sets given by (3); by taking the $(2^{\delta}-1)$ relations like (1), we obtain, in the same way, sets like (3), giving terms of all the $(2^{\delta}-1)$ varieties of perpetuant products. If the coefficients of M satisfy the relation (1),

then M is annihilated by

$$O \equiv \frac{\partial}{\partial x_{r(\epsilon)}} - \frac{\partial}{\partial x_{r(\epsilon-1)}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r(1)}},$$

and there are $(2^\delta - 1)$ such linear partial differential operators, which we shall denote by

$$O_1, O_2, \ldots, O_{(2^\delta-1)}.$$

11. Now, in place of the operators of § 2, by means of which each relation is derived from the relation of order greater by unity, let us take these operators

$$O_1, O_2, \ldots, O_{(2^\delta-1)},$$

so that now $\lambda = 2^\delta - 1$.

Take the general relation of total order $\{(m+1)\lambda - 1\}$ in $(\delta - 1)$ variables as determined in § 7, and from it build up the relation in δ variables in the following way. Replace each form $[uvw\ldots]$ in that relation by $[suvw\ldots]'$, s being any one of $1, 2, 3, \ldots, \lambda$, where $[suvw\ldots]'$ is determined thus :—in the relations for δ variables, let

$$O_s \equiv \frac{\partial}{\partial x_{r(\epsilon)}} - \frac{\partial}{\partial x_{r(\epsilon-1)}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r(1)}}$$

be the operator which operating on the relation of total order $\{(m+1)\lambda - s\}$ gives a relation of the same form as that of total order $\{(m+1)\lambda - (s+1)\}$; and in $[uvw\ldots] \equiv x_1 + (u)\, x_2 + \ldots + (uvw\ldots)\, x_{\delta-1}$ replace $x_1, x_2, \ldots, x_{\delta-1}$ by $y_1, y_2, \ldots, y_{\delta-1}$ [given by (3)] in any definite order, which must be the same for all such linear forms ; the resulting linear form is the form $[suvw\ldots]'$ in question. If we do the same for each of the other operators, generalize the quantics by introducing x_δ, follow the same law for the indices of the linear forms, and take the sum of the $\lambda^{\delta-1}$ terms thus obtained, we shall, as before, obtain the relation of total order $\{(m+1)\lambda - 1\}$ in the δ variables $x_1, x_2, \ldots, x_\delta$. The remaining $(\lambda - 1)$ relations can be found from this in the same way as before, and from their constitution it follows that, if the relation of total order n is impossible, so also is the relation of total order $(n+1)$, for all values of n, and we need only consider the case of total order unity.

If we take $O_{\lambda-1} \equiv \dfrac{\partial}{\partial x_1}$ and $O_\lambda \equiv \dfrac{\partial}{\partial x_2}$, then the forms $[\lambda - 1 \ldots]'$, $[\lambda \ldots]'$ are free of x_1 and x_2 respectively, and the proof that each quantic (constant) in the relation of total order unity is identically zero proceeds on exactly similar lines to that in § 8. The corresponding theorem relating to the unique expression of any form follows at once from § 1.

12. Make the preceding substitutions for $x_1, x_2, \ldots, x_\delta$ in any such unique expression, and multiply by the proper (infinite) powers of $a_{1_s}, a_{2_s}, \ldots, a_{\delta+1_s}$ respectively; in the unique expression of total order $\{(m+1)\lambda-s\}$ put $m=0$; then every quantic on the right-hand side either vanishes or becomes a constant, while each linear form gives rise to terms which are perpetuant products, by § 10, and therefore any perpetuant of weight $(\lambda-s)$, where s is one of $1, 2, 3, \ldots, \lambda$, is reducible. Hence :—

The minimum weight of an irreducible perpetuant of degree $(\delta+1)$ is

$$2^\delta-1.$$

Again, taking the form of unique expression of a quantic of order $\{(m+1)\lambda-s\}$ in this way, every linear form involves $(\delta-1)$ symbolical determinants, and has an index $\{(m+1)(\lambda-1)-s+1\}$ at least; hence, if g denote the grade of a perpetuant of degree $\delta+1$ and weight

$$\omega\,[=(m+1)\lambda-s],$$

$$g \geqslant \frac{(m+1)(\lambda-1)-s+1}{\delta-1}, \quad \text{where} \quad \lambda = 2^\delta-1,$$

or
$$g \geqslant \frac{(\omega+s)\dfrac{\lambda-1}{\lambda}-s+1}{\delta-1} \geqslant \frac{\omega(\lambda-1)+\lambda-s}{\delta-1};$$

and therefore
$$g \geqslant \frac{\omega(\lambda-1)}{\lambda(\delta-1)}, \quad \text{since} \quad s \leqslant \lambda.$$

Hence :—

The grade of any perpetuant of weight ω and degree $\delta \geqslant \dfrac{2^{\delta-1}-2}{2^{\delta-1}-1}\,\dfrac{\omega}{\delta-2}$.

If $\delta=3$, the grade $\geqslant \frac{2}{3}\omega$, which is the result of the Jordan lemma.

When δ is even, a higher value for the grade can be found by taking $\lambda=2$ in the result of § 8.

Interchanging x_1 and x_2 in that result, and taking three variables x_1, x_2, x_3, the linear forms occurring in the relation are

$$x_2+(1)x_1+(11)x_3, \qquad x_2+(2)x_1+(21)x_3,$$

$$x_2+(1)x_1+(12)x_3, \qquad x_2+(2)x_1+(22)x_3,$$

and we may take

$$(1)=(11)=(21)=0 \quad \text{and} \quad (2)=(12)=(22)=1;$$

so that the linear forms are

$$x_2, \qquad x_2+x_3, \qquad x_1+x_2, \qquad x_1+x_2+x_3,$$

or $$(a_2a_3), \qquad (a_2a_4), \qquad (a_1a_3), \qquad (a_1a_4);$$

each involving *one* symbolical determinant.

In the general case of a perpetuant of degree 2ϵ, take the unique expression for a form in $(2\epsilon-1)$ variables and interchange x_1 and x_2; then take every $(\ldots 1)$ as zero, and every $(\ldots 2)$ as unity: with these values of the coefficients each linear form will involve $(\epsilon-1)$ determinantal factors, for we have seen that it is true for $\epsilon=2$, and, assuming that it is true for $(2\epsilon-1)$ variables, we shall show that it is true for $(2\epsilon+1)$ variables. For let P be a linear form thus chosen in $(2\epsilon-1)$ variables, and so involving only $(\epsilon-1)$ determinantal factors; each of the linear forms in $(2\epsilon+1)$ variables can be written in one of the four ways

$$P+(\ldots 1)\,x_{2\epsilon}+(\ldots 11)\,x_{2\epsilon+1},$$

$$P+(\ldots 1)\,x_{2\epsilon}+(\ldots 12)\,x_{2\epsilon+1},$$

$$P+(\ldots 2)\,x_{2\epsilon}+(\ldots 21)\,x_{2\epsilon+1},$$

and $$P+(\ldots 2)\,x_{2\epsilon}+(\ldots 21)\,x_{2\epsilon+1},$$

or P, $P+x_{2\epsilon+1}$, $P+x_{2\epsilon}$, and $P+x_{2\epsilon}+x_{2\epsilon+1}$, each of which involves only ϵ determinantal factors. Hence such a choice for the linear forms is always possible, and therefore any perpetuant of weight ω and degree 2ϵ is of grade $\geqslant \omega/(2\epsilon-2)$. Combining these two results we have

If g is the grade of any perpetuant of weight ω and degree δ, then

$$g \geqslant \frac{\omega}{\delta-2}, \quad \text{if } \delta \text{ is even,}$$

and $$g \geqslant \left(\frac{\omega}{\delta-2}\right)\frac{2^{\delta-1}-2}{2^{\delta-1}-1}, \quad \text{if } \delta \text{ is odd.}$$

The values of the grade for $\delta = 3, 4, \ldots, 8$ are

$$\text{Weight } \omega \left\{\begin{array}{lcccccc} \text{Degree:} & 3, & 4, & 5, & 6, & 7, & 8 \\ \text{Grade:} & \frac{2\omega}{3}, & \frac{\omega}{2}, & \frac{14\omega}{45}, & \frac{\omega}{4}, & \frac{62\omega}{315}, & \frac{\omega}{6} \end{array}\right\}.$$

The grade decreases with the degree, as it should, for

$$\text{(i.)} \quad \frac{\omega}{2\epsilon-2} > \left(\frac{\omega}{2\epsilon-1}\right)\frac{2^{2\epsilon}-2}{2^{2\epsilon}-1} \quad \text{obviously,}$$

and $$\text{(ii.)} \quad \frac{\omega}{2\epsilon-2} < \left(\frac{\omega}{2\epsilon-3}\right)\frac{2^{2\epsilon-2}-2}{2^{2\epsilon-2}-1},$$

if $$(2\epsilon-3)(2^{2\epsilon-2}-1) < (2\epsilon-2)(2^{2\epsilon-2}-2),$$

i.e., if $2^{2\epsilon-2} > 2\epsilon-1$, which is true if $\epsilon > 1$.

It is obvious that a much more general choice may be made for the linear forms and the corresponding operators; so that the preceding principles and *arithmetical* results still hold good.

The essential features of the method, apart from the numerical results which necessarily persist in all such relations, are:—

(i.) The form of relation of order n is derivable from the relation of order $(n+1)$ by operation with a suitable linear differential operator, which is, in some degree, arbitrary.

(ii.) Those linear forms in any of the relations which are annihilated by the operator used to reduce the order of that relation themselves yield terms, which constitute the standard form of relation of the same order in variables, whose number is less by unity.

SOME ILLUSTRATIONS OF MODES OF DECAY OF VIBRATORY MOTIONS

By A. E. H. LOVE.

[Received and Read March 10th, 1904.]

1.

The cause of decay of vibratory motion which is to be considered here is the communication of a disturbance from the vibrator to the surrounding medium. The energy of the vibrator is gradually transmitted to the distant parts of the medium by wave-motion. When the vibrator is a solid elastic body and the medium is air there are customary certain processes for estimating the rate of decay of the vibrations. In one of these processes the vibrator is assumed to have its natural free period, the motion of the air is assumed to be that progressive wave-motion of simple harmonic type in the same period which would be forced if the free motion in question were maintained for an indefinitely long time in opposition to the reaction of the air, the rate of transmission of the energy across a surface surrounding the vibrator is calculated, and this is taken to be the rate at which the energy of the actual vibratory motion is diminishing owing to the presence of the air. In another process it is assumed that waves of simple harmonic type are propagated in air outwards from the vibrator, the reaction of the air on the vibrator is calculated in accordance with this assumption, the equation of motion of the vibrator subject to this reaction is formed, and, with a certain interpretation of symbols, it is found to be of the ordinary type of damped harmonic vibrations.* The rate of decay of the actual vibratory motion is taken to be that indicated by this equation. These two processes lead to the same result when the rate of decay is slow, and when this is the case there can be no doubt of their correctness as regards the motion of the vibrator; but it is manifest that they do not give a complete account of the motion of the medium. This motion cannot be a progressive wave-motion of simple harmonic type because in the neighbourhood of the vibrator its amplitude is continually diminishing. Further, the motion of the vibrator must have a beginning in time, and

* See Lord Rayleigh, *Theory of Sound*, Vol. II., § 302, and other Articles.

consequently the waves that are generated must have a front or boundary at any instant; beyond this moving boundary there is no disturbance. The existence of such a boundary implies conditions which the disturbance must satisfy, but the processes which are indicated above take no account of any such conditions. So far as I am aware, the only problems concerning such motions which have been solved completely relate to systems in one dimension,* and it will be found that three-dimensional systems present new features.

Even in their application to the motion of the vibrator it is clear that the success of the above described methods depends upon the possibility of free vibrations of the vibrator when isolated from the medium, and upon the smallness of the part played by the medium in the actual motion. In the case of a sounding body the density of the body must be much greater than that of the air. Otherwise the vibrations of the body in air will not be approximately the same as in a vacuum. This remark becomes of great importance when it is sought to extend the methods to electrical vibrations. In this case the essential phenomenon is the wave-motion excited in the æther, and there is in general no meaning in electrical vibrations independent of the surrounding medium. Exceptional cases are the vibrations of a condenser with or without a small aperture,† and vibrations within an insulating body of enormous specific inductive capacity.‡ These are examples of systems in which electrical vibrations that approximate to free vibrations are possible.§ The nearly dead-beat oscillations of a Hertzian vibrator differ essentially from those that occur in the above mentioned exceptional cases. The vibrator is not in any sense isolated from the medium; and the disturbance that takes place is much more accurately described as a change of state of the medium than as a change of state of the vibrator.

The fundamental tone of acoustical resonators is given out by a mode of vibration which depends essentially upon the neck making communication with the external medium. Air contained in a cavity within a rigid boundary having no aperture has definite modes of free vibration, but none of them is the same as the mode of vibration that is characteristic of the resonator made by producing an aperture in the boundary. The system acquires through the existence of the aperture a new mode of

* H. Lamb, *Proceedings*, Vol. XXXII., p. 208 (1900).

† J. Larmor, *Ibid.*, Vol. XXVI. (1895).

‡ H. Lamb, *Cambridge Phil. Soc. Trans.*, Vol. XVIII. (1900).

§ A conductor outside which the space is doubly or multiply connected—*e.g.*, an infinite cylinder or a ring—admits, when thin, of electrical oscillations which are very nearly free oscillations.

vibration, which decays through the transmission of energy to the external air. A system with a permanent vibration of nearly the same period and type, which shall differ from the system consisting of the air in the cavity only by having an additional degree of freedom, can be devised by imagining a piston in the aperture to be held nearly in position by a constant force equal to that exerted upon it by the pressure of the air inside when at rest, and neglecting the air outside. The mode of vibration characteristic of the resonator is that in which the piston oscillates to and fro within the walls of the aperture. When the external air is present and there is no piston a slightly damped harmonic vibration with nearly the same period is possible, and such vibrations are excited by any causes which vary the pressure over the aperture, just as the oscillation of the piston would be excited by varying the force applied to it. The electrical analogue of the air in the cavity would appear to be the dielectric plate of a condenser of which the conducting surfaces are closed. Making an aperture in the outer conductor appears in this case not to introduce any new electrical degrees of freedom, and the analogue of the aperture in the acoustical problem appears to be a wire joining the two conducting surfaces.* But the analogue is imperfect, inasmuch as opening a communication with the external medium is no longer the process by which the new degree of freedom is introduced.

In what follows there will be investigated some problems concerning the generation of sound waves in air and of electrical vibrations in free æther. It will be seen that the customary methods represent well the motion of a sounding body, but that the nature of the sound waves generated by the body is in general different from that assumed in these methods. For electrical vibrations the case chosen will be that of a spherical conductor over which a surface distribution of electricity variable from point to point is produced. For the sake of simplicity the sphere will be taken to be a perfect conductor.

In the ordinary method of treating this problem,† the disturbance is assumed to be of exponential type, and the possible exponents are determined by the condition that the electric force at the surface of the conductor is normal to the conductor. The exponents may be real and negative or complex with negative real parts. Thus the solutions that are found

---

* H. M. Macdonald, *Electric Waves* (Cambridge, 1902), p. 57.

† J. J. Thomson, *Recent Researches*, pp. 361 et seq. The problem was treated by the same author in *Proceedings*, Vol. xv. (1884), and by H. Lamb in *Phil. Trans. Roy. Soc.*, Vol. CLXXIV. (1883).

contain factors of the forms

$$e^{-p(ct-r)} \quad \text{and} \quad e^{-p(ct-r)} \sin q\,(ct-r+\epsilon),$$

which tend to become infinite with r. These solutions cannot represent unlimited trains of waves propagated outwards. The waves that are actually propagated have a boundary which moves outwards with the velocity c. The effects due to the boundary of the waves are usually left out of account, and the disturbances of exponential type are also ignored. They will be found to represent an essential part of the disturbance. Whenever they can occur they are necessary to the continued advance of the wave-boundary.

In addition to problems of the decay of vibrations that are consequences of an initial state, some examples will be discussed of vibrations that are maintained for a time and are then left to decay when the cause that maintains them ceases to operate. These examples bring out the result that there is no essential difference in the modes of subsidence that are exhibited in the two cases.

2. *Introduction of Arbitrary Functions.*

In Prof. Lamb's paper* to which reference has been made there is given an illustration of the decay of vibratory motion by transmission of the energy to a distance. The system considered is a massive body attached to an infinitely long stretched string and capable of vibrating transversely under the action of a spring. The waves that are propagated along the string must be expressible by a function of the form $f(at-x)$. The initial state of the system being one of equilibrium, the body is struck transversely to the string, and the initial conditions together with the equation of motion of the body suffice to determine completely the value of the function f for all values of x and t. It appears that f is of the form $Ae^{-p(at-x)} \sin q\,(at-x)$ when $x < at$, but $f = 0$ when $x > at$. The step by which we can advance beyond the more customary and less satisfactory method of assuming that the waves in the string are of simple harmonic type is the substitution of an arbitrary function $f(at-x)$ for a function of the form $A \sin n\,(t-x/a)$. The solutions of problems connected with spherical boundaries which will be discussed below contain arbitrary functions of $t-r/a$, where r denotes distance from the centre of the sphere and a is the velocity of wave-propagation. In the case of sound waves the velocity potential ϕ satisfies the equation $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$,

* *Proceedings*, Vol. XXXII., p. 208.

and the most general solution that can express waves travelling outwards and be proportional to a spherical surface harmonic S_n is of the form

$$r^n S_n \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \left\{\frac{1}{r}\chi(at-r)\right\}.$$

In the case of electric waves, we take c for the velocity of radiation and obtain solutions of equal generality* by assuming a vector (ξ, η, ζ) to be given by means of equations of the form

$$\xi = \left(y\frac{\partial \omega_n}{\partial z} - z\frac{\partial \omega_n}{\partial y}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{\chi(ct-r)}{r}, \tag{1}$$

in which ω_n is a spherical solid harmonic of order n, and η, ζ are obtained from ξ by cyclical interchanges of the letters x, y, z, while ω_n and χ remain unaltered. Then we may write at pleasure either

$$\left.\begin{aligned}(\alpha, \beta, \gamma) &= \frac{\partial}{\partial t}(\xi, \eta, \zeta)\\ (X, Y, Z) &= \mathrm{c}\,\mathrm{curl}(\xi, \eta, \zeta)\end{aligned}\right\} \tag{2}$$

or

$$\left.\begin{aligned}(X, Y, Z) &= \frac{\partial}{\partial t}(\xi, \eta, \zeta)\\ (\alpha, \beta, \gamma) &= -\mathrm{c}\,\mathrm{curl}(\xi, \eta, \zeta)\end{aligned}\right\}. \tag{3}$$

Here (X, Y, Z) represents electric force measured electrostatically, (α, β, γ) represents magnetic force measured electromagnetically, and the axes of (x, y, z) are a right-handed system. In the case expressed by (1) and (2) the normal component of (X, Y, Z) at the surface of a sphere of radius r can be shown to be

$$-\mathrm{c}\,n(n+1)\frac{\omega_n}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n \frac{\chi(ct-r)}{r}; \tag{4}$$

so that this form is suitable for representing oscillating electric charges on a sphere, the distribution of surface density being proportional to the surface harmonic contained in ω_n.

3. *Conditions to be satisfied at Wave-Fronts.*†

For the determination of the arbitrary functions χ which occur in such solutions as the above we may have, in addition to the conditions which

* These solutions were given by the author in *Phil. Trans. Roy. Soc.*, Vol. CXCVII. (1901), as generalizations of the well-known forms in which χ is an exponential function of its argument.

† The results here stated for the case of waves advancing into a previously undisturbed portion of the medium have been given by the author in *Proceedings* (Ser. 2), Vol. 1, p. 37, and the extensions to cases in which the medium beyond the advancing wave-front is disturbed can be made without any difficulty.

must hold near to the vibrating nucleus, certain conditions which must be satisfied at the fronts of waves. In the case of a sound wave advancing into still air the wave-front advances with velocity a, and the velocity potential just behind the front must satisfy the condition

$$(\partial\phi/\partial t)+a(\partial\phi/\partial N) = 0,$$

where N denotes the direction of the normal to the front in the sense of advance of the front. If there is motion in the region into which the waves advance, we may denote by ϕ' the velocity potential just behind the wave-front and by ϕ'' that just ahead of the same surface, and then the condition to be satisfied is

$$(\partial\phi'/\partial t)+a(\partial\phi'/\partial N) = (\partial\phi''/\partial t)+a(\partial\phi''/\partial N). \tag{5}$$

The conditions $\phi'-\phi'' =$ const. in this case and $\phi =$ const. in the previous case also hold at the wave-front, but they will be found to be satisfied of themselves in the problems that we shall consider.

When a train of electric waves advances into a region in which the electric and magnetic forces are null the wave-front advances with the velocity c of radiation, and the electric force (X, Y, Z) and magnetic force (α, β, γ) just behind the advancing front are connected by the equations

$$\left.\begin{array}{rl} X = & \cos(z, N)\beta-\cos(y, N)\gamma \\ \ldots & \ldots \quad \ldots \quad \ldots \\ -\alpha = & \cos(z, N)Y-\cos(y, N)Z \\ \ldots & \ldots \quad \ldots \quad \ldots \end{array}\right\}, \tag{6}$$

where N is the normal to the wave-front drawn in that direction in which this front advances. If the magnetic and electric forces in the region into which the waves advance are not null, we may denote by (X', Y', Z') and $(\alpha', \beta', \gamma')$ the forces just behind the advancing front and by (X'', Y'', Z'') and $(\alpha'', \beta'', \gamma'')$ the forces just ahead of the same surface. Then the process by which equations (6) are established in the case where $X'', \ldots, \alpha'', \ldots$ are zero leads to two systems of equations, viz., three of the type

$$(X'-X'') = \cos(z, N)(\beta'-\beta'')-\cos(y, N)(\gamma'-\gamma'') \tag{7}$$

and three of the type

$$-(\alpha'-\alpha'') = \cos(z, N)(Y'-Y'')-\cos(y, N)(Z'-Z''). \tag{8}$$

The equations of types (7) and (8) are not independent. For example, the three equations of (8) show that

$$(\alpha'-\alpha'')\cos(x, N)+(\beta'-\beta'')\cos(y, N)+(\gamma'-\gamma'')\cos(z, N) = 0;$$

and, if this condition is satisfied, the three equations of (8) can be deduced from (7). The geometrical interpretation of the conditions is given in my paper already cited.

4. *Sphere Vibrating Radially in Air.*

The sphere will be treated as an elastic membrane of mass M and surface density σ, which is maintained nearly at a definite radius r_0 by springs. It will be supposed that, in the absence of the air, the frequency of vibration of the sphere would be $n/2\pi$. If ρ is the density of the air, r_0 the radius of the sphere when in equilibrium under the pressure of the air, $r_0+\xi$ its radius at time t, δp the excess of pressure above that in equilibrium, the equation of motion of the sphere is

$$M(\ddot{\xi}+n^2\xi) = -4\pi r_0^2\delta p,$$

where dots denote differentiation with respect to t.

With the ordinary approximations the velocity of the air and δp can be expressed in terms of the velocity potential ϕ. The above equation may be written

$$\ddot{\xi}+n^2\xi = \frac{\rho}{\sigma}\left(\frac{\partial\phi}{\partial t}\right)_{r=r_0}. \tag{9}$$

The velocity of sound in air being denoted by a, ϕ must satisfy the equation $\partial^2\phi/\partial t^2 = a^2\nabla^2\phi$ outside the sphere and the condition

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=r_0} = \dot{\xi}. \tag{10}$$

The conditions of the problem being symmetrical about the centre of the sphere, ϕ must have the form $r^{-1}\chi(at-r)$, where χ is an unknown function. Equations (9) and (10) may be written

$$\ddot{\xi}+n^2\xi = \frac{a\rho}{\sigma r_0}\chi', \qquad \dot{\xi} = -\frac{1}{r_0^2}(\chi+r_0\chi'), \tag{11}$$

where accents denote differentiation of the function $\chi(at-r_0)$ with respect to its argument. The system of differential equations (11) is of the third order, and we may solve it by eliminating ξ and forming a differential equation for χ or *vice versa*. We should get a linear differential equation of the third order with constant coefficients. The three arbitrary constants of the solution of the equation for χ are definite multiples of the constants of the solution of the equation for ξ. Instead of proceeding in this way, we can obtain the complete primitive of the system of equations by assuming the forms

$$\chi(at-r) = Ae^{\lambda(at-r+r_0)}, \qquad \xi = Be^{\lambda at}, \tag{12}$$

the constant factor $e^{\lambda r_0}$ being inserted in the form of χ. Then

$$(n^2+\lambda^2 a^2) B = \frac{a\rho}{\sigma r_0}\lambda A, \qquad \lambda a B = -\frac{1}{r_0^2} A(1+\lambda r_0). \tag{13}$$

It follows that λ satisfies the equation

$$(n^2+\lambda^2 a^2)(1+\lambda r_0)+\frac{\rho r_0}{\sigma}\lambda^2 a^2 = 0, \tag{14}$$

and that, if λ_1, λ_2, λ_3 are the roots of this equation, the complete primitive of the system of equations (11) leads to the following forms for ϕ and ξ:—

$$\phi = \frac{1}{r}\left[A_1 e^{\lambda_1(at-r+r_0)}+A_2 e^{\lambda_2(at-r+r_0)}+A_3 e^{\lambda_3(at-r+r_0)}\right], \tag{15}$$

$$\xi = -\frac{1+\lambda_1 r_0}{r_0^2 a\lambda_1}A_1 e^{\lambda_1 at}-\frac{1+\lambda_2 r_0}{r_0^2 a\lambda_2}A_2 e^{\lambda_2 at}-\frac{1+\lambda_3 r_0}{r_0^2 a\lambda_3}A_3 e^{\lambda_3 at}. \tag{16}$$

5. *Symmetrical Sound Wave produced by Initial Impulse.*

In the simplest case the system is set in motion by an impulse delivered at the instant $t=0$. Then ϕ vanishes when t is negative, and ξ vanishes when $t=0$, but $\dot{\xi}$ has a given value $\dot{\xi}_0$ when $t=0$. The condition that ϕ vanishes for all negative values of t requires that $\chi(\zeta)=0$ for all values of ζ which are less than $-r_0$. Hence the solution expressed by (15) holds only for values of r which are less than $at+r_0$. For greater values of r, $\chi(at-r)=0$. Hence we have a wave with a boundary $r=at+r_0$ travelling outwards with velocity a. The values of $\partial\phi/\partial r$ and $\partial\phi/\partial t$ at the front of the wave must satisfy the condition expressed in § 3, viz.,

$$\frac{\partial\phi}{\partial t} = -a\frac{\partial\phi}{\partial r},$$

or $$\frac{A_1\lambda_1 a}{r}+\frac{A_2\lambda_2 a}{r}+\frac{A_3\lambda_3 a}{r} = a\left[\frac{A_1\lambda_1}{r}+\frac{A_2\lambda_2}{r}+\frac{A_3\lambda_3}{r}+\frac{A_1+A_2+A_3}{r^2}\right],$$

or $$A_1+A_2+A_3 = 0.$$

The initial conditions in regard to ξ and $\dot{\xi}$ give the equations

$$\left(\frac{1}{\lambda_1}+r_0\right)A_1+\left(\frac{1}{\lambda_2}+r_0\right)A_2+\left(\frac{1}{\lambda_3}+r_0\right)A_3 = 0,$$

$$(1+\lambda_1 r_0)A_1+(1+\lambda_2 r_0)A_2+(1+\lambda_3 r_0)A_3 = -r_0^2\dot{\xi}_0.$$

To determine the constants A_1, A_2, A_3 we have therefore the equations

$$\Sigma A = 0, \qquad \Sigma A/\lambda = 0, \qquad \Sigma A\lambda = -r_0\dot{\xi}_0, \tag{17}$$

and the complete solution of the problem is expressed by the equations

$$\xi = \frac{1}{ar_0}\dot{\xi}_0\left\{\frac{(1+\lambda_1 r_0)e^{\lambda_1 at}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{(1+\lambda_2 r_0)e^{\lambda_2 at}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}+\frac{(1+\lambda_3 r_0)e^{\lambda_3 at}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right\}, \tag{18}$$

$$\phi = -\frac{r_0}{r}\dot{\xi}_0\left\{\frac{\lambda_1 e^{\lambda_1(at-r+r_0)}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}+\frac{\lambda_2 e^{\lambda_2(at-r+r_0)}}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}+\frac{\lambda_3 e^{\lambda_3(at-r+r_0)}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right\}, \tag{19}$$

of which the latter holds when $at+r_0 > r > r_0$. When $r > at+r_0$ we must put $\phi = 0$. Here λ_1, λ_2, λ_3 are the roots of the equation (14).

The solution represents a composite system of waves. To interpret it we consider the case where the ratio $\rho r_0/\sigma$ is small. The roots of equation (14) are to a first approximation

$$\lambda_1 = -1/r_0, \qquad \lambda_2 = \iota n/a, \qquad \lambda_3 = -\iota n/a.$$

To this order of approximation the first term in ξ vanishes and ξ becomes $n^{-1}\dot{\xi}_0 \sin nt$. To the same order of approximation ϕ becomes

$$\frac{r_0^2 a^2}{a^2+n^2r_0^2}\dot{\xi}_0\frac{1}{r}\left[e^{-(at+r_0-r)/r_0}-\cos n\left(t-\frac{r-r_0}{a}\right)-\frac{nr_0}{a}\sin n\left(t-\frac{r-r_0}{a}\right)\right]. \tag{20}$$

Hence, to this order of approximation, the motion of the sphere is the same as it would be in the absence of the air, and the motion of the air consists of two wave-motions: one of simple harmonic type which would be forced by the maintenance for an indefinite time of this motion of the sphere, the other of exponential type. Near the sphere the latter is damped rapidly, but near the front of the wave it is of the same degree of importance as the simple harmonic wave. The wave of exponential type is practically confined to a small region near the front of the advancing wave, but, in this region, it is comparable with the simple harmonic wave and the coexistence of the two is required for the continued advance of the front.

When we proceed to a second approximation we find that, to the first order in $\lambda r_0/\sigma$,

$$\left.\begin{aligned}\lambda_1 &= -\frac{1}{r_0}-\frac{\rho}{\sigma}\frac{a^2}{a^2+n^2r_0^2}\\ \lambda_2 &= -\frac{\rho r_0}{2\sigma}\frac{n^2r_0}{a^2+n^2r_0^2}+\iota\left(\frac{n}{a}-\frac{\rho r_0}{2\sigma}\frac{na}{a^2+n^2r_0^2}\right)\end{aligned}\right\}, \tag{21}$$

and λ_3 is the imaginary conjugate to λ_2. It follows that the motion of

the sphere consists of a motion of exponential type which decays very rapidly, compounded with a motion of the ordinary damped harmonic type. Since the coefficient of $e^{\lambda_1 at}$ in (18) is small of the order $\rho r_0/\sigma$, the former motion is small compared with the latter, and, since this coefficient is negative, the effect of this component of the motion is to make the maximum displacement of the surface slightly less than it would be in the absence of the air. The modulus of decay of the damped harmonic oscillations is $\frac{\rho r_0}{2\sigma}\frac{n^2 r_0 a}{a^2+n^2 r_0^2}$, which is the value that would be found by the customary methods. In the motion of the air the simple harmonic wave-trains obtained by the first approximation become damped harmonic wave-trains, so that the motion near the sphere subsides gradually in the same way as the motion of the sphere; but, since all the exponentials in (19) have the value unity at the wave-front, there is no damping at the front, and the motion at the front of the wave is subject to diminution through the law of spherical divergence only. As before, the co-existence of the exponential wave-train and the slightly damped harmonic wave-train is necessary to the continued advance of the wave-front.

When there is initial displacement as well as initial velocity the problem is but slightly more complicated. The second of equations (17) must then be replaced by $\Sigma A/\lambda = -r_0^2 a\xi_0$. *Keeping the first approximation only, we find that when t is positive and r is less than $at+r_0$ the forms for ξ and ϕ are

$$\xi = \xi_0 \cos nt + n^{-1}\dot{\xi}_0 \sin nt,$$

$$\phi = \frac{A}{r}\left[\sin\left\{n\left(t-\frac{r-r_0}{a}\right)+\alpha\right\} - \sin\alpha\, e^{-(at-r+r_0)/r_0}\right],$$

where A and α are given in terms of ξ_0 and $\dot{\xi}_0$ by the equations

$$A^2 = \frac{a^2 r_0^4(\dot{\xi}_0^2+n^2\xi_0^2)}{a^2+n^2 r_0^2}, \qquad \tan\alpha = \frac{a\dot{\xi}_0+n^2 r_0\xi_0}{n(r_0\dot{\xi}_0-a\xi_0)}.$$

The form of $r\phi$, as the sum of a simple harmonic function of $n(t-r/a)$ and an exponential function of $(at-r)/r_0$, is determined by the conditions which hold at the surface of the vibrating sphere. The form of the ratio of the coefficients of these two terms, viz., $-\sin\alpha$, is determined by the conditions which hold at the front of the wave. The actual value of this

* A small addition (placed between asterisks) has here been made to the paper (April 17th, 1904). Prof. Larmor called my attention to the special case noted in equation (21a).

ratio is determined by the initial conditions. As in the case of initial velocity without initial displacement, the wave is in general composite. In one case it can be simple. This happens when ξ_0 and $\dot{\xi}_0$ are connected by the equation $\dot{\xi}_0 = -n^2 r_0 a^{-1} \xi_0$. In this case α vanishes and ϕ has the form

$$\phi = A r^{-1} \sin n \{t-(r-r_0)/a\}. \tag{21a*}$$

6. *Decay of Vibrations that have been maintained for a time.*

We may extend the method of Art. 5 to the case where the system is set in motion by forces which operate for a finite time. It will be sufficient to consider the motion due to periodic forces acting in the interval $t_1 > t > 0$, and to suppose that when $t < 0$ the sphere and the air surrounding it are at rest. Taking the force acting on the sphere to be proportional to $e^{\iota\kappa at}$, equation (9) is replaced by an equation of the form

$$\ddot{\xi}+n^2\xi = Fe^{\lambda_0 at}+\frac{\rho}{\sigma}\left(\frac{\partial\phi}{\partial t}\right)_{r=r_0}, \tag{22}$$

where λ_0 is written for $\iota\kappa$. Equation (10) is unaltered, and the form of ϕ is the same as before, viz., $r^{-1}\chi(at-r)$. The system of equations (10) and (22) will possess a particular solution of the form

$$\phi = r^{-1}A_0 e^{\lambda_0(at-r+r_0)}, \qquad \xi = B_0 e^{\lambda_0 at}, \tag{23}$$

where

$$\left.\begin{aligned} B_0(n^2+\lambda_0^2a^2)-\frac{a\rho}{\sigma r_0}\lambda_0 A_0 &= F \\ B_0 a\lambda_0+\frac{1}{r_0^2}(1+\lambda_0 r_0)A_0 &= 0 \end{aligned}\right\}. \tag{24}$$

Since equation (14) has not any pure imaginary roots, λ_0 cannot be a root of it, and the equations (24) determine A_0 and B_0 in terms of F. We shall therefore take A_0 and B_0 to be known. The complete expressions for ϕ and ξ are to be determined by adding to the right-hand members of (23) expressions of the forms given by (15) and (16), in which the constants A_1, A_2, A_3 are to be determined by the conditions that ξ and $\dot{\xi}$ vanish when $t = 0$ and that $(\partial\phi/\partial t)+a(\partial\phi/\partial r)$ vanishes at $r = at+r_0$. These conditions give

$$\sum_0^3\left(\frac{1}{\lambda_s}+r_0\right)A_s = 0, \quad \sum_0^3(1+r_0\lambda_s)A_s = 0, \quad \sum_0^3 A_s = 0, \tag{25}$$

and these equations determine A_1, A_2, A_3 in terms of A_0. It follows

that we may put

$$\left.\begin{aligned}\xi &= -\frac{1}{ar_0^2}\sum_0^3\left(\frac{1}{\lambda_s}+r_0\right)A_s e^{\lambda_s at} \\ \phi &= \frac{1}{r}\sum_0^3 A_s e^{\lambda_s(at-r+r_0)}\end{aligned}\right\}, \tag{26}$$

in which the A's are known in terms of F, λ_0 is $\iota\kappa$, and λ_1, λ_2, λ_3 are the roots of the equation (14). This solution holds for ξ when t is in the interval $t_1 > t > 0$, and it holds for ϕ when $at+r_0 > r > r_0$ and t is in the same interval. The motion of the sphere is compounded of three motions :—(1) a simple harmonic motion of the same period as the force and having a definite phase-relation to the force, (2) a motion of exponential type which is relatively very small when the sphere is massive, (3) a motion of slightly damped harmonic type. The second and third of these motions are of the same types as those which are consequent upon an initial disturbance. The motion of the air is compounded of three wave-motions of types corresponding exactly with the three motions of the sphere. When the force has been in action for a sufficiently long time the motion of the sphere is practically a simple harmonic motion, and the motion of the air near the sphere is practically a simple harmonic wave-train. These motions are represented by the particular solutions (23). But the motion of the air near the front of the waves never has this simple character. The co-existence of the three types of waves is necessary to the continued advance of the wave-front.

The mode of decay of the vibratory motion after the force has ceased to act will be determined by taking a new solution of the equations (9) and (10) in the forms

$$\left.\begin{aligned}\phi &= \frac{1}{r}\sum_1^3 A'_s e^{\lambda_s(at-at_1-r+r_0)} \\ \xi &= -\frac{1}{ar_0^2}\sum_1^3\left(\frac{1}{\lambda_s}+r_0\right)A'_s e^{\lambda_s(at-at_1)}\end{aligned}\right\}, \tag{27}$$

in which constant factors $e^{\lambda_s at_1}$ are absorbed in the constants A'_s. The constants A'_1, A'_2, A'_3 are determined by the conditions that ξ and $\dot{\xi}$ have given values when $t = t_1$, and that $(\partial\phi/\partial t)+a(\partial\phi/\partial r)$ is continuous at the surface $r = at-at_1+r_0$. The solution expressed by (27) will hold in the interval $t > t_1$ and in the region $r_0 < r < at-at_1+r_0$. The equations by

which the constants A'_s are determined are accordingly

$$\left.\begin{aligned}\sum_1^3\left(\frac{1}{\lambda_s}+r_0\right)A'_s &= \sum_0^3\left(\frac{1}{\lambda_s}+r_0\right)A_s e^{\lambda_s a t_1}\\ \sum_1^3(1+\lambda_s r_0)\,A'_s &= \sum_0^3(1+\lambda_s r_0)\,A_s e^{\lambda_s a t_1}\\ \sum_1^3 A'_s &= \sum_0^3 A_s e^{\lambda_s a t_1}\end{aligned}\right\}. \tag{28}$$

The results show that the simple harmonic motion of the sphere with the period of the force ceases at once, and the subsequent motion of the sphere is of the same kind as the motion consequent upon given initial displacements and velocities. The motion of the air near the sphere is of the same kind as that determined by initial conditions. The two types of motion—exponential and slightly damped harmonic—must co-exist in order that the waves sent out in the subsequent motion may be continuous with the waves sent out by the maintained vibrations.

7. *Rigid Sphere vibrating in Air.*

As a second example, we may consider the vibrations of a rigid sphere of mass M controlled by a spring of such strength that in the absence of the air the frequency would be $n/2\pi$. The surface of the sphere at any time may be taken to be expressed by the equation $r = r_0+\xi P_1$, where P_1, or more fully $P_1(\cos\theta)$, is the zonal surface harmonic of degree unity referred to the line of motion of the centre as axis. The motion of the air will be expressed by a velocity potential ϕ of the form given by the equation

$$\phi = rP_1\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi(at-r)}{r} = -P_1 r^{-2}(\chi+r\chi'). \tag{29}$$

The function χ is connected with the displacement ξ by two equations which hold at $r = r_0$. One of these is the condition of continuity of velocity normal to the surface, viz.,

$$\dot{\xi} = -\frac{\partial}{\partial r}\{r^{-2}\chi+r^{-1}\chi'\}_{r=r_0}, \tag{30}$$

and the other is the equation of motion of the sphere, viz.,

$$M(\ddot{\xi}+n^2\xi) = \int_0^\pi\left(-\rho\frac{\partial\phi}{\partial t}\right)_{r=r_0}(-\cos\theta)\,2\pi r_0^2\sin\theta\,d\theta, \tag{31}$$

where $\cos\theta$ is the argument of P_1. Taking σ for the density of the

sphere, we may write this equation:—

$$\ddot{\xi}+n^2\xi = -\frac{\rho}{\sigma}\frac{a}{r_0^3}\{\chi'(at-r_0)+r_0\chi''(at-r_0)\}. \tag{32}$$

Equation (30) is

$$\dot{\xi} = r_0^{-3}\{2\chi(at-r_0)+2r_0\chi'(at-r_0)+r_0^2\chi''(at-r_0)\}. \tag{33}$$

To solve these equations we assume

$$\chi(at-r_0) = Ae^{\lambda at}, \qquad \xi = Be^{\lambda at}, \tag{34}$$

a factor $e^{-\lambda r_0}$ being absorbed in A. Then we have

$$\left.\begin{aligned} B(n^2+a^2\lambda^2) &= -\frac{\rho}{\sigma}\frac{a}{r_0^3}(\lambda+r_0\lambda^2)A \\ B\lambda a &= \frac{1}{r_0^3}(2+2r_0\lambda+r_0^2\lambda^2)A \end{aligned}\right\}, \tag{35}$$

so that λ must satisfy the equation

$$(n^2+a^2\lambda^2)(2+2r_0\lambda+r_0^2\lambda^2)+\frac{\rho}{\sigma}a^2\lambda^2(1+r_0\lambda) = 0. \tag{36}$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the roots of this equation, the forms for ξ and ϕ are

$$\xi = \sum_1^4 \frac{2+2r_0\lambda_s+r_0^2\lambda_s^2}{r_0^3 a\lambda_s} A_s e^{\lambda_s at}, \tag{37}$$

$$\phi = -\frac{P_1}{r^2}\sum_1^4 (1+r\lambda_s)A_s e^{\lambda_s(at-r+r_0)}. \tag{38}$$

This solution holds when $r < at+r_0$ and $t > 0$. It can be adapted, as before, to represent motions due to given initial values of ξ and $\dot{\xi}$. This adaptation yields two equations connecting the A's. The condition at the front of an advancing wave, viz., $(\partial\phi/\partial t)+a(\partial\phi/\partial r) = 0$, gives rise to the relation

$$\sum_1^4 \left(\frac{2}{r^3}+\frac{\lambda_s}{r^2}\right)A_s = 0,$$

which must be satisfied when $r = at+r_0$. This condition, therefore, is equivalent to the two equations

$$\sum_1^4 A_s = 0, \qquad \sum_1^4 \lambda_s A_s = 0. \tag{39}$$

Hence all the constants are determinate when the initial state is given.

When ρ/σ is small two roots of the equation (36) are approximately

$$\lambda_1 = \iota n/a, \qquad \lambda_2 = -\iota n/a, \tag{40}$$

and the other two are approximately

$$\lambda_3 = \tfrac{1}{2}(-1+\iota)/r_0, \qquad \lambda_4 = \tfrac{1}{2}(-1-\iota)/r_0. \tag{41}$$

The coefficients of $e^{\lambda_3 at}$ and $e^{\lambda_4 at}$ in (37) are very small, and the motion of the sphere is very nearly a simple harmonic vibration of frequency $n/2\pi$. The motion of the air is compounded of two wave-motions: one wave-train is very nearly simple harmonic, with the same period as the motion of the sphere; and the other is very rapidly damped. Near the sphere the motion of the air is practically that belonging to the simple harmonic wave-train. Near the front of the wave the rapidly damped harmonic motion has the same degree of importance as the nearly simple harmonic motion, and the co-existence of the two is necessary to the continued advance of the front.

When we make a second approximation to the roots λ_1 and λ_2 we find

$$\lambda_1 = -\frac{\rho}{2\sigma}\,\frac{n^4 r_0^3}{4a^4+n^4 r_0^4} + \iota\frac{n}{a} - \iota\frac{\rho}{2\sigma}\,\frac{an(2a^2+n^2 r_0^2)}{4a^4+n^4 r_0^4}, \tag{42}$$

and λ_2 is the imaginary conjugate to λ_1. This approximation gives the same results as regards the effective inertia and the decay of the motion as are obtained by Lord Rayleigh (*Theory of Sound*, Vol. II., § 325).

Similar methods may be employed when the motion of the sphere is maintained periodic for a time and then allowed to decay, with results of the same kind as those for radial vibrations. Further, no essentially new feature is introduced when the normal displacement of the sphere depends upon a surface harmonic of order higher than unity.

8. *Electric Vibrations of Order Unity.*

The first case of electric vibrations to be discussed is that in which electrification is distributed over the surface of a conducting sphere with surface density proportional to the first zonal harmonic P_1. We shall suppose that before the instant $t = 0$ the electrostatic field of this electrification is established through all space outside the sphere $r = r_0$. The initial state of the medium outside this sphere is that expressed by the equations

$$\left.\begin{aligned} (X, Y, Z) &= E\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, -\frac{1}{r^3}+\frac{3z^2}{r^5}\right) \\ (\alpha, \beta, \gamma) &= 0 \end{aligned}\right\}, \tag{43}$$

in which E is a constant. The initial surface density on the sphere is then $EP_1/2\pi r_0^3$.

At the instant $t = 0$ the cause which previously maintained the field expressed by (43) is supposed to cease to operate. Thereafter the surface

$r = r_0$ is to be taken to be that of a perfect conductor. It is required to determine the subsequent state of the medium in accordance with the conditions: (i.) that the initial field is that expressed by (43), (ii.) that the tangential electric force vanishes at $r = r_0$.

It is clear that a new state of the medium arises, for the surface condition at $r = r_0$ is not satisfied by (43). It is clear also that the disturbed state of the medium cannot at any instant t have extended to the part of the medium beyond the sphere $r = ct+r_0$. Further, it is known that the problem can have only one solution.* The form of solution which suggests itself naturally involves the assumptions (i.) that the surface density on the sphere is always distributed so as to be proportional to P_1, (ii.) that the spherical surface $r = ct+r_0$ is the front of an advancing wave. We can show that the solution obtained by means of these assumptions satisfies all the conditions of the problem. In accordance with what has been said in § 2, we ought to take (X, Y, Z) and (α, β, γ) in the region $r < ct+r_0$ to be given by (1) and (2) with $n = 1$ and $\omega_n = z$. We take them, therefore, to be given by the equations

$$\left.\begin{aligned} (X, Y, Z) &= c(xz,\ yz,\ -x^2-y^2)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r} \\ &\quad + c(0,\ 0,\ -2)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi(ct-r)}{r} \\ (\alpha, \beta, \gamma) &= c(y,\ -x,\ 0)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi'(ct-r)}{r} \end{aligned}\right\}, \tag{44}$$

which are the same as

$$(X, Y, Z) = c\left(\frac{\partial^2}{\partial x\,\partial z},\ \frac{\partial^2}{\partial y\,\partial z},\ -\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)\frac{\chi(ct-r)}{r},$$

$$(\alpha, \beta, \gamma) = \left(\frac{\partial^2}{\partial y\,\partial t},\ -\frac{\partial^2}{\partial x\,\partial t},\ 0\right)\frac{\chi(ct-r)}{r},$$

while (43) is the same as

$$(X, Y, Z) = \left(\frac{\partial^2}{\partial x\,\partial z},\ \frac{\partial^2}{\partial y\,\partial z},\ \frac{\partial^2}{\partial z^2}\right)\frac{E}{r}.$$

We then show that we can adjust the unknown function χ so as to satisfy the surface condition at the conductor $r = r_0$, and also to satisfy the conditions of the types (7) and (8) which must hold at the front of the advancing wave.

* For the proof of this theorem in the case where there is a moving surface of discontinuity, see my paper already cited in *Proceedings* (Ser. 2), Vol. 1.

To express the condition that the tangential component of (X, Y, Z) vanishes at $r = r_0$, denote by R the radial component of (X, Y, Z), so that

$$Rr = Xx + Yy + Zz.$$

Then, at this surface, R is the resultant of (X, Y, Z) and the direction of (X, Y, Z) is the same as that of r. Hence, at this surface, we have

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = \frac{R}{r}$$

or

$$X - Rx/r = 0, \quad \ldots.$$

I form therefore the vector $(X - Rx/r,\ Y - Ry/r,\ Z - Rz/r)$, and express the condition that it vanishes at $r = r_0$. I find

$$\left(X - R\frac{x}{r},\ Y - R\frac{y}{r},\ Z - R\frac{z}{r}\right)$$
$$= c(xz,\ yz,\ -x^2-y^2)\left[\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 \frac{\chi(ct-r)}{r} + \frac{2}{r^3}\frac{\partial}{\partial r}\frac{\chi(ct-r)}{r}\right], \tag{45}$$

and it vanishes at $r = r_0$, provided

$$\chi(ct-r_0) + r_0\chi'(ct-r_0) + r_0^2\chi''(ct-r_0) = 0. \tag{46}$$

This holds for all positive values of t.

To deduce the form of χ let ζ stand for $ct - r_0$. Then $\chi(\zeta)$ is a function of ζ which, for all values of ζ that are $> -r_0$, satisfies the equation

$$\chi''(\zeta) + r_0^{-1}\chi'(\zeta) + r_0^{-2}\chi(\zeta) = 0;$$

and therefore, for all such values, $\chi(\zeta)$ has the form

$$\chi(\zeta) = e^{-\frac{1}{2}\zeta/r_0}\left[A_1 \cos\left(\tfrac{1}{2}\sqrt{3}\,\zeta/r_0\right) + B_1 \sin\left(\tfrac{1}{2}\sqrt{3}\,\zeta/r_0\right)\right].$$

It follows that, for all values of r and t which satisfy the inequality $ct > r - r_0$, $\chi(ct-r)$ may be written in the form

$$\chi(ct-r) = A\,e^{-\frac{1}{2}(ct-r+r_0)/r_0} \sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0) + \epsilon\right\}, \tag{47}$$

where A and ϵ are arbitrary constants. It follows that a damped harmonic train of waves is propagated outwards; the period is $4\pi r_0/c\sqrt{3}$, and the modulus of decay is $c/2r_0$. The forms obtained by substituting this value of χ in (44) are those which are generally taken as the solution of the problem. This solution holds, however, only when $r < ct + r_0$.

The conditions which have to be satisfied at the wave-front $r = ct + r_0$ are three of the type

$$r(X - X_0) = z\beta - y\gamma \tag{48}$$

and three of the type $\quad -r\alpha = z(Y-Y_0)-y(Z-Z_0), \qquad (49)$

where X_0, Y_0, Z_0 are the X, Y, Z expressed in (43), and (X, Y, Z) and (α, β, γ) are given by (44). The first of equations (48) is

$$\frac{xz}{r^4}(3\chi+3r\chi'+r^2\chi'')-\frac{E}{c}\,\frac{3xz}{r^4}=\frac{xz}{r^3}(\chi'+r\chi''), \qquad (50)$$

the second differs from this only by having yz in place of xz, and the third is

$$-\frac{x^2+y^2}{r^4}(3\chi+3r\chi'+r^2\chi'')+\frac{2}{r^2}(\chi+r\chi')-\frac{E}{c}\left\{\frac{2}{r^2}-\frac{3(x^2+y^2)}{r^4}\right\}$$
$$=-\frac{x^2+y^2}{r^3}(\chi'+r\chi''). \qquad (51)$$

In these χ, χ', χ'' must have their values at the surface $r = ct+r_0$ and r must have this value. It follows that the value of χ' vanishes at this surface, and that the value of χ at this surface is E/c. When these conditions are satisfied, it appears that equations (49) are satisfied identically. Now we have

$$\chi'(ct-r) = -\tfrac{1}{2}Ar_0^{-1}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\Big[\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)+\epsilon\right\}$$
$$-\sqrt{3}\cos\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)+\epsilon\right\}\Big],$$

and this vanishes at the surface $r = ct+r_0$, if $\epsilon = \frac{1}{3}\pi$. Thus we have

$$\chi = \frac{2E}{c\sqrt{3}}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)+\frac{\pi}{3}\right\}, \qquad (52)$$

and the constants A and ϵ are determined. With this form of χ we find

$$\left.\begin{aligned}\chi' &= -\frac{1}{r_0}\,\frac{2E}{c\sqrt{3}}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)\right\}\\ \chi'' &= \quad\frac{1}{r_0^2}\,\frac{2E}{c\sqrt{3}}e^{-\frac{1}{2}(ct-r+r_0)/r_0}\sin\left\{\frac{\sqrt{3}}{2r_0}(ct-r+r_0)-\frac{\pi}{3}\right\}\end{aligned}\right\}. \qquad (53)$$

It may be observed that with the above determination of A and ϵ the magnetic force along a circle of latitude is

$$-\frac{\sin\theta}{r}\,\frac{2E}{r_0^2\sqrt{3}}\left(1-\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{\frac{1}{2}}e^{-\vartheta}\cos(\sqrt{3}\,\vartheta+\delta),$$

where $\qquad \vartheta = \frac{1}{2}(ct-r+r_0)/r_0 \quad$ and $\quad \tan\delta = (r-2r_0)/r\sqrt{3}$;

also the radial electric force is

$$\frac{\cos\theta}{r^2}\frac{4E}{r_0\sqrt{3}}\left(1-\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{\frac{1}{2}}e^{-\vartheta}\cos\left(\sqrt{3}\,\vartheta+\delta+\frac{\pi}{3}\right),$$

and the tangential electric force is

$$-\frac{\sin\theta}{r}\frac{2E}{r_0^2\sqrt{3}}\left(1-\frac{r_0}{r}\right)\left(1+\frac{r_0}{r}+\frac{r_0^2}{r^2}\right)^{\frac{1}{2}}e^{-\vartheta}\cos(\sqrt{3}\,\vartheta+\delta'),$$

where $\tan\delta' = (r-r_0)/(r+r_0)\sqrt{3}$. These results differ from those obtained by J. J. Thomson (*loc. cit.*) as regards phase. The phases given by him are not determined by the conditions which hold at the front of the wave-train.

It appears from the above analysis that the damped harmonic wave-train represented by the customary form of solution can advance into a region in which the electric field is expressed by (43). The same analysis can be applied at once to show that it cannot advance into a region free from electric and magnetic forces; it can also be applied to determine the mode of decay of the external field due to maintained electrical oscillations of the same surface harmonic type on the sphere. Exactly as in the problem of sound waves it appears that the forced wave must be accompanied by a wave of the type (47), and that the wave that is propagated outwards after the system is left to itself is also of the type expressed by (47); and the constants A and ϵ of these two waves can be adjusted so as to satisfy the conditions that hold at the front of the forced wave and at the common boundary of the two waves. The concurrent existence of a wave of type (47) along with the forced wave is necessary to the continued advance of the wave-front. Exactly as in the sound problems the damped harmonic wave-train is not damped at the front, but is subject only to the kind of diminution by spherical divergence that is appropriate to the spherical surface harmonic.

9. *Redistribution of the Energy.*

In the initial state the æther in the region between the spheres r and $r+dr$ possesses electric energy of amount $\dfrac{1}{8\pi}\,4\pi r^2 dr E^2\,\dfrac{2}{r^6}$ or $r^{-4}E^2dr$, and the total energy of the field is $\frac{1}{3}E^2r_0^{-3}$. In the subsequent state of wave-disturbance the same portion of the medium possesses magnetic energy of amount

$$\frac{1}{8\pi}\,4\pi r^2 dr\,\tfrac{2}{3}c^2r^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\frac{\chi'(ct-r)}{r}\right\}^2, \tag{54}$$

and it possesses electric energy of amount

$$\frac{1}{8\pi}\, 4\pi r^2 dr \left[\tfrac{2}{3} c^2 r^4 \left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 \frac{\chi(ct-r)}{r}\right\}^2 + 4c^2 \left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right) \frac{\chi(ct-r)}{r}\right\}^2 \right.$$
$$\left. + \tfrac{8}{3} c^2 r^2 \left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right) \frac{\chi(ct-r)}{r}\right\} \left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2 \frac{\chi(ct-r)}{r}\right\}\right], \tag{55}$$

where χ has the value already determined. The amount of the magnetic energy in the region is

$$\tfrac{4}{9} E^2 \frac{dr}{r_0^4 r^2} \left\{ r_0 \sin \sqrt{3}\, \vartheta - r \sin \left(\sqrt{3}\, \vartheta - \frac{\pi}{3}\right)\right\}^2 e^{-2\vartheta}, \tag{56}$$

where ϑ is written for $\frac{1}{2}(ct-r+r_0)/r_0$. As soon as the wave-front has travelled to a distance from the conducting surface which is at all large compared with the radius of this surface the factor $e^{-2\vartheta}$ will be small except in the immediate neighbourhood of the wave-front, and thus we see that the energy of the wave-motion will be accumulated near the wave-front. Also when r is large compared with r_0 the above expression may approximately be replaced by

$$\tfrac{2}{9} \frac{E^2}{r_0^4} e^{-2\vartheta} \left\{1 - \cos\left(2\sqrt{3}\, \vartheta - \frac{2\pi}{3}\right)\right\} dr. \tag{57}$$

We may calculate the energy between the wave-front and a spherical surface within it, and not far from it, by integrating this expression. Consider the case where the inner of these surfaces is at a distance of half a wave-length behind the front, *i.e.*, at a distance $2\pi r_0/\sqrt{3}$. The magnetic energy between the surfaces is approximately equal to

$$\tfrac{2}{9} \frac{E^2}{r_0^4} \int_0^{\pi/\sqrt{3}} e^{-2\vartheta} \left\{1 + \tfrac{1}{2}\cos(2\sqrt{3}\, \vartheta) - \frac{\sqrt{3}}{2} \sin(2\sqrt{3}\, \vartheta)\right\} 2r_0\, d\vartheta,$$

which is $\qquad \tfrac{1}{6} E^2 r_0^{-3} (1 - e^{-2\pi/\sqrt{3}}).$

If we had taken the first wave-length of the advancing wave instead of the first half wave-length, we should have found $\frac{1}{6}E^2 r_0^{-3}(1-e^{-4\pi/\sqrt{3}})$ as the approximate value of the magnetic energy between the surfaces. If we calculate the electric energy in the same way and to the same order of approximation, we find the same values, so that the total energy between the two surfaces is approximately equal to $\frac{1}{3}E^2 r_0^{-3}(1-e^{-2\pi/\sqrt{3}})$ when the surfaces are half a wave-length apart, and to $\frac{1}{3}E^2 r_0^{-3}(1-e^{-4\pi/\sqrt{3}})$ when they are a wave-length apart. The terms omitted in the calculation are small compared with those retained in the order r_0/r and higher powers of r_0/r, r denoting the radius of the wave-front. It appears therefore that the

energy of the initial electrostatic field is propagated outwards with the wave in such a way that the energy that was initially within a spherical surface surrounding the conductor is the energy of the wave-motion when that surface is the wave-front, and it is gathered up close behind the wave-front.* When the wave-front is at a great distance from the conductor the accumulation of energy at the front is so great that all but about $\frac{2}{75}$ of the total initial static energy of the field is gathered up in the first half-wave-length, and all but about $\frac{1}{1400}$ of it is gathered up in the first wave-length.

10. *Electrical Vibrations of Order* 2.

We suppose that the initial electrification has surface density proportional to the second zonal harmonic P_2, or to the solid harmonic $2z^2-x^2-y^2$, which is $2r_0^2P_2$, on the sphere of radius r_0, and we take the initial field to be given by the equation

$$(X, Y, Z) = E\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{x^2+y^2-2z^2}{r^5}, \tag{58}$$

which gives a surface density $(3/2\pi)Er_0^{-4}P_2$. The appropriate forms of (X, Y, Z) and (α, β, γ) in the ensuing wave-disturbance are expressed by (1) and (2), in which $n=2$ and $\omega_n = 2z^2-x^2-y^2$, and we have

$$\left.\begin{aligned}
(\alpha, \beta, \gamma) &= 6c(yz, -zx, 0)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi'(ct-r)}{r}\\
X &= 6c\left\{x\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}+z^2x\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\frac{\chi(ct-r)}{r}\right\}\\
Y &= 6c\left\{y\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}+z^2y\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\frac{\chi(ct-r)}{r}\right\}\\
Z &= 6c\left\{-2z\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}-(x^2+y^2)z\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3\frac{\chi(ct-r)}{r}\right\}
\end{aligned}\right\} \quad 59)$$

With these forms we find

$$X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r} = -6c\frac{2z^2-x^2-y^2}{r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\frac{\chi(ct-r)}{r}, \tag{60}$$

and the (x, y, z)-components of a vector which has the same tangential

* The remark that it is the static electric energy of the field which is propagated with the waves was made by Prof. Larmor in a letter to the author before this paper began to be written.

components as (X, Y, Z), and no radial component can be written down in the forms

$$\left.\begin{aligned} X-\frac{x}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right) &\\ = 6cxz^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3+\frac{3}{r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\right\}&\frac{\chi(ct-r)}{r} \\ Y-\frac{y}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right) &\\ = 6cyz^2\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3+\frac{3}{r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\right\}&\frac{\chi(ct-r)}{r} \\ Z-\frac{z}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right) &\\ = -6cz(x^2+y^2)\left\{\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^3+\frac{3}{r^2}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^2\right\}&\frac{\chi(ct-r)}{r} \end{aligned}\right\} \quad (61)$$

Hence the condition that the surface $r=r_0$ may be that of a perfect conductor is

$$\chi'''(ct-r_0)+3r_0^{-1}\chi''(ct-r_0)+6r_0^{-2}\chi'(ct-r_0)+6r_0^{-3}\chi(ct-r_0)=0, \quad (62)$$

and it follows that $\chi(ct-r)$ may be expressed in the form

$$\chi(ct-r)=A_1e^{\lambda_1(ct-r+r_0)}+A_2e^{\lambda_2(ct-r+r_0)}+A_3e^{\lambda_3(ct-r+r_0)}, \quad (63)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the equation

$$r_0^3\lambda^3+3r_0^2\lambda^2+6r_0\lambda+6=0, \quad (64)$$

and A_1, A_2, A_3 are constants. The real root is approximately $-(1{\cdot}6)\,r_0^{-1}$, and the imaginary roots* are approximately $(-0{\cdot}7 \pm 1{\cdot}8\iota)\,r_0^{-1}$.

To determine the constants A_1, A_2, A_3 we have the conditions at the front of the wave, *i.e.*, at the surface $r=r_0+ct$. We use the same equations (48) and (49) as in the previous problem, but now X_0, Y_0, Z_0 are given by the equations

$$\left.\begin{aligned} X_0 &= -E\frac{3x}{r^5}\left(1-\frac{5z^2}{r^2}\right) \\ Y_0 &= -E\frac{3y}{r^5}\left(1-\frac{5z^2}{r^2}\right) \\ Z_0 &= -E\frac{3z}{r^5}\left(3-\frac{5z^2}{r^2}\right) \end{aligned}\right\}, \quad (65)$$

---

* J. J. Thomson, *Recent Researches*, p. 371, gives the imaginary roots.

and (X, Y, Z) and (α, β, γ) are given by (59). Now we have

$$\left.\begin{aligned} rX-z\beta+y\gamma &= 6c\left[\frac{3x}{r^4}\left\{\left(1-\frac{5z^2}{r^2}\right)\chi+\left(1-\frac{4z^2}{r^2}\right)r\chi'\right\}+\frac{x}{r^2}\left(1-\frac{3z^2}{r^2}\right)\chi''\right] \\ rY-x\gamma+z\alpha &= 6c\left[\frac{3y}{r^4}\left\{\left(1-\frac{5z^2}{r^2}\right)\chi+\left(1-\frac{4z^2}{r^2}\right)r\chi'\right\}+\frac{y}{r^2}\left(1-\frac{3z^2}{r^2}\right)\chi''\right] \\ rZ-y\alpha+x\beta &= 6c\left[\frac{3z}{r^4}\left\{\left(3-\frac{5z^2}{r^2}\right)\chi+2\left(1-\frac{2z^2}{r^2}\right)r\chi'\right\}+\frac{z}{r^2}\left(1-\frac{3z^2}{r^2}\right)\chi''\right] \end{aligned}\right\}, \tag{66}$$

so that equations (48) require that at the surface $r = ct+r_0$ we should have

$$\chi' = 0, \quad \chi'' = 0, \quad \chi = -E/6c,$$

and it will be found that when these equations are satisfied equations (49) are satisfied identically. Hence the constants A_1, A_2, A_3 are determined by the equations

$$\sum_1^3 A_s = -E/6c, \quad \sum_1^3 A_s\lambda_s = 0, \quad \sum_1^3 A_s\lambda_s^2 = 0. \tag{67}$$

The important result is that the A's are definite multiples of E/c, and, in particular, that the A that corresponds with the real value of λ is not small in comparison with the other A's.

Hence in this case the wave-motion that ensues when the initial statical field is left to subside is compounded of two wave-motions—one of exponential type determined by the real value of λ and the corresponding value of A, and the other of damped harmonic type determined by the conjugate complex values of λ and the conjugate complex values of A that belong to them. Neither of these waves can be propagated except in company with the other, for the co-existence of the two is requisite to the continued advance of the wave-front. Near the conductor the field subsides very rapidly, but near the common front of the waves the fields that belong to them are not subject to damping, but merely diminish according to the law of spherical divergence which is appropriate to the spherical surface harmonic concerned.

11. *Generalization of the Results for "Electrical Vibrations on a Spherical Conductor."*

We may proceed to discuss waves that correspond with surface harmonics of higher orders. In any case we have to form the condition

which is to be satisfied at the surface of the conductor. Taking the forms (1) and (2), we may show that

$$X-\frac{x}{r}\left(X\frac{x}{r}+Y\frac{y}{r}+Z\frac{z}{r}\right)$$
$$=-\frac{c}{2n+1}\left\{(n+1)\frac{\partial\omega_n}{\partial x}-nr^{2n+1}\frac{\partial}{\partial x}\left(\frac{\omega_n}{r^{2n+1}}\right)\right\}$$
$$\times\left\{(n+1)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n\frac{\chi}{r}+r^2\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{n+1}\frac{\chi}{r}\right\}, \tag{69}$$

and so on. Thus the condition is expressed by the equation

$$\frac{\partial}{\partial r}\left\{r^{n+1}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^n\frac{\chi(ct-r)}{r}\right\}=0, \tag{70}$$

which must hold when $r=r_0$. This is a linear differential equation of the $(n+1)$-th order with constant coefficients satisfied by $\chi(ct-r_0)$, and it serves for the determination of $\chi(ct-r)$ in the form

$$\chi(ct-r)=\sum_1^{n+1}A_s e^{\lambda_s(ct-r+r_0)}, \tag{71}$$

where the values of λr_0 are roots of an equation of the $(n+1)$-th degree with determinate numerical coefficients. When n is even one root is real, and we may expect it to be negative; we may also expect the remaining roots to be complex with negative real parts, and when n is odd we may expect all the roots to have this character.* The constants A will be determined by the conditions which hold at the wave-front $r=ct+r_0$. In general these conditions can be shown to lead to the equations

$$\chi'=0,\quad \chi''=0,\quad \ldots,\quad \chi^{(n)}=0,$$

and $\chi=$ a given constant. These hold at $r=ct+r_0$, and they suffice to determine the constants A_1, A_2, ..., A_{n+1}. It follows that, in general, with an initial distribution of surface density proportional to a definite surface harmonic of order $2m$ a wave of exponential type and m waves of damped harmonic type are propagated together with a common front, and that when the harmonic is of order $2m+1$ the waves propagated are all of damped harmonic type and are in number $m+1$. In both cases the field near the sphere subsides rapidly, nearly all its energy being transferred to a relatively thin spherical shell near the wave-front. The field of each of the waves near their common front diminishes only through the kind of

---

* J. J. Thomson (*loc. cit.*) gives the roots in case $n=3$.

spherical divergence that is appropriate to the spherical harmonic concerned. In all cases the co-existence of the various waves of the system is requisite for the continued advance of the wave-front.

12. *General Conclusion.*

In our problems of sound waves we found that, besides the slightly damped harmonic wave-trains which have periods nearly equal to the natural periods of the vibrator, there must be others of exponential or rapidly damped harmonic type, which accompany the former as they travel outwards and serve to establish continually the front of the advancing waves. These subsidiary wave-trains have little influence on the motion of the vibrator, but they play a large part in the motion of the medium, especially in the region near the wave-front. The number of them increases with the complexity of the mode of vibration (expressed in the case of a sphere by the order of a surface harmonic), and all those that correspond with a particular mode of vibration must co-exist along with the slightly damped harmonic wave-train that is characteristic of the mode. They cannot exist independently. The motion of the medium that belongs to any particular vibration of the nucleus may be analysed as above into a system of damped harmonic and exponential wave-trains, but the analysis is entirely mathematical and does not correspond with a possible physical analysis into motions that can be executed independently of each other.

When a distribution of charge, which would not be possible for a free charge, is maintained over a conductor, and suddenly released, electric waves travel out into the medium. The waves may be expected to fall into classes determined by the modes of distribution of the charge, and the number of waves in a class may be expected to be definite. The different waves in a class are of exponential or damped harmonic types, and they are distinguished from each other by the exponents and periods.* They can have no physical existence independently of each other; all the waves in a class must co-exist.† The composite system of electric waves which consists of all the waves in a class advances into the statical field due to the initial distribution of the charge, and the co-existence of the

* It is possible that the types may involve a dependence on time of a more complicated character than exponential or damped harmonic when the conductor is not spherical.

† Mr. Macdonald has called my attention to the fact that a similar result was found by Heaviside, *Electrical Papers*, Vol. II., p. 408.—*April 17th*, 1904.

several waves is necessary to the continued advance of the front. As the wave advances, it transforms into electromagnetic energy the excess of the statical energy of the initial field over that of a free charge of the same total amount on the same conductor; and this electromagnetic energy is transferred continually towards the front of the advancing wave, in such a way that at a distance from the conductor the wave practically passes as a pulse. The electric waves that are thus generated appear to have little analogy to the sound waves sent out from a vibrator, and having nearly the natural period of the vibrator, but to be analogous rather to the subsidiary sound waves which accompany these and serve to establish the advancing wave-fronts without having a sensible influence upon the vibrator. This theory should be applicable to all cases in which the space outside the conductor is simply-connected; there may be exceptions when the space is multiply-connected or when this condition is nearly realized—for example, when a gap is made in a conducting ring.

ON A PLANE QUINTIC CURVE

By F. MORLEY.

[Received March 13th, 1904.—Read April 14th, 1904.]

IN a memoir in the *American Journal* (Vol. x., "On Critic Centres") I pointed out that a curve of order 5, through the nine flexes of a cubic curve and the twelve other intersections of the lines of flexes, would have itself flexes at the nine points, the stationary lines thereat meeting on the quintic. And I mentioned that a curve of order 4, through the twelve points, would have them as flexes. This quartic appears in a paper by Caporali, published first in his *Works* (p. 336), whence it appears that the twenty-four flexes of the quartic fall into two such sets of twelve. Proofs of Caporali's statements are supplied in a memoir by Ciani (*Nap. Rendiconti*, Ser. 3, Vol. II., p. 126, 1896).

The object of the present paper is now clear—to prove that the forty-five flexes of the quintic fall into five sets of nine, or rather that *the forty-five stationary lines pass by nines through five points on the curve;* and to discuss the quintic with reference to this set of five points, which are singular points of a novel kind.

1. *First Equation of the Curve.*

If we denote a cubic by $(ax)^3$, its Hessian by $(hx)^3$, and draw from a point y tangents to the pencil $a+\lambda h = 0$, we have as locus of points of contact the quintic in question

(1 . 1) $$Q \equiv (ax)^3 (hx)^2 (hy) - (hx)^3 (ax)^2 (ay) = 0;$$

the polar quartic of y is

(1 . 2) $$Q_1 \equiv (ax)^3 (hx) (hy)^2 - (hx)^3 (ax) (ay)^2 = 0;$$

the polar cubic of y is either

(1 . 3) $$Q_{11} \equiv (ax)^3 (hy)^3 - (hx)^3 (ay)^3 = 0$$

or

(1 . 3′) $$(ax)^2 (ay) (hx) (hy)^2 - (hx)^2 (hy) (ax) (ay)^2 = 0,$$

the two being equivalent by Salmon's identity (*Higher Plane Curves*, p. 206).

The polar conic of y is

$$(1.4)\qquad Q_{111} \equiv (ax)^2(ay)(hy)^3-(hx)^2(hy)(ay)^3 = 0,$$

and the polar line of y is

$$(1.5)\qquad Q_{1111} \equiv (ax)(ay)^2(hy)^3-(hx)(hy)^2(ay)^3 = 0.$$

It appears from these that (1) the points where a meets h are on Q, Q_1, Q_{11}; whence they are flexes on Q, the stationary lines meeting at y. (2) The points t, where $(ax)^2(ay)$ meets $(hx)^2(hy)$, are on Q, Q_{11}, Q_{111}. (3) The point where $(ax)(ay)^2$ meets $(hx)(hy)^2$ is on Q_1, Q_{11}, Q_{1111}. Thus, in particular, *y and any of the four points t are each on the polar conic of the other.*

2. *Mutuality of the Five Points.*

Use now the canonic form of Hesse, and let a term in parentheses be subjected to permutation of the suffixes 1, 2, 3, and then to summation of distinct terms. Thus let (x_1^3) stand for $x_1^3+x_2^3+x_3^3$, $(x_1y_2t_3)$ for the sum of six terms, and so on.

The pencil of cubics is $(x_1^3)+6mx_1x_2x_3 = 0$, the polar conic of y is $(y_1x_1^2)+2m(y_1x_2x_3) = 0$. These give on elimination of m the quintic Q:

$$(2.1)\qquad (x_1^3)(y_1x_2x_3) = 3x_1x_2x_3(y_1x_1^2).$$

The polar cubic of a point t as to Q is

$$(2.2)\qquad (y_1t_2t_3)(x_1^3)+3(y_1t_2x_3)(t_1x_1^2)+3(y_1x_2x_3)(t_1^2x_1)$$
$$= 3(t_1t_2x_3)(y_1x_1^2)+6(t_1x_2x_3)(y_1t_1x_1)+3x_1x_2x_3(y_1t_1^2).$$

I say that this equation holds when x and t are distinct points such that

$$(2.3)\qquad (y_1x_1^2) = 0,\quad (y_1x_2x_3) = 0,\quad (y_1t_1^2) = 0,\quad (y_1t_2t_3) = 0.$$

With these suppositions, (2.2) reduces at once to

$$(2.2')\qquad (y_1t_2x_3)(t_1x_1^2) = 2(t_1x_2x_3)(y_1t_1x_1).$$

But, from (2.3), $\qquad \lambda y_1 = t_2^2x_3^2-t_3^2x_2^2,$

$$\mu y_1 = t_1x_1(t_2x_3-t_3x_2);$$

whence $\qquad \dfrac{\lambda}{\mu} = \dfrac{t_2x_3+t_3x_2}{t_1x_1} = \dfrac{(y_1t_2x_3)}{(y_1t_1x_1)}$

and also $\qquad = \dfrac{2(t_1x_2x_3)}{(t_1x_1^2)};$

whence (2.2′) follows. That is to say, the four points t_a are related among themselves as each was with y. The five points are mutual—we call them all t_a ($a = 1$ to 5); the polar conic of any one goes through the rest, or, in other words, $t_a^2 t_\beta^3$ is apolar with Q; or, again, the join of any two cuts from Q three other points of which the two are the Hessian.

It follows that what was true for the one point is true for all; in particular, that *the forty-five stationary lines pass, by nines, through the five points.*

3. *The Equation in terms of the Five Points.*

The quintic depended on a Hesse configuration (eight numbers) and a point (two numbers). Hence we may expect the five points to determine Q, and we next obtain a symmetric equation.

Let C be the conic on the five points, t_a; it touches the quintic at each point. Taking for coordinates of t_a

$$x_1 = t_a^2, \quad x_2 = 2t_a, \quad x_3 = 1,$$

let the conic be

(3.1) $$C \equiv x_1 x_3 - x_2^2/4 = 0,$$

and let the tangent at t_a be

(3.2) $$C_a \equiv x_1 - x_2 t_a + x_3 t_a^2 = 0.$$

Also let $$C_{a\beta} = (t_a - t_\beta)^2,$$

and let the conic in lines be

$$\Gamma \equiv \xi_1 \xi_3 - \xi_2^2 = 0.$$

The polar of Γ as to $C_1 C_2 C_3 C_4 C_5$ is the cubic

(3.3) $$\Sigma C_{12} C_3 C_4 C_5 = 0,$$

which cuts out from the conic the Hessian of the five points; and the second polar of Γ as to the five lines is the line

(3.4) $$2\Sigma C_{12} C_3 C_{45} = 0,$$

which cuts out the fourth transvectant of the five points.

Assume now

(3.5) $$Q \equiv C_1 C_2 C_3 C_4 C_5 - \lambda C \overset{10}{\Sigma} C_{12} C_3 C_4 C_5 + \mu C^2 \Sigma C_{12} C_3 C_{45}.$$

Excluding C_1 from the summations, we have

$$Q = C_1 C_2 C_3 C_4 C_5 - \lambda C C_1 \overset{6}{\Sigma} C_2 C_3 C_{45} - \lambda \overset{4}{\Sigma} C_{12} C_3 C_4 C_5$$
$$+ \mu C^2 C_1 \overset{3}{\Sigma} C_{23} C_{45} + \mu C^2 \overset{12}{\Sigma} C_{12} C_3 C_{45}.$$

Operating with $\frac{1}{2}(t_1^2\xi_1+2t_1\xi_2+\xi_3)^2$, we get the second polar of t_1,

$$Q_{11} = C_1 \overset{6}{\Sigma} C_{12}C_{13}C_4C_5 - \lambda C_1^2 \overset{12}{\Sigma} C_{12}C_3C_{45} - 2\lambda C_1 \overset{6}{\Sigma} C_{12}C_{13}C_4C_5$$
$$+\mu C_1^3 \overset{3}{\Sigma} C_{23}C_{45} + \mu C_1^2 \overset{12}{\Sigma} C_{12}C_3C_{45}$$
$$+\text{terms with } C \text{ as factor.}$$

This vanishes at t_2 if

$$(1-2\lambda)\, C_{12}^2 \overset{3}{\Sigma} C_{13}C_{24}C_{25} + (\mu-\lambda)\, C_{12}^2 \{C_{12} \overset{3}{\Sigma} C_{23}C_{45} + 2 \overset{3}{\Sigma} C_{13}C_{24}C_{25}\}$$
$$+\mu C_{12}^3 \overset{3}{\Sigma} C_{23}C_{45} = 0;$$

hence $1-2\lambda+2(\mu-\lambda) = 0$ and $2\mu-\lambda = 0$, whence $\lambda = \frac{1}{3}$, $\mu = \frac{1}{6}$. Thus the *symmetric equation of the quintic* is

$$(3.6) \qquad C_1C_2C_3C_4C_5 - \tfrac{1}{3}C\Sigma C_{12}C_3C_4C_5 + \tfrac{1}{6}C^2\Sigma C_{12}C_3C_{45} = 0,$$

or, in a convenient notation,

$$(3.7) \qquad (1-\tfrac{1}{3}C.\Gamma+\tfrac{1}{12}C^2.\Gamma^2)\, C_1C_2C_3C_4C_5 = 0.$$

4. *Common Lines of the Quintic and the Five-fold Polar Conic.*

In the symmetric equation (3.6), let $t_4 = 0$, $t_5 = \infty$, $x_1/2x_2 = x$, $x_3/2x_2 = y$, so that x and y will be conjugate coordinates if the conic be a circle and t_4 and t_5 the points at infinity on it. The equation becomes

$$3xy\,\Pi(x+yt_1^2-2t_1)$$
$$-(xy-1)\{\Sigma(t_2-t_3)^2(x+yt_1^2-2t_1)xy$$
$$+\Sigma(x+yt_1^2)(x+yt_2^2-2t_2)(x+yt_3^2-2t_3)+\Pi(x+yt_1^2-2t_1\}$$
$$+\tfrac{1}{2}(xy-1)^2\{\Sigma(t_2-t_3)^2(x+yt_1^2-2t_1)+\Sigma(t_2^2+t_3^2)(x+yt_1^2-2t_1)$$
$$+\Sigma(t_2-t_3)^2(x+yt_1^2)\} = 0;$$

or, if $\qquad \overset{3}{\Pi}(t-t_1) = t^3-s_1t^2+s_2t-s_3,$

$$(4.1) \qquad -xy(x^3+y^3s_3^2)-6x^2y^2s_3+4(x^3+y^3s_3^2)+6xy(xs_2+ys_1s_3)$$
$$-6(x^2s_1+y^2s_2s_3)-4xy(s_1s_2+s_3)+3(xs_1^2+ys_2^2)$$
$$-2(s_1s_2-2s_3) = 0.$$

From this it appears immediately that, if τ be a cube root of s_3^2, the real asymptotes are $x+y\tau^2 = 2\tau$. These are lines of the circle. Hence :

The tangents of the quintic at the points where the join of two of the five points meets it again are also tangents of the conic.

There are then, in addition to the common tangents at the five points, thirty other common tangents, or forty in all. Hence the quintic is of class 20, and *is of full genus* 6.

Selecting a zero of direction such that $s_3 = 1$, the polar cubic of t_4 and t_5 as to (4 . 1) is

$$x^3+y^3+6xy-3(xs_2+ys_1)+s_1s_2+1 = 0. \tag{4.2}$$

If we operate on this (written homogeneously) with

$$(\xi t_1^2+\eta+\zeta t_1)(\xi t_2^2+\eta+\zeta t_2),$$

the result is zero. Hence :

Any four of the five points are apolar with the quintic ; or, otherwise expressed, the polar conic of three of the five points is the double join of the other two.

In the notation of § 3, this double line is $C_{12}C = C_1C_2$.

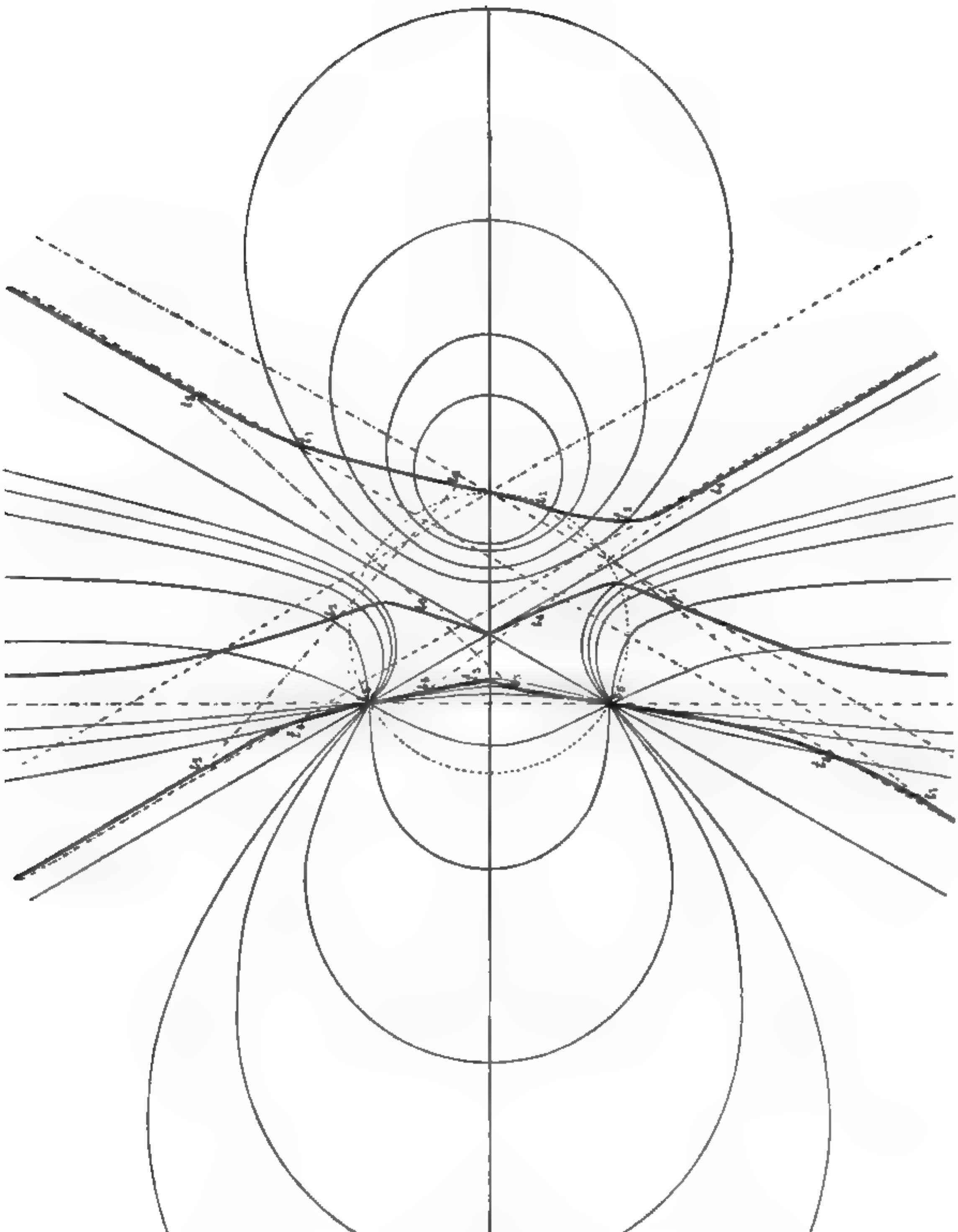
A number of conics and cubics associated with the curve might be written down; I mention only the conic $xy = 4$, which from (4 . 1) osculates the quintic at t_4 and t_5, and passes through the intersections of the tangents where their join meets Q again.

5. *Nature of the Five Points.*

In the general plane quintic, through a point of the curve can be drawn six lines meeting the curve again in a self-apolar set of four points. This six-line for the quintic replaces the four tangents from a point of a cubic, which Salmon showed to have constant double ratios. And it may be—to this point I expect to return—that among the three six-lines from three points of a quintic there is a special trilineation.

At certain points of the curve specified invariants of the six-line will vanish: for instance, the six-line will be self-apolar in general at forty points.

The peculiarity of the points t, in the quintic of this memoir, is that *their six-lines are arbitrary ;* any line through a point t cuts out a self-apolar four-point. The proof is simply that, if in (4 . 1) we hold y,



we have a quartic in x, say $(ax)^4$, for which $|\alpha\beta|^4 = 0$. This is verified at once.

The same could be proved from the fact that the pencil of cubics

$$(ax)^3+\lambda\,(hx)^3 = 0$$

cuts a line in an involution any two triads of which are apolar; whence the four points where cubics of the pencil touch a line are self-apolar, and the four points in which these cubics cut the line again are also self-apolar, the two sets of four forming a cube-configuration.

6. *The Look of the Curve.*

The five points t may be all real, three real, or one real, so that we may have fifteen, nine, or three real flexes, and respectively 0, 3, 6 real isolated double lines, by Klein's rule.

The figure (drawn by Mr. J. F. Messick) indicates the most difficult case of fifteen real flexes, and may be otherwise of use as showing the pencil of cubics. In obtaining this figure I write the pencil, in conjugate coordinates,

$$xy\,(x+y)+\mu\,(x^2+xy+y^2)+1 = 0;$$

two flexes are at the circular points, the two real finite flexes are the complex cube roots of unity.

MATHEMATICAL ANALYSIS OF WAVE PROPAGATION IN ISOTROPIC SPACE OF p DIMENSIONS

By T. H. HAVELOCK.

[Received March 16th, 1904.—Read April 14th, 1904.]

CONTENTS.

1. *Introductory.*

The object of abstract wave theory has generally been to examine solutions of $c^2\nabla^2\phi = \ddot{\phi}$, in three dimensions, in the form of surface integrals of the Kirchhoff or Poisson type; and from these the principle of Huygens in actual wave motions has been deduced.

Prof. Love has lately considered the modification of these results in the case of wave motions with discontinuities at wave fronts; without considering these latter cases, the object of the present paper is to extend the methods of abstract wave theory to the general equation

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\ldots+\frac{\partial^2\phi}{\partial x_p^2}=\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2}.$$

An extension is first made of Whittaker's form of solution, and from this are deduced the various types of ϕ representing hyper-spherical diverging waves; with the help of these forms a general solution of the Kirchhoff type is examined; and, finally, from a more general point of view, solutions both of the Kirchhoff and Poisson types are obtained and discussed.

2. *Extension of Whittaker's Solution.*

It has been shown that the general solution, regular at the origin, of the equation

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\ldots+\frac{\partial^2\phi}{\partial x_p^2}=\frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2},\tag{1}$$

when $p = 2$, has the form

$$\phi = \int_0^{2\pi} f(x_1 \sin u + x_2 \cos u + ct,\ u)\,du, \tag{2}$$

and, when $p = 3$, has the form

$$\phi = \int_0^{2\pi}\int_0^{\pi} f(x_1 \sin u \sin v + x_2 \sin u \cos v + x_3 \cos u + ct,\ u,\ v)\,du\,dv. \tag{3}$$

This form of solution can evidently be generalized for any value of p, and a formal proof would follow the same lines as that for the simple cases.*

We find then, in the general case,

$$\phi = \int_0^{2\pi}\int_0^{\pi} \dots \int_0^{\pi} f(\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_p x_p + ct,\ u,\ v_1,\ \dots,\ v_{p-2})\,du\,dv_1 \dots dv_{p-2}, \tag{4}$$

where

$$\left.\begin{aligned} \lambda_1 &= \sin u \sin v_1 \sin v_2 \dots \sin v_{p-2} \\ \lambda_2 &= \sin u \sin v_1 \sin v_2 \dots \cos v_{p-2} \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \lambda_{p-1} &= \sin u \cos v_1 \\ \lambda_p &= \cos u \end{aligned}\right\}. \tag{5}$$

3. *Relations among Hyper-spherical Harmonics.*

If in the general solution the time occurs as a factor e^{ict}, the equation takes the form

$$(\nabla_p^2 + 1)\phi = 0, \tag{6}$$

and the general solution becomes

$$\phi = \int_0^{2\pi}\int_0^{\pi} \dots \int_0^{\pi} e^{i(\lambda_1 x_1 + \dots + \lambda_p x_p)} f(u,\ v_1,\ \dots,\ v_{p-2})\,du\,dv_1 \dots dv_{p-2}. \tag{7}$$

Particular solutions of (6) are known in the form of hyper-spherical harmonics, and these can be shown to be included under (7) by means of a general relation

$$\int_0^{\pi} e^{ir\cos u} P_n(p, \cos u) \sin^{p-2} u\,dv = \text{const.}\ \frac{J_{n+\frac{1}{2}p-1}(r)}{r^{\frac{1}{2}p-1}}, \tag{8}$$

where $P_n(p, \cos u)$ is the zonal harmonic of order n and rank p.

---

* E. T. Whittaker, *Math. Ann.*, Vol. LVII., p. 333.

4. *Symmetry round the Origin.*

But for our present purpose we require the form of the general solution (4) when ϕ is symmetrical round the origin. Since ϕ only depends upon r, we find its form by taking the mean of (4) over the boundary of a hyper-sphere of radius r round the origin. Thus, if

$$x_1 = r\sin\theta\sin\phi_1\ldots\sin\phi_{p-2} = \mu_1 r, \quad \ldots, \quad x_p = r\cos\theta = \mu_p r,$$

and $$d\omega = \sin^{p-2}\theta\sin^{p-3}\phi_1\ldots\sin\phi_{p-3}\,d\theta\,d\phi_1\ldots d\phi_{p-2},$$

we have

$$\phi = \int d\omega\int_0^{2\pi}\ldots\int_0^{\pi} f\{(\lambda_1\mu_1+\ldots+\lambda_p\mu_p)r+ct,\, u,\, v_1,\,\ldots,\, v_{p-2}\}\,du\,dv_1\ldots dv_{p-2}. \tag{9}$$

Then, changing the (θ, ϕ) variables so that $(\lambda_1\mu_1+\ldots+\lambda_p\mu_p) = \cos w$, say, we are evidently left with an expression of the form

$$\phi = \int_0^{\pi}\psi(r\cos w+ct)\sin^{p-2}w\,dw. \tag{10}$$

This then is the general solution, regular at the origin, of the differential equation (1), which reduces to

$$\frac{\partial^2\phi}{\partial r^2}+\frac{p-1}{r}\frac{\partial\phi}{\partial r} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}. \tag{11}$$

Suppose first that p is odd and equal to $2n+3$. Then

$$\phi = \int_0^{\pi}\psi(ct+r\cos w)\sin^{2n+1}w\,dw = \int_{-1}^{1}\psi(ar+ct)(1-a^2)^n\,da. \tag{12}$$

Now let U be a function given by

$$U = \int_{-1}^{1}F(ar+ct)\,da. \tag{13}$$

Then $\dfrac{\partial^2 U}{\partial r^2} = \displaystyle\int_{-1}^{1}a^2F''(ar+ct)\,da$, where $F'' = \dfrac{\partial^2 U}{\partial y^2}$, if $y = ar+ct$. Hence, from (12), we have

$$\phi = \left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial r^2}\right)^n U. \tag{14}$$

Again, if p is even and equal to $2n+2$, we have

$$\phi = \int_0^{\pi}\psi(r\cos w+ct)\sin^{2n}w\,dw, \tag{15}$$

and, if $$V = \int_0^{\pi}F(r\cos w+ct)\,dw, \tag{16}$$

we have, as before,
$$\phi = \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n V. \tag{17}$$

When the time occurs as a factor e^{ict}, these expressions reduce to the known solutions of
$$\frac{\partial^2\phi}{\partial r^2} + \frac{p-1}{r}\,\frac{\partial\phi}{\partial r} + \phi = 0. \tag{18}$$

In (14), U becomes $\frac{\sin r}{r}$, and $\frac{\partial^2}{\partial y^2} = 1$; thus we have
$$\phi = \left(1 + \frac{\partial^2}{\partial r^2}\right)^n \frac{\sin r}{r} = \text{const. } r^{-(n+\frac{1}{2})} J_{n+\frac{1}{2}}(r). \tag{19}$$

Similarly, from (17), we have
$$\phi = \left(1 + \frac{\partial^2}{\partial r^2}\right)^n J_0(r) = \text{const. } r^{-n} J_n(r). \tag{20}$$

5. *Diverging Waves in Two and Three Dimensions.*

In all these results ϕ has been assumed to be regular at the origin, but we require to remove this limitation.

The general expression to be substituted in (13) for U is, of course,
$$U = \frac{f(ct-r)}{r} + \frac{F(ct+r)}{r}. \tag{21}$$

Or we might write it in the form of an integral to compare with the case of two dimensions as
$$U = \int_0^\pi f(ct + r\cos w)\sin w\,dw + \int_0^\pi F(ct + r\cos w)\,\frac{1}{r\sin^2 w}\,\sin w\,dw. \tag{22}$$

The corresponding general form for V in (16) is that given by Poisson,
$$V = \int_0^\pi f(ct + r\cos\omega)\,d\omega + \int_0^\pi F(ct + r\cos\omega)\log(r\sin^2\omega)\,d\omega. \tag{23}$$

These integrals can easily be transformed into others more suitable for our purpose. Thus V is the solution of the equation
$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\,\frac{\partial V}{\partial r} = \frac{1}{c^2}\,\frac{\partial^2 V}{\partial t^2}. \tag{24}$$

Consider a function of the form
$$V = \int f(ct + ar)\,U(a)\,da. \tag{25}$$

Substituting in the differential equation, we have

$$\int \{r(a^2-1)f''+af'\}\, U da = 0.$$

Now we choose U so that

$$\{r(a^2-1)f''+af'\}\, U = \frac{\partial}{\partial a}\{Wf'\}.$$

Thus $\qquad W' = aU, \qquad W = (a^2-1)\, U.$

Hence $\qquad W = \sqrt{a^2-1} \quad$ and $\quad U = (a^2-1)^{-\frac{1}{2}}.$

Thus

$$V = \int f(ct+ar)\frac{da}{\sqrt{a^2-1}} \tag{26}$$

is a solution of (24), provided that $\sqrt{a^2-1}\, f'(ct+ar)$ vanishes at the two limits of the path considered.

Taking paths along the real axis for a from 1 to ∞ and from -1 to $-\infty$, and changing the variable, we get the general solution as

$$V = \int_0^\infty f(ct-r\cosh v)\, dv + \int_0^\infty F(ct+r\cosh v)\, dv, \tag{27}$$

with suitable limitations on the forms of f and F, involving the convergence of the integrals.

6. *Diverging Waves in p Dimensions.*

Finally, then, we obtain the general expression for ϕ which satisfies the differential equation (11) and represents symmetrical diverging waves in space of p dimensions.

In three dimensions we have

$$\phi = r^{-1} f(ct-r), \tag{28}$$

and in space of an odd number of dimensions ($p = 2n+3$)

$$\phi = \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)\frac{f(y)}{r}, \qquad y = ct-r. \tag{29}$$

In two dimensions we have

$$\phi = \int_0^\infty f(ct-r\cosh v)\, dv, \tag{30}$$

and in space of an even number of dimensions ($p = 2n+2$)

$$\phi = \int_0^\infty \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n f(y)\, dv, \qquad y = ct-r\cosh v. \tag{31}$$

7. *Strength of the "Source" in these cases.*

For the value of $\lim_{r=0}\left(r^{p-1}\frac{\partial\phi}{\partial r}\right)$ in the different cases we have first, from (29),

$$\lim_{r=0}\left(r^{2n+2}\frac{\partial\phi_{2n+3}}{\partial r}\right) = \text{const.}\, f(ct).$$

Also, from (31),

$$\lim_{r=0}\left(r^{2n+1}\frac{\partial\phi_{2n+2}}{\partial r}\right)$$

$$= \lim_{r=0} r^{2n+1}\int_0^\infty \left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial r^2}\right)^n \frac{\partial}{\partial r} f(ct-r\cosh v)\,dv$$

$$= \lim_{r=0} r^{2n+1}\int_0^\infty \frac{\partial}{\partial r}\,\frac{2n-1}{r}\,\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial r^2}\right)^{n-1} f(ct-r\cosh v)\,dv$$

$$= \lim_{r=0} r^{2n+1}\frac{\partial}{\partial r}\,\frac{2n-1}{r}\,\frac{\partial}{\partial r}\,\frac{2n-3}{r}\cdots\frac{\partial}{\partial r}\,\frac{1}{r}\,\frac{\partial}{\partial r}\int_0^\infty f(ct-r\cosh v)\,dv$$

$$= \text{const.}\lim_{r=0} r\frac{\partial}{\partial r}\int_0^\infty f(ct-r\cosh v)\,dv = \text{const.}\, f(ct).$$

8. *Discussion of the Forms for ϕ.*

Returning now to the general equation, putting c equal to unity for convenience,

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\dots+\frac{\partial^2\phi}{\partial x_p^2}=\frac{\partial^2\phi}{\partial t^2}. \tag{32}$$

It has been pointed out before* that for solutions symmetrical round the origin the case of $p=3$ is somewhat unique in that it is the only case in which there is a solution of the form

$$\phi = \psi(r)f(t\pm r). \tag{33}$$

Duhem argues from this that it is only in three dimensions that solutions of (32) are possible in the form of surface integrals of the Kirchhoff and Poisson types; from another point of view Lamb considers that the cases $p = 1, 2, 3$ form a sequence, with a regular gradation of properties due to the increasing mobility of the medium. However, from the preceding results we see that the real distinction lies between the cases in which p

* Duhem, *Hydrodynamique*, t. II., p. 138; Volterra, *Acta Mathematica*, Vol. XVIII., p. 220; Lamb, *Proc. London Math. Soc.*, Vol. XXXV., p. 141.

is odd and those in which p is even, the comparative simplicity of the former being also expressed in the fact that for harmonic variation with the time the solutions are expressible in terms of Bessel functions of half an odd integer. From (29) we see that for p odd there is a solution given by

$$\phi = \psi_1(r) f_1(t+r) + \psi_2(r) f_2(t+r) + \text{similar terms.} \tag{34}$$

However, the case of $p = 3$ is distinguished by being the only case in which, if we regard ϕ as propagated from a point source, it is propagated without change of type.

9. *Extension of Kirchhoff Solution.*

We may now consider directly the question of solutions of the Kirchhoff form in the general case.

Let ϕ and ϕ_0 be functions of $x_1, x_2, \ldots, x_p, t$ satisfying equation (32) and regular within a domain V in p dimensions whose frontier is denoted by S. Then, extending Green's theorem, we have

$$\int_S \left(\phi_0 \frac{\partial\phi}{\partial\nu} - \phi\frac{\partial\phi_0}{\partial\nu}\right) dS = \int_V (\phi\nabla^2\phi_0 - \phi_0\nabla^2\phi)\, dV$$
$$= \frac{\partial}{\partial t}\int_V \left(\phi\frac{\partial\phi_0}{\partial t} - \phi_0\frac{\partial\phi}{\partial t}\right) dV. \tag{35}$$

First, let p be odd and equal to $2n+3$, and take

$$\phi_0 = \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n \frac{f(y)}{r}, \quad y = t+r, \tag{36}$$

where $$r^2 = (x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 + \ldots + (x_p - x_p^0)^2.$$

Then, cutting out the point $(x_1^0, x_2^0, \ldots, x_p^0)$ in the usual way by a hyperspherical surface of small radius, we obtain

$$A\,\phi(x_1^0, \ldots, x_p^0, t) f(t) + \int_S \left(\phi_0\frac{\partial\phi}{\partial\nu} - \phi\frac{\partial\phi_0}{\partial\nu}\right) dS = \frac{\partial}{\partial t}\int_V \left(\phi\frac{\partial\phi_0}{\partial t} - \phi_0\frac{\partial\phi}{\partial t}\right) dV, \tag{37}$$

where $$A = 2\pi^{\frac{1}{2}p}\,\Gamma(p-1)/\Gamma(\tfrac{1}{2}p).$$

Now denoting the operator $\dfrac{\partial^2}{\partial y^2} - \dfrac{\partial^2}{\partial r^2}$ by D, we have

$$\phi\frac{\partial\phi_0}{\partial\nu} = \phi D^n\left\{f(y)\frac{\partial r^{-1}}{\partial\nu} + \frac{1}{r}\frac{\partial r}{\partial\nu} f'(y)\right\}$$
$$= \phi D^n f(y)\frac{\partial r^{-1}}{\partial\nu} - \frac{\partial\phi}{\partial t} D^n\frac{1}{r} f(y)\frac{\partial r}{\partial\nu} + \frac{\partial}{\partial t}\left\{\phi D^n\frac{1}{r} f(y)\frac{\partial r}{\partial\nu}\right\}. \tag{38}$$

Hence

$$A\,\phi(x_1^0, \ldots, x_p^0, t)\,f(t)+\int_S \left[\frac{\partial\phi}{\partial\nu} D^n \frac{f(y)}{r} - \phi D^n f(y)\,\frac{\partial r^{-1}}{\partial\nu} + \frac{\partial\phi}{\partial t} D^n \frac{f(y)}{r}\,\frac{\partial r}{\partial\nu}\right] dS$$

$$= \frac{\partial}{\partial t}\left[\int_S \phi D^n \frac{f(y)}{r}\,\frac{\partial r}{\partial\nu}\,dS + \int_V \left\{ \phi \frac{\partial}{\partial t} D^n \frac{f(y)}{r} - \frac{\partial\phi}{\partial t} D^n \frac{f(y)}{r} \right\} dV\right]. \quad (39)$$

Now we multiply each term of this equation by dt and integrate with respect to t between the limits $-\infty$ and $+\infty$. Then, if the function $f(t)$ is taken so as to vanish at the limits, and if ϕ is also subject to this limitation, the right-hand side of (39) gives zero to the result; also the terms of the second member on the left can easily be seen to be transformable into corresponding terms with the functions f and ϕ interchanged, but with y now equal to $t-r$ instead of $t+r$.

Thus we have

$$\int_{-\infty}^{\infty} f(t)\,dt \left[A\phi(x_1^0, \ldots, x_p^0, t)+\int_S \left\{ D^n \frac{1}{r}\,\frac{\partial\phi(y)}{\partial\nu} - D^n\phi(y)\,\frac{\partial r^{-1}}{\partial\nu} \right.\right.$$

$$\left.\left. + D^n \frac{1}{r}\,\frac{\partial\phi(y)}{\partial t}\,\frac{\partial r}{\partial\nu}\right\} dS\right] = 0. \quad (40)$$

Now the function $f(t)$ is arbitrary; hence we infer that the factor under the integral sign is always zero; and, writing

$$\frac{d}{d\nu} = \sum_{i=1}^{p} \frac{\partial x_i}{\partial\nu}\,\frac{\partial}{\partial x_i}, \qquad \frac{\delta}{\delta\nu} = \frac{\partial r}{\partial\nu}\,\frac{\partial}{\partial r}, \quad (41)$$

this can be put in the form

$$A\phi(x_1^0, \ldots, x_p^0, t) = \int_S \left[\frac{d}{d\nu}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n \frac{\phi(x_1, \ldots, x_p, y)}{r}\right.$$

$$\left. - \frac{\delta}{\delta\nu}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n \frac{\phi(x_1, \ldots, x_p, y)}{r}\right] dS, \quad (42)$$

where $\qquad y = t - r.$

Secondly, let p be even and equal to $2n+2$; then we take

$$\phi_0 = \int_0^{\infty} \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n f(y)\,dv, \qquad y = t + r\cosh v.$$

And by a similar transformation we find in this case

$$A\phi(x_1^0, \ldots, x_p^0, t) = \int_S \left[\frac{d}{d\nu}\int_0^{\infty} \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n \phi(x_1, \ldots, x_p, y)\,dv\right.$$

$$\left. - \frac{\delta}{\delta\nu}\int_0^{\infty} \left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial r^2}\right)^n \phi(x_1, \ldots, x_p, y)\,dv\right] dS, \quad (43)$$

where $\qquad y = t - r\cosh v.$

10. *More general view of the Kirchhoff-Poisson Solutions.*

Before discussing these results we shall obtain them by a more general method, which is also capable of giving solutions of the Poisson type.

Consider the equation

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\ldots+\frac{\partial^2\phi}{\partial x_p^2}-\frac{\partial^2\phi}{\partial t^2}=0. \tag{44}$$

We require to find a solution ϕ finite and continuous in a certain domain, and taking given values over a given frontier.

Let ϕ_0 be a solution of (44) finite and continuous along with its derivatives of the first two orders in a domain in $p+1$ dimensions, whose frontier is denoted by ω_{p+1}; then, writing

$$\sum_{i=1}^{p}\frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial \nu}-\frac{\partial U}{\partial t}\frac{\partial t}{\partial \nu}=D_\nu^p U, \tag{45}$$

we have, by an extension of Green's theorem,*

$$\int_{\omega_{p+1}}(\phi_0 D_\nu^p\phi-\phi D_\nu^p\phi_0)\,d\omega_{p+1}=0. \tag{46}$$

Let $(x_1^0, \ldots, x_p^0, t^0)$ be the coordinates of a point in space of $p+1$ dimensions; and let

$$r^2=(x_1-x_1^0)^2+\ldots+(x_p-x_p^0)^2,$$

$$u=(t_0-t)/r.$$

A function ϕ_0 satisfying the conditions in the region for which $u>1$ is given by

$$\phi_0=\int_1^u (u^2-1)^{\frac{1}{2}(p-3)}\,du. \tag{47}$$

Of this region consider the part bounded by the frontier ω_{p+1} made up of the following parts:—

(i.) The cone C given by $u=1=(t_0-t)/r$.

(ii.) The cylinder R given by $r=\epsilon$, where ϵ is to decrease indefinitely.

(iii.) The surface Ω on which the function ϕ and its derivatives are supposed known.

---

* Coulon, *Proc.-Verb.*, Bordeaux, 1898-99, p. 85.

Now, if we take Ω to be a general surface in $p+1$ dimensions, we shall obtain ϕ in the form of a combination of solutions of the Kirchhoff and Poisson types; but to separate the two types we first suppose Ω to be a cylindrical surface with its generators parallel to the axis of t. Thus on Ω we have

$$\frac{\partial t}{\partial \nu} = 0 \quad \text{and} \quad d\omega_{p+1} = dS\,dt,$$

where $dS =$ element of surface of a section of Ω by a plane $t =$ constant. For convenience the figure is drawn for the case $p = 2$.

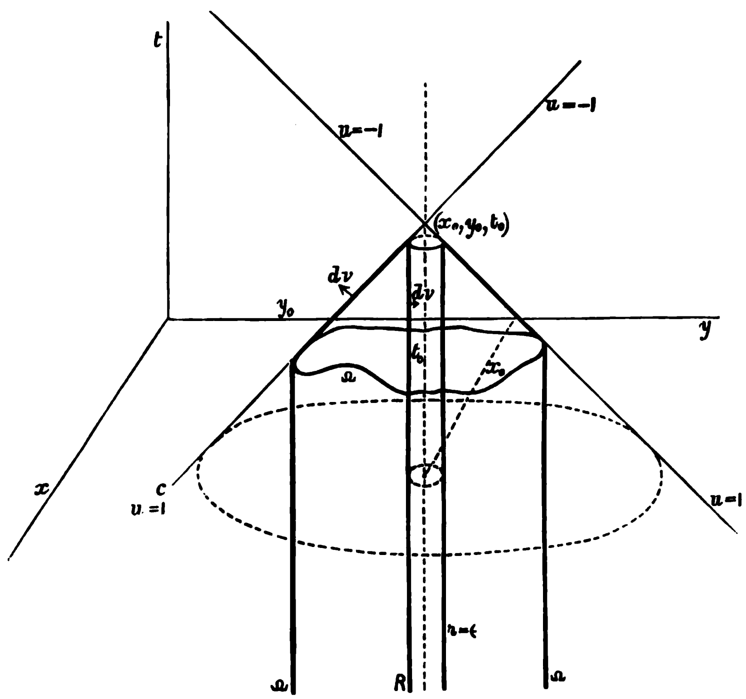


FIG. 1.

Then (46) becomes

$$\int_{\Omega+R+C} (\phi_0 D_\nu^p \phi - \phi D_\nu^p \phi_0)\, d\omega_{p+1} = 0. \tag{48}$$

It can soon be verified that the integral over the cone vanishes, and that over the cylinder R it becomes in the limit

$$-A \int_{t_0}^{-\infty} (t_0 - t)^{p-2}\, \phi\,(x_1^0, \ldots, x_p^0, t)\, dt,$$

where A is the surface of the hyper-sphere of unit radius in p dimensions. Thus we obtain, from (48),

$$A\int_{t_0}^{-\infty}(t_0-t)^{p-2}\,\phi(x_1^0,\,\ldots,\,x_p^0,\,t)\,dt=\int dS\int_{t_0-r}^{-\infty}\left(\phi_0\frac{\partial\phi}{\partial\nu}-\phi\frac{\partial\phi_0}{\partial\nu}\right)dt. \quad (49)$$

Now, if we differentiate this equation $p-1$ times with regard to t_0, we get on the left-hand side—supposing ϕ to vanish for infinitely large negative values of t—

$$-A\,(p-2)!\,\phi(x_1^0,\,\ldots,\,x_p^0,\,t^0). \quad (50)$$

Also the expressions on the right-hand side can be reduced by the aid of the following results.

Using the definition of ϕ_0 given in (47), we find

$$\frac{\partial^s}{\partial t_0^s}\int_{t_0-r}^{-\infty}\psi(t)\,\phi_0\,dt=\int_{t_0-r}^{-\infty}\psi\,\frac{\partial^s\phi_0}{\partial t_0^s}\,dt=\int_{t_0-r}^{-\infty}\frac{\partial^s\psi}{\partial t^s}\,\phi_0\,dt,$$

provided that ψ vanishes for $t=-\infty$, and also that

(i.) if p be odd, then $s \leqslant \frac{1}{2}(p-1)$;

(ii.) if p be even, $s \leqslant \frac{1}{2}p$;

and in both cases

$$\frac{\partial^{p-1}}{\partial t_0^{p-1}}\int_{t_0-r}^{-\infty}\psi\phi_0\,dt=\int_{t_0-r}^{-\infty}\frac{\partial^{p-2}\psi}{\partial t^{p-2}}\,\frac{\partial\phi_0}{\partial t_0}\,dt$$

$$=\frac{1}{r}\int_{t_0-r}^{-\infty}\frac{\partial^{p-2}\psi}{\partial t^{p-2}}\left\{\left(\frac{t_0-t}{r}\right)^2-1\right\}^{\frac{1}{2}(p-3)}dt.$$

Hence, from the right-hand side of (49), we get

$$\int dS\int_{t_0-r}^{-\infty}dt\left[\frac{1}{r}\left\{\left(\frac{t_0-t}{r}\right)^2-1\right\}^{\frac{1}{2}(p-3)}\frac{\partial^{p-2}}{\partial t^{p-2}}\,\frac{\partial\phi}{\partial\nu}\right.$$
$$\left.+\frac{1}{r}\,\frac{\partial r}{\partial\nu}\left\{\left(\frac{t_0-t}{r}\right)^2-1\right\}^{\frac{1}{2}(p-3)}\frac{\partial^{p-1}\phi}{\partial t^{p-1}}\left(\frac{t_0-t}{r}\right)\right]. \quad (51)$$

Writing $t=t_0-r\cosh v$, and using our previous notation, we find that the first part of (51) becomes

$$-\int dS\,\frac{d}{d\nu}\int_0^{\infty}\phi^{(p-2)}\,(x_1^0,\,\ldots,\,x_p^0,\,t^0-r\cosh v)\sinh^{p-2}v\,dv,$$

and the second part of (51) is

$$\int dS\,\frac{\delta}{\delta\nu}\int_0^{\infty}\phi^{(p-2)}(x_1^0,\,\ldots,\,x_p^0,\,t^0-r\cosh v)\sinh^{p-2}v\,dv.$$

Hence we have, finally,

$$(p-2)!\, A\, \phi(x_1^0, \ldots, x_p^0, t^0)$$

$$= \int dS \left(\frac{d}{d\nu} - \frac{\delta}{\delta\nu}\right) \int_0^\infty \phi^{(p-2)}(x_1^0, \ldots, x_p^0, t^0 - r\cosh v) \sinh^{p-2} v\, dv. \quad (52)$$

When p is odd the integral with respect to v can be evaluated, and in fact we find that for the two cases—p odd and p even—this can be put into two forms (42) and (43) given previously.

11. *Discussion of the Results.*

Returning to these expressions, we notice at once the similarity of the results to the ordinary form in three dimensions: if we regard the value of ϕ due to a simple source, varying with the time according to the function $f(t)$, as given by the expressions in (29) and (31), we see that we may consider the general form of ϕ at a point as due to a distribution of simple and double sources over the given frontier. On the other hand, the difference between (42) and (43) is marked. For when p is odd we require to know the values over the frontier S at a particular time for each point of S, namely the time $t_0 - r$; but when p is even we require to know these values not only at the time $t_0 - r$, but for all time previous to this.*

12. *Extension of the Poisson Solution.*

We shall find the difference expressed in another way if we proceed to obtain solutions of the Poisson type. For this purpose we return to the integral given in (46), and now suppose Ω to be the plane $t = 0$. The figure (Fig. 2) is drawn for two dimensions; then Ω is the plane of x, y.

As before, the integral over the cone vanishes; and the integral over the cylinder is the same as before, but with different limits for t; in fact it is

$$A \int_0^{t_0} (t_0 - t)^{p-2} \phi(x_1^0, \ldots, x_p^0, t)\, dt. \quad (53)$$

Now Ω is the part of the plane $t = 0$ which is cut out by the cone $u = 1$; that is, it is a hyper-sphere of radius t_0 in space of p dimensions. Also

* Cf. Volterra, *Rend. R. Ac. Lincei*, Ser. 5, Vol. I., p. 161; Hadamard, *Bulletin de la Soc. Math.*, T. XXVIII., p. 69.

over Ω we have

$$\frac{\partial t}{\partial \nu} = -1, \qquad \frac{\partial x_i}{\partial \nu} = 0, \qquad i = 1, 2, \ldots, p.$$

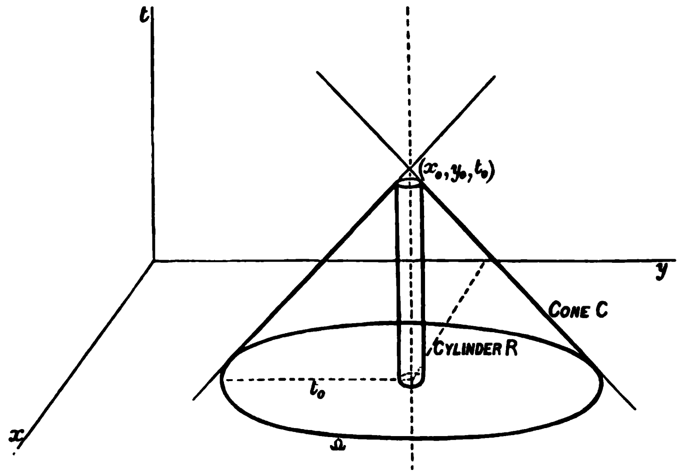


FIG. 2.

Hence, from (45) and (46), we see that the integral over Ω is

$$-\int \left(\phi_0 \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi_0}{\partial t}\right) d\Omega.$$

Thus we have

$$A \int_0^{t_0} (t_0 - t)^{p-2} \, \phi(\overset{0}{x_1}, \ldots, \overset{0}{x_p}, t) \, dt = \int_\Omega \left(\phi_0 \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi_0}{\partial t}\right) d\Omega, \tag{54}$$

where in the second integral t is put equal to zero after differentiating, and the integration extends throughout the hyper-sphere of radius t_0.

Remembering the definition of ϕ_0, and denoting by ϕ_i and $\dot{\phi}_i$ the values of ϕ and $\dot{\phi}$ when $t = 0$, (54) becomes

$$A \int_0^{t_0} (t_0 - t)^{p-2} \, \phi(\overset{0}{x_1}, \ldots, \overset{0}{x_p}, t) \, dt$$
$$= \int d\Omega \left[\frac{1}{r} (u^2 - 1)^{\frac{1}{2}(p-3)} \phi_i + \dot{\phi}_i \int_1^u (u^2 - 1)^{\frac{1}{2}(p-3)} du\right], \tag{55}$$

where $$u = t_0 / r.$$

Further, A is the surface of the hyper-sphere of unit radius; then

$$d\Omega = r^{p-1} dA \, dr.$$

Let $\overline{\phi_i(r)}$ and $\overline{\dot{\phi}_i(r)}$ denote the mean values of ϕ_i and $\dot{\phi}_i$ taken over the hyper-spherical surface of radius r; then (55) may be written

$$\int_0^{t_0} (t_0-t)^{p-2}\,\phi(\overset{0}{x}_1, \ldots, \overset{0}{x}_p, t)\,dt$$

$$= \int_0^{t_0} r^{p-1}\,dr\left[\frac{1}{r}(u^2-1)^{\frac{1}{2}(p-3)}\,\overline{\phi_i(r)}+\overline{\dot{\phi}_i(r)}\int_1^u (u^2-1)^{\frac{1}{2}(p-3)}\,du\right]. \quad (56)$$

Now, if we differentiate this equation $p-1$ times with respect to t_0, we get on the left-hand side

$$(p-2)!\,\phi(\overset{0}{x}_1, \ldots, \overset{0}{x}_p, t^0), \quad (57)$$

and, applying the operator to the right-hand side, the result may be written

$$(p-2)!\,\phi(\overset{0}{x}_1, \ldots, \overset{0}{x}_p, t^0)$$

$$= \frac{\partial^{p-1}}{\partial t_0^{p-1}}\int_0^{t_0} (t_0^2-r^2)^{\frac{1}{2}(p-3)}\,r\,\overline{\phi_i(r)}\,dr+\frac{\partial^{p-2}}{\partial t_0^{p-2}}\int_0^{t_0} (t_0^2-r^2)^{\frac{1}{2}(p-3)}\,r\,\overline{\dot{\phi}_i(r)}\,dr; \quad (58)$$

or, if we put $$F = \frac{\partial^{p-2}}{\partial t_0^{p-2}}\int_0^{t_0} (t_0^2-r^2)^{\frac{1}{2}(p-3)}\,r\,\overline{\dot{\phi}_i(r)}\,dr,$$

and f = similar quantity with $\overline{\phi_i(r)}$ instead of $\overline{\dot{\phi}_i(r)}$, we have

$$(p-2)!\,\phi(\overset{0}{x}_1, \ldots, \overset{0}{x}_p, t^0) = F+\frac{\partial f}{\partial t_0}. \quad (59)$$

13. *Comparison of different Cases of the Extended Poisson Form.*

The results given in (58) or (59) are solutions of the required Poisson type. We get the ordinary form by taking $p = 3$; then

$$F = \frac{\partial}{\partial t_0}\int_0^{t_0} r\,\overline{\dot{\phi}_i(r)}\,dr = t_0\,\overline{\dot{\phi}_i(t_0)}.$$

Hence $$\phi(\overset{0}{x}_1, \overset{0}{x}_2, \overset{0}{x}_3, t^0) = t_0\,\overline{\dot{\phi}_i(t_0)}+\frac{\partial}{\partial t_0}\left\{t_0\,\overline{\phi_i(t_0)}\right\}. \quad (60)$$

In two dimensions we have

$$F = \int_0^{t_0} \frac{r\,\overline{\dot{\phi}_i(r)}\,dr}{\sqrt{t_0^2-r^2}},$$

and $$\phi(\overset{0}{x}_1, \overset{0}{x}_2, t^0) = \int_0^{t_0} \frac{r\,\dot{\phi}_i(r)\,dr}{\sqrt{t_0^2 - r^2}} + \frac{\partial}{\partial t_0}\int_0^{t_0} \frac{r\,\overline{\phi_i(r)}\,dr}{\sqrt{t_0^2 - r^2}}, \tag{61}$$

which is the Poisson-Parseval solution for this case.*

These two cases are typical, as before, of the two classes—p odd and p even. It can be verified that in the former case F always reduces to a sum of expressions from which the integral with respect to r has disappeared; so that the values of the quantities are only required to be known on the frontier $r = t_0$. On the other hand, for p even, the initial values have to be known throughout the hyper-sphere $r = t_0$.

The application of the formula (60)—say to sound waves in three dimensions—is well known. Suppose the initial disturbance is confined to a limited region T, and let r_1, r_2 be the radii of the least and greatest spheres described about an external point O, so as just to cut T. Then ϕ at O is zero for any time $t < r_1$, and is also zero for any time $t > r_2$.

Now apply the result (61) to the similar problem in two dimensions. If r_1 be the radius of the least circle to cut T, we see that ϕ at O is zero for any time $t < r_1$. But, from the fact that the integrals are taken over the area of the circle $r = t$, we see that in general, at any time $t > r_1$, ϕ is not zero, but only falls asymptotically to zero. This seems to be the simplest way of showing the existence of the "tail" in the case of cylindrical waves.

14. *The Residual Integral.*

The results we have obtained can be expressed in another way by means of what has been called the residual integral of partial differential equations.†

Consider the simplest equation in two independent variables with real characteristics; that is, the case of $p = 1$,

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = 0,$$

which can be reduced to $$\frac{\partial^2\phi}{\partial u\,\partial v} = 0.$$

Suppose the values of ϕ and its first derivatives are given on an arc AB; then ϕ is defined in the region ABC formed by AB and parallels through

* Cf. Rayleigh, *Theory of Sound*, Vol. II., p. 103.

† Hadamard, *loc. cit.*

A and B to the axes of u and v respectively. On AB let there be two points α, β, such that on $A\alpha$ and $B\beta$ the given values of ϕ and its derivatives are zero; but on $\alpha\beta$ let these values be arbitrarily given, with the condition of being zero at α and β in order to preserve continuity. Then the area ABC can be divided as shown in the figure into three kinds of regions; it is the value of ϕ at any point O in the region numbered 3 which is defined as the residual integral. In this case it is easily seen to be a constant.

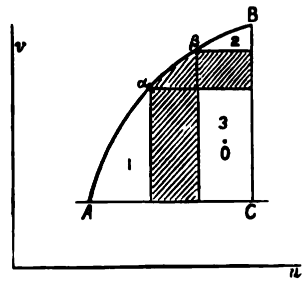


FIG. 3.

Now suppose that we are representing wave-motions—say sound waves in an unlimited medium—by the equations we have used; then, if the initial disturbance be confined to a limited region, we see that the medium at any point will come to rest after the passage of the disturbance across it, if the residual integral is in general a constant—which may in particular be zero.

Hence we may infer that in the general equation

$$\frac{\partial^2\phi}{\partial x_1^2}+\frac{\partial^2\phi}{\partial x_2^2}+\ldots+\frac{\partial^2\phi}{\partial x_p^2}-\frac{\partial^2\phi}{\partial t^2}=0$$

the residual integral is constant if p be odd, but is not so if p be even.

NOTE ON A SYSTEM OF LINEAR CONGRUENCES

By J. CULLEN.

[Received April 7th, 1904.—Read April 14th, 1904.]

GAUSS (*Werke*, Vol. II., pp. 507–509) has given a table from which the linear forms of H in the equation

$$N = lH^2 + mG^2$$

are easily obtained. In a previous communication* the writer has given a general process for solving any linear system. When, however, H is ambiguous in sign the work in applying this process can be considerably reduced. The object of the present note is to explain this briefly.

[The results in the present paper are numbered with accented figures, thus: 1′, 2′, &c. References with unaccented figures (as 1, 2, &c.) relate to the numbered results in the original paper.]

1. In the system § 2 of the original paper it is clear that the solution arising out of the selection of one residue in each row is quite independent of the value of any of the other residues.

Thus, for instance, in

$$H \equiv a_t \pmod{P} \equiv \beta_s \pmod{Q} \equiv \rho'_t \pmod{p'},$$

H in no way depends on the other residues.

Hence we may introduce

$$a_0 \equiv 0 \pmod{P}, \qquad \beta_0 \equiv 0 \pmod{Q},$$

provided (i.) the redundant cases are subsequently excluded, (ii.) the actual cases in which 0 occurs are retained.

Since the symbol κ in the set row of each strip refers to the case a_κ, and ϖ in the arrangement row of each strip to the case β_ϖ, the case a_0 is excluded or retained, according as the symbol 0 is not written or is written in the set rows. In like manner, β_0 is excluded by not using the arrangement in which 0 occurs in the arrangement rows, though 0 is always written in the initial division of the arrangement row of each strip.

2. The effect of introducing 0 in each row of the given system, where it is wanting, is that

(i.) The preliminary work is much simplified, for $a_0 \equiv 0$, $\beta_0 \equiv 0$ now take the place of a_1, β_1 in the determination of the λ's and the r's in p. 325;

---

* *Proc. London Math. Soc.*, Vol. XXXIV., pp. 323–334.

(ii.) The equation (10) becomes simply

$$H = xPQ + r_{\varpi, 0} + r_{0, \kappa}, \qquad \text{since} \qquad r_{0,0} = 0;$$

(iii.) The headings of the columns of the elements table, p. 333, become $r_{0,0}$, $r_{0,1}$, $r_{0,2}$, ..., $r_{1,0}$, $r_{2,0}$, $r_{3,0}$, ..., with 0 throughout the first column; and hence, because 0 is written in the initial division of the arrangement rows, this division is now always placed under the 0 in each line of the base sheet when finding the arrangement numbers, instead of different numbers for different strips, as given by Rule III.

In short, we proceed exactly as in the paper, with this difference: that the subscript 0 is substituted for 1 in the preliminary work and elements table, and the question of the retention or rejection of α_0, β_0 (it rarely happens that these are to be retained) is to be considered.

3. The remarks of the foregoing paragraphs apply to any linear system whatever. In what follows we consider a linear system in which there exists an ambiguity in sign for every residue, as is always the case in H where

$$aH^2 \pm bG^2 = N.$$

The same ambiguity arises also in the residues obtained by combining different moduli. In fact, we always have

$$a(\pm H)^2 \pm b(\pm G)^2 = N.$$

4. In a system of the given type,

$$\begin{aligned} H &\equiv \pm\alpha_1, \ \pm\alpha_2, \ \pm\ldots, \ \pm\alpha_m \ (\text{mod } P) && (1') \\ &\equiv \pm\beta_1, \ \pm\beta_2, \ \pm\ldots, \ \pm\beta_n \ (\text{mod } Q) && (2') \\ &\equiv \pm\rho_1', \ \pm\rho_2', \ \pm\ldots, \ \pm\rho_f' \ (\text{mod } p') \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ &\equiv \pm\rho_1^{(\sigma)}, \ \pm\rho_2^{(\sigma)}, \ \pm\ldots, \ \pm\rho_s^{(\sigma)} \ (\text{mod } p^{(\sigma)}), \end{aligned}$$

we are merely concerned with the congruences (1′) and (2′). As to the others, we need only supply 0 where it is absent, and proceed as in the original paper (pp. 323–334), dotting only the given 0's.

If we make all the residues in (1′) positive, and arrange them according to the order of magnitude, we have, on taking α_1, the smallest, and α_{2m}, the greatest,

$$H \equiv \alpha_1, \ \alpha_2, \ \ldots, \ \alpha_m, \ \alpha_{m+1}, \ \ldots, \ \alpha_{2m} \ (\text{mod } P).$$

Now, call α_{ϖ}, $\alpha_{\varpi'}$ complementary residues where

$$\alpha_{\varpi} + \alpha_{\varpi'} = P. \qquad (3')$$

We then have also

$$\varpi + \varpi' = 1 + 2m. \qquad (4')$$

5. It is now easy to show that $r_{\varpi,0}$, $r_{\varpi',0}$ are complementary with respect to the modulus PQ if $\varpi+\varpi' = 2m+1$. For, by (3), (5), (7), p. 324, we have

$$r_{\varpi,0} = P\lambda_{\varpi,0}+a_{\varpi} \equiv uPa_{\varpi}+a_{\varpi} \pmod{PQ} \equiv Qva_{\varpi} \pmod{PQ}.$$

Also $$r_{\varpi',0} \equiv Qva_{\varpi'} \pmod{PQ}.$$

Therefore, by (3′), $r_{\varpi,0}+r_{\varpi',0} \equiv 0$; so that, taking $r_{\varpi,0}$, $r_{\varpi',0}$ as least positive residues, then
$$r_{\varpi,0}+r_{\varpi',0} = PQ. \tag{5'}$$

In a precisely similar manner we find that, if $\kappa+\kappa' = 2n+1$, then

$$r_{0,\kappa}+r_{0,\kappa'} = PQ. \tag{6'}$$

6. From (5′) and (6′) it follows that, having found $r_{\varpi,0}$ and $r_{0,\kappa}$, it is not necessary to apply the congruences, p. 325, to find $r_{\varpi',0}$ and $r_{0,\kappa'}$, ϖ' ranging from $m+1$ to $2m$, and κ' from $n+1$ to $2n$. A similar remark applies to $\theta_{\varpi',0}$ and $\theta_{0,\kappa'}$ in the elements table for every row (1 to σ), since we have for any prime $p^{(\tau)}$, by (5′) and (6′),

$$\theta^{(\tau)}_{\varpi',0} \equiv t^{(\tau)}-\theta^{(\tau)}_{\varpi,0} \pmod{p^{(\tau)}}, \qquad \theta^{(\tau)}_{0,\kappa'} \equiv t^{(\tau)}-\theta^{(\tau)}_{0,\kappa} \pmod{p^{(\tau)}}.$$

It is not even necessary to complete the elements table in this manner; for all we really require are the residues (θ) and (t) for the columns under $r_{0,0}, r_{0,1}, \ldots, r_{0,n}$; $r_{1,0}, r_{2,0}, \ldots, r_{m,0}, PQ$. This is easily seen, as follows.

7. To find the arrangement numbers for the strip p', say, write 0 in the initial division and place this division under the 0 of the p' line of the base sheet, writing 1, 2, ..., m under the t''s that equal $\theta'_{1,0}, \theta'_{2,0}, \ldots, \theta'_{m,0}$, and $m+1$, $m+2$, ..., $2m$ under the t''s that equal $(t'-\theta'_{m,0})$, $(t'-\theta'_{m-1,0})$, ..., $(t'-\theta'_{1,0})$ respectively. This is nothing more than writing the corresponding subscripts of the complementary residues. [Cf. (5′).]

It will be noticed that ϖ and ϖ' are equally distant from the centre of the strip; in fact, since $\theta'_{\varpi,0} \equiv xt'$ and $\theta'_{\varpi',0} \equiv x't'$, we have, by (5′),

$$xt+x't' \equiv t' \pmod{p'} \qquad \text{or} \qquad x+x' \equiv 1 \pmod{p'};$$

so that $x+x' = 1+p'$ if $x > 0 < p'$, $x' > 0 < p'$, x and x' being the numbers in the base line over $\theta'_{\varpi,0}$, $\theta'_{\varpi',0}$.

[*Ex. gr.*—In the base sheet facing p. 334, for $p = 17$ say, and $\theta_{\varpi,0} \equiv 15$, then $\theta_{\varpi',0} \equiv 11-15 \equiv 13$, the numbers in the base line over 15 and 13 are $x = 6$, $x' = 12$, and $6+12 = 1+17$. Similarly for other cases.]

This serves as a useful check in writing down the arrangement numbers.

8. Having seen that the number of columns of the elements table giving the arrangement numbers may consist simply of m columns,

instead of $2m$, we can now show by (6′) that a similar result holds for the columns giving the set numbers, viz., that we need only n instead of $2n$ columns* (since $r_{0,0} = 0$).

For suppose that a solution is

$$H = xPQ + r_{\varpi',0} + r_{0,\kappa'};$$

we can, by searching to the left of the 0 of the base line, make x negative, and hence, if $x = -y$ $(y > 0)$, then

$$H = -(yPQ - r_{\varpi',0} - r_{0,\kappa'}) = -\{(y-2)PQ + r_{\varpi,0} + r_{0,\kappa}\};$$

and, as H is itself ambiguous in sign, there must also be the positive solution

$$H = (y-2)PQ + r_{\varpi,0} + r_{0,\kappa}.$$

In other words, since x and y are numerically equal, we have the result that, if κ' appears throughout the strips in the arrangement ϖ' under x on the left of the 0 of the base line (thus: ..., 5, 4, 3, 2, 1, 0, 1, 2, 3, 4, 5, ...), then κ appears throughout in the arrangement ϖ under $x-2$ on the right of the base line's 0, where

$$\varpi + \varpi' = 2m+1, \qquad \kappa + \kappa' = 2n+1.$$

As these solutions are numerically equal (in fact, complementary with respect to the modulus that is the product of all the moduli), we may obviously exclude one, which is done at once by excluding the set numbers κ' that range from $n+1$ to $2n$.

9. In brief, therefore, in the linear system arising from the equation

$$N = lH^2 + mG^2$$

we may neglect all ambiguity in sign in the α's and β's (so that no two complements ever occur in the system), and proceed as in the original paper, with the addition of reading to the left, and, if κ appears throughout the arrangement ϖ under x on the left, then the solution is

$$H = -xPQ + r_{\varpi,0} + r_{0,\kappa}.$$

10. The case $N = H^2 - G^2$, where N is a factor of $a^z \pm 1$, requires special treatment, since it is known that every factor of $a^z \pm 1$ is of the form $Rz+1$. It is further known that, generally, $H = z^2L + M$; hence we have a row

$$H = a_1, a_2, a_3, \ldots, a_m \pmod{z^2P},$$

in which none of the a's are complementary. It is, however, easy to see that all that is required is to proceed in the manner explained above in searching to the right of the 0 of the base line, but in searching to the left we take the complementary arrangements and deal only with these.

* A considerable saving in the labour of drawing up the strips. It also extends the scope of the process.

AN EXTENSION OF SYLOW'S THEOREM

By G. A. MILLER.

[Received April 7th, 1904.—Read April 14th, 1904.]

FROBENIUS extended Sylow's theorem by proving that every group (G) whose order (g) is divisible by p^a, p being any prime number, contains $1+kp$ sub-groups of order p^a.* The present note is devoted to the theorem that *the number of cyclic sub-groups of order* p^a ($a > 1$, $p > 2$) *in* G *is always of the form* kp *whenever the Sylow sub-groups* † *of order* p^m *in* G *are non-cyclic.* In particular, every non-cyclic group of order p^m contains just lp cyclic sub-groups of order p^a, and hence the number of its non-cyclic sub-groups of this order is of the form $1+kp$. We shall first prove this particular case; that is, we shall first assume that $g = p^m$.

When G is Abelian the number of its operators of order p^a is

$$p^{m_1+m_2+\ldots+m_a} - p^{m_1+m_2+\ldots+m_{a-1}}$$

where m_β ($\beta = 1, 2, \ldots, a$) is the number of the invariants of G which $\geqslant p^\beta$.‡ The number of the cyclic sub-groups of order p^a is the quotient obtained by dividing the number of the operators of this order by $p^a - p^{a-1}$. As $m_1+m_2+\ldots+m_{a-1} > a-1$,§ it follows that the given theorem is true for any Abelian group of order p^m. In the next three paragraphs it is assumed that G is a non-Abelian group of order p^m, $p > 2$.

Since the number of the non-invariant cyclic sub-groups of order p^a in G is a multiple of p, it remains only to show that the number of its invariant cyclic sub-groups of this order is also a multiple of p. Let P_a be such an invariant sub-group. It is contained in some invariant sub-group of each of the orders $p^{a+1}, p^{a+2}, \ldots, p^m$. Hence we may assume that there is an invariant non-cyclic sub-group of order $p^{a+\gamma}$, $\gamma > 0$, which includes an operator of order $p^{a+\gamma-1}$ which generates P_a. As this

* Frobenius, *Berliner Sitzungsberichte*, 1895, p. 984.

† *Bulletin of the American Mathematical Society*, Vol. IX., 1903, p. 543.

‡ Cf. Netto, *Vorlesungen über Algebra*, Vol. II., 1900, p. 247.

§ It will always assumed that G is non-cyclic. When G is cyclic

$$m_1 + m_2 + \ldots + m_{a-1} = a-1,$$

since $m_\beta = 1$.

non-cyclic group contains just p cyclic sub-groups of order p^{α},* being conformal with the Abelian group of type $(\alpha+\gamma-1, 1)$, G must contain at least p cyclic invariant sub-groups of order p^{α} whenever it contains one such sub-group. These p sub-groups generate a group (H_1) of $p^{\alpha+1}$.

If G contains another invariant cyclic sub-group P'_{α} of order p^{α}, it must contain another invariant sub-group (H_2) of order $p^{\alpha+1}$ which involves just p such invariant cyclic sub-groups. If none of these is contained in H_1, we have found just $2p$ cyclic invariant sub-groups of order p^{α}. If one of them is in H_1, the $2p-1$ distinct cyclic invariant sub-groups which are found in H_1 and H_2 have just $p^{\alpha-1}$ common operators. We proceed to show that the group generated by H_1 and H_2, $\{H_1, H_2\}$, is conformal with an Abelian group whenever H_1 and H_2 have a common cyclic group of order p^{α}.

The commutator sub-group of $\{H_1, H_2\}$ is clearly the sub-group of order p contained in P_{α}, since a generator of P'_{α} transforms each one of a set of generators of H_1 into itself multiplied by an operator of this sub-group of order p. Moreover, if an operator (s_1) transforms an operator s_2 into itself multiplied by an operator of order p, which is commutative with s_1 and s_2, then $(s_2 s_1)^p = s_2^p s_1^p$.† Hence $\{H_1, H_2\}$ is conformal with the Abelian group of type $(\alpha, 1, 1)$. This process can be continued until the cyclic invariant sub-groups of order p^{α} which have $p^{\alpha-1}$ operators in common with P_{α} are exhausted. All of these generate a group which is conformal with an Abelian group of type $(\alpha, 1, 1, \ldots)$. As all the invariant cyclic sub-groups of order p^{α} can be divided into one or more sets, it is proved that every non-cyclic group of order p^m contains lp cyclic sub-groups of order p^{α}, $p > 2$, $\alpha > 1$.‡

That the theorem is also true when g is not a power of a prime follows directly from the fact that every Sylow sub-group of order p^m transforms any sub-group of order p^{α} which is found in G, but not in the Sylow sub-group, into p^{γ} conjugates. Hence the number of sub-groups of any particular type which are found in G, but not in a given Sylow sub-group, is a multiple of p. It is clear that the given theorem could also have been stated as follows:—If G contains a non-cyclic sub-group of order p^{α}, the number of its *non-cyclic* sub-groups of this order is of the form $1+kp$.

---

* Burnside, *Theory of Groups of Finite Order*, 1897, p. 76.

† *Bulletin of the American Mathematical Society*, Vol. VII., 1901, p. 350.

‡ It is easy to prove that all the operators of order p^{α} in G which have at most p conjugates under G generate a characteristic sub-group which is conformal with an Abelian group. In particular, the number of sub-groups of order p is of the form $1+p+kp^2$. Bauer proved that the number of sub-groups of order p^{m-1} is of the form $1+p+p^2+\ldots+p^{r}$. *Nouvelles Annales*, Vol. XIX., 1900, p. 508.

PERPETUANT SYZYGIES OF DEGREE FOUR

By P. W. WOOD.

[Received March 29th, 1904.—Read April 14th, 1904.—Revised May 17th, 1904.]

1.

It is known that*

All perpetuants linear in the coefficients of each of the quantics $a_{1_x}^{n_1}, a_{2_x}^{n_2}, \ldots, a_{\delta_x}^{n_\delta}$, where $n_1, n_2, \ldots, n_\delta$ are all infinite, can be expressed linearly in terms of

(i.) symbolical products of the form

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_{\delta-1} a_\delta)^{\lambda_\delta},$$

where $\lambda_1 \geqslant 2^{\delta-2}$, $\lambda_2 \geqslant 2^{\delta-3}$, $\ldots$, $\lambda_{\delta-1} \geqslant 1$, and the sequence of the letters $a_1, a_2, \ldots, a_\delta$ is fixed beforehand; and

(ii.) products of perpetuants of lower degrees.

In a previous paper† I have proved this theorem to be exact, so that the type forms of class (i.) are actually irreducible; and this fact makes it worth while to search for the syzygies connecting the product forms of class (ii.) above.

2.

The number of linearly independent perpetuants of weight ω and degree δ is $\binom{\omega+\delta-2}{\delta-2}$; the number of types of class (i.) above is, since each symbolical product must be of weight $(2^{\delta-1}-1)$ at least, $\binom{\omega+\delta-2^{\delta-1}-1}{\delta-2}$; if P is the number of distinct product forms chosen in accordance with some definite convention, then the $\left\{\binom{\omega+\delta-2^{\delta-1}-1}{\delta-2}+P\right\}$ types and product forms are equivalent to $\binom{\omega+\delta-2}{\delta-2}$ linearly independent

* Grace, *Proc. London Math. Soc.*, Vol. XXXV.
† Wood, *ibid.*, Vol. 1, Ser. 2.

forms. Hence there must be S syzygies between these types and products where

$$S \equiv \binom{\omega+\delta-2^{\delta-1}-1}{\delta-2}+P-\binom{\omega+\delta-2}{\delta-2};$$

so, since the theorem of § 1 is exact, there must be exactly

$$\binom{\omega+\delta-2^{\delta-1}-1}{\delta-2}+P-\binom{\omega+\delta-2}{\delta-2}$$

syzygies among the product forms alone.

Here we have defined P as the number of distinct product forms; if we introduce Q other distinct product forms, then we have to find Q further syzygies; it is therefore contrary to our purpose to introduce any product form which is obviously expressible in terms of those already chosen; and in general we should ensure that none of the product forms are in this sense redundant.

3. *Syzygies of Degree* 3.

The only product forms of weight ω are

$$(bc)^\omega,\ (ca)^\omega,\ (ab)^\omega.$$

If $\omega > 1$, we have $S = (\omega-2)+3-(\omega+1) = 0,$

and, if $\omega = 1$, there are no type forms, and the number of syzygies is 1. Hence there are no syzygies of degree 3, save for weight unity, for which there is a single syzygy $(bc)+(ca)+(ab) \equiv 0.$

4. *Syzygies of Degree* 4.

Using the symbol $\{a_1a_2\ldots a_r\}^\omega$ to denote a covariant of weight ω in the symbolical letters $a_1, a_2, \ldots, a_r$, the products of degree 4 are of the following classes:—

$$\left.\begin{matrix}\{ab\}^\omega\{c\}\{d\}, & \{ac\}^\omega\{b\}\{d\}, & \{ad\}^\omega\{b\}\{c\} \\ \{bc\}^\omega\{a\}\{d\}, & \{bd\}^\omega\{a\}\{c\}, & \{cd\}^\omega\{a\}\{b\}\end{matrix}\right\} \text{Degree } 2\times\text{Degree } 1 \times\text{Degree } 1.$$

$$\left.\begin{matrix}\{a\}\{bcd\}^\omega \\ \{b\}\{acd\}^\omega \\ \{c\}\{abd\}^\omega \\ \{d\}\{abc\}^\omega\end{matrix}\right\} \text{Degree } 3\times\text{Degree } 1.$$

$$\left.\begin{matrix}\{ab\}^\lambda\{cd\}^{\omega-\lambda}, & \lambda=1, 2, \ldots, \omega-1 \\ \{ac\}^\lambda\{bd\}^{\omega-\lambda}, & \lambda=1, 2, \ldots, \omega-1 \\ \{ad\}^\lambda\{bc\}^{\omega-\lambda}, & \lambda=1, 2, \ldots, \omega-1\end{matrix}\right\} \text{Degree } 2\times\text{Degree } 2.$$

Consider the products $\{a\}\{bcd\}^{\omega}$; the covariants $\{bcd\}^{\omega}$ admit of expression in a number of ways; in the present paper we shall write any such covariant in the form $(bc)^{\omega-\lambda}(cd)^{\lambda}$, where λ may take any value from 1 to $\omega-1$. In permitting λ to take the value $\omega-1$, we are disregarding the results for degree 3, for we know by the theorem of § 1 that the covariant $(bc)(cd)^{\omega-1}$ is expressible in terms of the other products already included. [Covariants $(cd)^{\omega-\lambda}(db)^{\lambda}$, $(db)^{\omega-\lambda}(bc)^{\lambda}$ are obviously redundant.] Similarly, in writing further

$$\left.\begin{array}{l} \{b\}\{acd\}^{\omega} \text{ in the form } (cd)^{\omega-\lambda}(da)^{\lambda} \\ \{c\}\{abd\}^{\omega} \text{ in the form } (da)^{\omega-\lambda}(ab)^{\lambda} \\ \{d\}\{abc\}^{\omega} \text{ in the form } (ab)^{\omega-\lambda}(bc)^{\lambda} \end{array}\right\}, \quad \lambda = 1, 2, \ldots, \omega-1,$$

we are including four forms $(bc)(cd)^{\omega-1}$, $(cd)(da)^{\omega-1}$, $(da)(ab)^{\omega-1}$, $(ab)(bc)^{\omega-1}$, which are, by the theorem of § 1, expressible in terms of other product forms. We may omit these forms as being reducible, and the only effect will be to remove from our final result four of the syzygies; for the sake of symmetry the four forms are here retained.

The product forms we are considering are

$$(ab)^{\omega},\quad (ac)^{\omega},\quad (ad)^{\omega},\quad (bc)^{\omega},\quad (bd)^{\omega},\quad (cd)^{\omega},$$

$$\left.\begin{array}{ll} (ab)^{\omega-\lambda}(bc)^{\lambda}, & (ac)^{\omega-\lambda}(bd)^{\lambda} \\ (bc)^{\omega-\lambda}(cd)^{\lambda}, & (ab)^{\omega-\lambda}(cd)^{\lambda} \\ (cd)^{\omega-\lambda}(da)^{\lambda}, & (ad)^{\omega-\lambda}(bc)^{\lambda} \\ (da)^{\omega-\lambda}(ab)^{\lambda}, & \end{array}\right\}, \quad \lambda = 1, 2, 3, \ldots, \omega-1.$$

The number of product forms is therefore

$$6+7(\omega-1) = 7\omega-1;$$

and therefore the number of syzygies between them is, if $\omega \geqslant 5$,

$$\binom{\omega-5}{2}+7\omega-1-\binom{\omega+2}{2} = 13.$$

If $\omega \leqslant 4$, the number of syzygies is

$$S \equiv 7\omega-1-\binom{\omega+2}{2};$$

and the values of S are given by

$$\omega = 1, 2, 3, 4; \qquad S = 3, 7, 10, 12 \quad \text{respectively.}$$

[In the same way, in the general case, if $\omega \leqslant 2^{\delta-1}-\delta$, then

$$S \equiv P-\binom{\omega+\delta-2}{\delta-2}\Bigg].$$

5.

Put $\alpha \equiv (ab)c_x d_x$, $\beta \equiv (bc)d_x a_x$, $\gamma \equiv (cd)a_x b_x$, $\delta \equiv (da)b_x c_x$; so that $\alpha+\beta+\gamma+\delta \equiv 0$. Then the product forms are

$$
\begin{array}{cccccc}
\alpha^\omega, & \beta^\omega, & \gamma^\omega, & \delta^\omega, & (\alpha+\beta)^\omega, & (\beta+\gamma)^\omega; \\
\beta^{\omega-1}\gamma, & \beta^{\omega-2}\gamma^2, & \ldots, & \beta^{\omega-r}\gamma^r, & \ldots, & \beta\gamma^{\omega-1}; \\
\gamma^{\omega-1}\alpha, & \gamma^{\omega-2}\alpha^2, & \ldots, & \gamma^{\omega-r}\alpha^r, & \ldots, & \gamma\alpha^{\omega-1}; \\
\alpha^{\omega-1}\beta, & \alpha^{\omega-2}\beta^2, & \ldots, & \alpha^{\omega-r}\beta^r, & \ldots, & \alpha\beta^{\omega-1}; \\
\alpha^{\omega-1}\delta, & \alpha^{\omega-2}\delta^2, & \ldots, & \alpha^{\omega-r}\delta^r, & \ldots, & \alpha\delta^{\omega-1}; \\
\beta^{\omega-1}\delta, & \beta^{\omega-2}\delta^2, & \ldots, & \beta^{\omega-r}\delta^r, & \ldots, & \beta\delta^{\omega-1}; \\
\gamma^{\omega-1}\delta, & \gamma^{\omega-2}\delta^2, & \ldots, & \gamma^{\omega-r}\delta^r, & \ldots, & \gamma\delta^{\omega-1};
\end{array}
$$

$$
(\alpha+\beta)^{\omega-1}(\beta+\gamma),\quad (\alpha+\beta)^{\omega-2}(\beta+\gamma)^2,\quad \ldots,
$$
$$
(\alpha+\beta)^{\omega-r}(\beta+\gamma)^r,\quad \ldots,\quad (\alpha+\beta)(\beta+\gamma)^{\omega-1}.
$$

We require thirteen linear relations among these forms by virtue of the relation

$$\alpha-\beta+\gamma+\delta \equiv 0.$$

Consider first those forms which are immediately expressible in terms of others: we have

$$
(\alpha+\beta)^\omega = \sum_{i=0}^{i=\omega}\binom{\omega}{i}\alpha^{\omega-i}\beta^i, \qquad (\beta+\gamma)^\omega = \sum_{i=0}^{i=\omega}\binom{\omega}{i}\beta^{\omega-i}\gamma^i;
$$

$$
\alpha^{\omega-1}\delta = -(\alpha^\omega+\alpha^{\omega-1}\beta+\alpha^{\omega-1}\gamma),
$$
$$
\beta^{\omega-1}\delta = -(\beta^\omega+\beta^{\omega-1}\gamma+\beta^{\omega-1}\alpha),
$$
$$
\gamma^{\omega-1}\delta = -(\gamma^\omega+\gamma^{\omega-1}\alpha+\gamma^{\omega-1}\beta);
$$

$$
\begin{aligned}
(\alpha+\beta)^{\omega-1}(\beta+\gamma) &= \beta(\alpha+\beta)^{\omega-1}+(-)^{\omega-1}\gamma(\gamma+\delta)^{\omega-1}, \\
&= \sum_{i=0}^{i=\omega-1}\binom{\omega-1}{i}\alpha^i\beta^{\omega-i}+(-)^{\omega-1}\sum_{i=0}^{i=\omega-1}\binom{\omega-1}{i}\gamma^{\omega-i}\delta^i,
\end{aligned}
$$

$$
(\alpha+\beta)(\beta+\gamma)^{\omega-1} = \sum_{i=0}^{i=\omega-1}\binom{\omega-1}{i}\beta^{\omega-i}\gamma^i+(-)^{\omega-1}\sum_{i=0}^{i=\omega-1}\binom{\omega-1}{i}\alpha^{\omega-i}\delta^i;
$$

$$
\delta^\omega = -(\alpha\delta^{\omega-1}+\beta\delta^{\omega-1}+\gamma\delta^{\omega-1}).
$$

Therefore the forms $(\alpha+\beta)^\omega$, $(\beta+\gamma)^\omega$, $(\alpha+\beta)^{\omega-1}(\beta+\gamma)$, $(\alpha+\beta)(\beta+\gamma)^{\omega-1}$, $\alpha^{\omega-1}\delta$, $\beta^{\omega-1}\delta$, $\gamma^{\omega-1}\delta$, δ^ω are immediately expressible in terms of the remaining forms.

Again, if $\omega \geqslant s$, we have two further syzygies

$$\sum_{i=0}^{i=\omega} \binom{\omega}{i} (\alpha+\beta)^{\omega-i}(\beta+\gamma)^i = (\alpha+2\beta+\gamma)^\omega = (\beta-\delta)^\omega$$
$$= \sum_{i=0}^{i=\omega} (-)^i \binom{\omega}{i} \beta^{\omega-i}\delta^i,$$

$$\sum_{i=0}^{i=\omega} (-)^i \binom{\omega}{i} (\alpha+\beta)^{\omega-i}(\beta+\gamma)^i = (\alpha-\gamma)^i = \sum_{i=0}^{i=\omega} \binom{\omega}{i} \alpha^{\omega-i}\gamma^i.$$

If $\omega = 4$, only one of these syzygies is a new one and is to be regarded as expressing $(\alpha+\beta)^2(\beta+\gamma)^2$ in terms of the remaining forms; if $\omega < 4$, there are no syzygies of this kind.

Finally we have

$$(\alpha+\delta)^\omega = (-)^\omega(\beta+\gamma)^\omega, \qquad (\beta+\delta)^\omega = (-)^\omega(\gamma+\alpha)^\omega,$$
$$(\gamma+\delta)^\omega = (-)^\omega(\alpha+\beta)^\omega;$$

and these three relations, together with

$$\delta^\omega = -(\alpha\delta^{\omega-1}+\beta\delta^{\omega-1}+\gamma\delta^{\omega-1}),$$

may be taken as equations for expressing δ^ω, $\alpha\delta^{\omega-1}\beta\delta^{\omega-1}\gamma\delta^{\omega-1}$ in terms of other forms, and are therefore independent of the preceding syzygies, provided $\omega \neq 3$.

Hence, if $\omega \geqslant 5$, the thirteen syzygies are, writing down only the determinantal factors,

$$(ab)^\omega+(ab)^{\omega-1}(bc)+(ab)^{\omega-1}(cd)+(ab)^{\omega-1}(da) \equiv 0, \tag{i.}$$

$$(bc)^\omega+(bc)^{\omega-1}(cd)+(bc)^{\omega-1}(da)+(bc)^{\omega-1}(ab) \equiv 0, \tag{ii.}$$

$$(cd)^\omega+(cd)^{\omega-1}(da)+(cd)^{\omega-1}(ab)+(cd)^{\omega-1}(bc) \equiv 0, \tag{iii.}$$

$$(da)^\omega+(da)^{\omega-1}(ab)+(da)^{\omega-1}(bc)+(da)^{\omega-1}(cd) \equiv 0, \tag{iv.}$$

$$(ac)^\omega - \sum_{i=0}^{i=\omega} \binom{\omega}{i} (ab)^{\omega-i}(bc)^i \equiv 0, \tag{v.}$$

$$(bd)^\omega - \sum_{i=0}^{i=\omega} \binom{\omega}{i} (bc)^{\omega-i}(cd)^i \equiv 0, \tag{vi.}$$

$$\sum_{i=0}^{i=\omega} \binom{\omega}{i} (ab)^{\omega-i}(da)^i - (-)^\omega \sum_{i=0}^{i=\omega} \binom{\omega}{i} (bc)^{\omega-i}(cd)^i \equiv 0, \tag{vii.}$$

$$\sum_{i=0}^{i=\omega} \binom{\omega}{i} (bc)^{\omega-i}(ab)^i - (-)^\omega \sum_{i=0}^{i=\omega} \binom{\omega}{i} (cd)^{\omega-i}(da)^i \equiv 0, \tag{viii.}$$

$$\sum_{i=0}^{i=\omega} \binom{\omega}{i} (cd)^{\omega-i}(ab)^i - (-)^\omega \sum_{i=0}^{i=\omega} \binom{\omega}{i} (bc)^{\omega-i}(da)^i \equiv 0, \tag{ix.}$$

$$(ac)^{\omega-1}(bd) - \sum_{i=0}^{i=\omega-1} \binom{\omega-1}{i} (ab)^i (bc)^{\omega-i} + (-)^{\omega} \sum_{i=0}^{i=\omega-1} \binom{\omega-1}{i} (cd)^{\omega-i} (da)^i \equiv 0, \tag{x.}$$

$$(ac)(bd)^{\omega-i} - \sum_{i=0}^{i=\omega-1} \binom{\omega-1}{i} (cd)^i (bc)^{\omega-i} + (-)^{\omega} \sum_{i=0}^{i=\omega-1} \binom{\omega-1}{i} (ab)^{\omega-i} (da)^i \equiv 0, \tag{xi.}$$

$$\sum_{i=0}^{i=\omega} \binom{\omega}{i} (ac)^{\omega-i} (bd)^i - \sum_{i=0}^{i=\omega} (-)^i \binom{\omega}{i} (bc)^{\omega-i} (da)^i \equiv 0, \tag{xii.}$$

$$\sum_{i=0}^{i=\omega} (-)^i \binom{\omega}{i} (ac)^{\omega-i} (bd)^i - \sum_{i=0}^{i=\omega} (-)^i \binom{\omega}{i} (ab)^{\omega-i} (cd)^i \equiv 0. \tag{xiii}$$

If $\omega = 4$, one of the syzygies (xii.), (xiii.) is involved in the others, so that there are twelve syzygies in all;

if $\omega = 3$, both the syzygies (xii.), (xiii.) are involved in the others, and also the four syzygies (iv.), (vii.), (viii.), (ix.) are equivalent to only three, so that there are ten syzygies in all;

if $\omega = 2$, both the syzygies (xii.), (xiii.) are involved in the others, the syzygies (x.) and (xi.) are identical, and also the syzygies (vii.), (viii.), (ix.) are involved in the others, so that there are seven syzygies in all;

if $\omega = 1$, the thirteen syzygies reduce to the following:—

$$(ab)+(bc)+(cd)+(da) \equiv 0, \quad (ac)-(ab)-(bc) \equiv 0, \quad (bd)-(bc)-(cd) \equiv 0.$$

ON SPHERICAL CURVES. PART II.

By HAROLD HILTON.

[Received March 29th, 1904. — Read April 14th, 1904.]

1. In a previous paper* I discussed the properties of curves on a sphere which can be stereographically projected into plane algebraic curves. Every n-ic (curve of the n-th degree) on a sphere defined in the usual manner as a curve cut by any plane in n points is an algebraic curve, for its stereographic projection on a plane is evidently cut by any straight line in n points and is therefore algebraic.

2. In my previous paper I showed that, if a spherical n-ic has δ nodes and κ cusps, and is such that m tangent circles pass through any two points and such that τ bitangent circles and ι osculating circles cut any given circle orthogonally,

$$m = \tfrac{1}{2}n^2 - 2\delta - 3\kappa,\dagger \qquad n = m(m-1) - 2\tau - 3\iota,$$

$$\iota = \tfrac{3}{2}n(n-2) - 6\delta - 8\kappa, \qquad \kappa = 3m(m-2) - 6\tau - 8\iota,$$

$$\text{deficiency} = \tfrac{1}{4}(n-2)^2 - \delta - \kappa \nless 0.$$

These results hold when a curve has ordinary singularities of any order, provided that a k-ple point‡ with superlinear branches of orders $r, s, t, \ldots$ $[\Sigma r = k]$ is considered equivalent to $\{\tfrac{1}{2}k(k-1) - \Sigma(r-1)\}$ nodes and $\Sigma(r-1)$ cusps. If $\delta + \kappa = \tfrac{1}{4}(n-2)^2$ and (ρ, θ) are the polar coordinates of any point of the curve, $\cos\rho$ and $\tan\theta$ can be expressed as rational algebraic functions of a single parameter.

---

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 267 (1904). To save space I shall assume that the reader has access to this paper.

† The number of circles through two given points normal to the curve $= m$; the number of circles touching the curve, orthogonal to a given circle, and cutting another given circle at a given angle $= 2m$; the number of circles touching the curve and cutting two given circles at given angles $= 4m$.

‡ A multiple point of order k. Multiple points involving compound singularities (*e.g.*, cusps of the second species) are excluded.

3. Let A be any point on the sphere. Let one of the imaginary generators through A meet a spherical n-ic* in P and let the other meet it in Q. Let the other generators through P and Q meet in B. Then B will be called *a satellite of A with respect to the curve:* A is, of course, a satellite of B. Each point A has p^2 satellites ($n = 2p$), p of which are real, unless A is a k-ple point of the curve, when it has $(p-k)^2$ satellites, $(p-k)$ of which are real.

If the generators through A touch the curve, so that A is a *focus* (or if each generator through A passes through a double point of the curve), two of the real and two of the imaginary satellites of A coincide in one real *4-ple satellite of A*. If the generators at A touch the curve at A, so that A is a *nodal focus*, A coincides with its multiple satellite. If the generators through A are inflexional tangents to the curve, so that A is an *inflexional focus* (or if each generator through A is a tangent at a double point of the curve), three of the real and six of the imaginary satellites of A coincide in one real 9-ple satellite of A.

4. It is easy to find the properties of the plane curve into which a spherical curve is projected from a point A on the sphere, when the stereographic projection of the spherical curve from some other point B is known: for we obtain the projection from A by inverting the projection from B with respect to a circle whose centre is the projection of A from B, and then reflecting it in a diameter of the sphere perpendicular to AB (*loc cit.*, § 5). Thus, when we project stereographically from A, the satellites of A project into the singular foci of the plane curve (the intersections of the tangents to the plane curve at the circules). If A is a nodal focus, the projection of the spherical curve is a plane $(n-2)$-ic touching the line at infinity at each circule. If A is the 4-ple satellite of a focus B, B projects into a singular inflexional focus of the plane curve (the intersection of tangents of three-point contact at the circules). If A is a focus whose 4-ple satellite is B, B projects into the intersection of tangents at the circules to cusps of the plane curve. If A is an ordinary k-ple point of the spherical curve, the plane curve is of degree $(n-k)$ and has the projections of the k circles of curvature at A as asymptotes.† If A is a cusp of the first species, the plane curve touches the line at infinity. If A is a cusp of the second species, the plane curve has a cusp at infinity, &c., &c.

* I shall assume that the spherical curve and vertex of projection are both real unless the contrary is stated. The stereographic projection is then a real algebraic plane curve.

† It follows that $2k \not> n$.

5. The following theorems will illustrate properties of spherical curves :—

(1) Given four fixed points A, B, C, D on a sphere ; all spherical 4-ics through A, C, D having B for a focus whose satellite is A pass through a fifth fixed point and have four-point contact with the osculating circle at A.

(2) If A is a point on a spherical 4-ic, the generators through A and the osculating circle at A meet the curve at four concyclic points (including A).

(3) The osculating circle at a fixed point B of a spherical 4-ic meets the curve again in A ; and on any circle meeting the curve in A, H, B, K a point P is taken such that the pencil $A(BP, HK)$ is harmonic : then the locus of P is a circle through A.

(4) If two points A, B on a spherical 4-ic are satellites of each other, the points of contact of the two circles through A and B which touch the curve lie on two circles, one touching the 4-ic at A and the other at B.

(5) A spherical 4-ic has a cusp C and a focus A. If B is the 4-ple satellite of A, the circle ABC touches the 4-ic at C.

(6) Given in a cuspidal spherical 4-ic the osculating circle at a fixed point A and the satellite B of A, the locus of the cusp is a spherical 4-ic touching the given circle and having a focus at B and a node at A. Each branch of the locus at A cuts the given circle at an angle of 30°.

(7) If one of the four nodes A of a spherical 6-ic is the 4-ple satellite of a focus, the points of contact of the four osculating circles through A are concyclic.

(8) If the node A of a spherical 6-ic is the 4-ple satellite of a focus B, the twenty-two osculating circles through A all touch a curve with four real foci, B being one.

(9) A spherical 6-ic has a node, a nodal focus A, and two cusps H, K; if the osculating circles at L and M pass through A, the circles AHK, ALM touch.

(10) If a spherical 6-ic has three nodes and a nodal focus A, the six points of contact of the three bitangent circles through A are concyclic.

The above theorems may be proved by projecting stereographically from A.

6. If P is any point on a spherical $2p$-ic and A is any fixed point, then

$$\sin^2 \tfrac{1}{2}PB_1 \sin^2 \tfrac{1}{2}PB_2 \ldots \sin^2 \tfrac{1}{2}PB_p,$$

$$\sin^2 \tfrac{1}{2}PC_1 \sin^2 \tfrac{1}{2}PC_2 \ldots \sin^2 \tfrac{1}{2}PC_{p-1} \sin^2 \tfrac{1}{2}PA,$$

$$\sin^2 \tfrac{1}{2}PD_1 \sin^2 \tfrac{1}{2}PD_2 \ldots \sin^2 \tfrac{1}{2}PD_{p-2} \sin^4 \tfrac{1}{2}PA,$$

$$\ldots,$$

$$\sin^{2p} \tfrac{1}{2}PA$$

are connected by a homogeneous linear relation, where the B's are the p real satellites of A and the C's, D's, ... are also fixed when A is fixed. Conversely, if the quantities are connected by a homogeneous linear relation when A and the B's, C's, D's, ... are fixed, the locus of P is a spherical $2p$-ic. The theorem leads to simple results in certain cases, *e.g.*:—

B_1, B_2 are the real satellites of A with respect to a spherical 4-ic; A, C_1 are the real satellites of B_1 and A, C_2 of B_2. If a circle orthogonal to the circles AB_1C_1, AB_2C_2 cuts the curve in two real points P_1 and P_2, then

$$\sin \tfrac{1}{2}P_1B_1 \sin \tfrac{1}{2}P_1B_2 \operatorname{cosec}^2 \tfrac{1}{2}P_1A = \sin \tfrac{1}{2}P_2B_1 \sin \tfrac{1}{2}P_2B_2 \operatorname{cosec}^2 \tfrac{1}{2}P_2A.$$

If A is a focus of the 4-ic, B_1 and B_2 coincide (at B say). We have then

$$\sin \tfrac{1}{2}P_1B \operatorname{cosec} \tfrac{1}{2}P_1A = \sin \tfrac{1}{2}P_2B \operatorname{cosec} \tfrac{1}{2}P_2A,$$

where P_1 and P_2 are two real intersections of the curve with any circle orthogonal to all circles through A and B.

7. Two points P, Q on a sphere are inverse with respect to a circle j when the line PQ passes through the pole of the plane of j with respect to the sphere. The locus of Q is the inverse of the locus of P with respect to j. The values of n, m, δ, κ, τ, ι and the deficiency are the same for two inverse curves. The inverses of the satellites of a point are the satellites of the inverse point with respect to the inverse curve. The inverse of a focus is a focus of the inverse curve, &c.

8. Let O, O' be two diametrically opposite points on a sphere, and let P be any point of a spherical n-ic. The envelope of circles on OP as diameter is the first (positive) *pedal* of the curve with respect to O, and the first *negative pedal* with respect to O'. The second (positive) pedal is the first pedal of the first pedal, the third pedal is the first pedal of the second pedal, and so on; similarly for the negative pedals. The p-th pedal of the q-th pedal is the $(p+q)$-th pedal, p and q being any positive or negative integers (the 0-th pedal being the curve itself). The r-th pedal with respect to O is the r-th negative pedal with respect to O'. Properties

of the pedals may be proved by projecting stereographically from O or O'.

If n_r, m_r, ... are the quantities corresponding to n, m, ... for the r-th pedal with respect to O, when O and O' have no special relation towards the given curve, then

$$\begin{array}{llll} n_1 = 2m, & n_2 = 4m, & \ldots, & n_r = 2rm; \\ m_1 = n+2m, & m_2 = 5m, & \ldots, & m_r = (3r-1)m; \\ \kappa_1 = \iota, & \kappa_2 = \iota+n+m, & \ldots, & \kappa_r = \iota+n+(r-1)m; \\ \iota_1 = \iota+3n, & \iota_2 = \iota+n+4m, & \ldots, & \iota_r = \iota+n+4(r-1)m;^* \end{array}$$

the deficiency of all pedals is the same as the deficiency of the curve; O is an rm-ple point of the r-th pedal at which there are m superlinear branches of order r; each of the generators through O' touches the first pedal at $\frac{1}{2}n$ places, has two-point contact with the second pedal at $(m-n)$ places and three-point contact at $\frac{1}{2}n$ places, has $(r-1)$-point contact with the r-th pedal at $\frac{1}{2}n$ places, r-point contact at $(m-n)$ places, and $(r+1)$-point contact at $\frac{1}{2}n$ places.†

If O or O' has a special relation towards the given curve, these results must be modified. Thus, if O is a focus of the curve, it is an $(m-2)$-ple point of the first pedal, which is only of degree $(2m-2)$;‡ if, in addition, O' is the 4-ple satellite of O,§ the same is true of all the pedals. If O is an inflexional focus of the curve, it is an ordinary focus of the first pedal. If O is the 4-ple satellite of a focus of the given curve, O' is the 4-ple satellite of a focus with respect to the first pedal. If O' is a focus of the given curve, it is an inflexional focus of the first pedal.‖ If O is a nodal focus of the curve, it is a nodal focus of the first pedal which has $(m-4)$ other branches through O and is of degree $(2m-2)$.

The properties of the negative pedals may be obtained by interchanging O and O' in the above.¶

---

* If a generator through O' cuts a circle j in P, the pair of generators through P counts as $(r-2)m$ of the ι_r circles osculating the r-th pedal and cutting j orthogonally.

† The first pedal has $\frac{1}{4}n^2$ foci besides O', but no other positive pedal has a focus other than the multiple focus O'.

‡ If the generators through O each touch the curve at k places, O is an $(m-2k)$-ple point of the first pedal which is of degree $2(m-k)$.

§ Or if O' is the focus and O its 4-ple satellite.

‖ If the generators through O′ have each r-point contact with the curve, they have $(r+1)$-point contact with the first pedal.

¶ These theorems hold when we take O and O' not diametrically opposite, if, in the definition of a pedal, we substitute for "the circle on OP as diameter" "the circle through O and P cutting the circle OPO' orthogonally."

9. The *evolute of a spherical curve with respect to a circle j* is the locus of the limiting points* of j and the osculating circles of the curve, *i.e.*, the envelope of circles orthogonal to the curve and to j.† It is anallagmatic (its own inverse) with respect to j. Let n', m', ... be the quantities for the evolute corresponding to n, m, ... Then $n' = 2\iota$; for any circle s orthogonal to j cuts the evolute only at the limiting points of j and the ι osculating circles orthogonal to s; the evolute meets j at the 2ι points of contact of j and osculating circles. Again, $m' = 2m+2\iota$;‡ for, taking any two points H, H' inverse with respect to j, we see that the circles through H, H' touching the evolute are the m circles through H, H' normal to the given curve (each of which has *double* contact with the evolute) and the 2ι intersections of the evolute with j.

Again, $\iota' = 2n+6\iota$, for the only osculating circles of the evolute orthogonal to j are those whose points of contact are at the intersections of the curve or of the evolute with j; for otherwise we should have two consecutive circles normal to the curve and j coinciding, which is not possible in general. Now, it may be readily shown that each intersection of the given curve with j is an inflexional focus of the evolute; and at each of the 2ι points where the evolute cuts j we have three consecutive osculating circles of the evolute orthogonal to j (*loc. cit.*, p. 270, § 14).

Again, $\kappa' = 3n'-3m'+\iota' = 2(3\iota-3m+n)$, showing that the number of osculating circles of four-point contact of a spherical n-ic is $(3\iota-3m+n)$ in general.

The above results must be modified if the given curve is anallagmatic with respect to j; see § 13.

Every focus of a curve is a focus of its evolute; for, if a generating line touches the curve at P and meets j at Q, the generator-pair through Q is a circle cutting j orthogonally at Q and the curve orthogonally at P.

10. Let V be the vertex of a given cone and j the circle in which the polar plane of V meets the sphere. The envelope of the polar planes with respect to the sphere of the points in which the given cone meets the plane of j is a cone with vertex V which is the *reciprocal of the given cone with respect to the sphere.* The reciprocal cones cut the plane of j in two curves which are polar reciprocals of each other with respect to j, and cut the sphere in two curves both anallagmatic with respect to j which are

* The limiting points of two circles are the two points through which pass all circles cutting the given circles orthogonally; *i.e.*, the points in which the sphere meets the line conjugate to the intersection of the planes of the circles.

† If the circle j is not mentioned, it is assumed to be the circle at infinity.

‡ Not m, as erroneously stated, *loc. cit.* p. 272, § 21.

called *reciprocal spherical curves.* The reciprocal of a circle whose plane passes through V is a pair of points collinear with V, and *vice versa.* The reciprocal of a pair of nodes on a curve is a circle orthogonal to j and touching the reciprocal curve in four places; the reciprocal of a pair of cusps is a circle orthogonal to j and osculating the reciprocal curve in two places. If N, M, Δ, K, T, I correspond in the reciprocal curve to n, m, δ, κ, τ, ι in the given curve, then

$$N = (m-n), \quad M = m, \quad \Delta = \delta + \tfrac{1}{4}(m-2n)(m-18),$$

$$\mathrm{K} = \kappa + 3(m-2n), \quad \mathrm{T} = \tau - \tfrac{1}{2}(m-2n), \quad \mathrm{I} = \iota,$$

$$\tfrac{1}{2}(N-2)^2 - (\Delta + \mathrm{K}) = \tfrac{1}{4}(n-2)^2 - (\delta+\kappa) + \tfrac{1}{2}(m-2n).*$$

If we are given a property of a certain type of anallagmatic spherical curve (the type being defined by the quantities n, m, ...), we can deduce by reciprocation a property of another type of anallagmatic spherical curve. If $m = 2n$, we have

$$n = N, \quad m = M, \quad \delta = \Delta, \quad \kappa = \mathrm{K}, \quad \tau = \mathrm{T}, \quad \iota = \mathrm{I};$$

so that all properties of an anallagmatic spherical curve for which $m = 2n$ may be duplicated.†

11. The $\frac{1}{2}n(n-1)$ circles each of which touches a spherical n-ic anallagmatic with respect to j at two of the n points where it meets j are called the *asymptotic arcs* of the curve. The pairs of points reciprocal to the asymptotic arcs are $n(n-1) = (M-N)^2 - (M-N)$ foci of the reciprocal curve, the remaining $(M-N)$ foci being the n intersections of j with the given curve. For let $P_1, P_2, \ldots, P_n$ be these intersections; then the tangent planes at $P_1, P_2, \ldots, P_n$ touch the cone reciprocal to the cone on which the given curve lies, so that $P_1, P_2, \ldots, P_n$ are foci of the reciprocal curve (*loc. cit.*, p. 275, line 12). Again, the generators through P_1 and P_2 meet again in a pair of foci of the reciprocal curve, the pair being the reciprocal of a circle cutting j orthogonally at P_1 and P_2. Therefore the generators through $P_1, P_2, \ldots, P_n$ meet again in $n(n-1)$ foci of the reciprocal curve which are reciprocal in pairs to the asymptotic arcs of the given curve.‡

* From these equations we can, of course, deduce

$$n = (M-N), \quad \delta = \Delta + \tfrac{1}{4}(M-2N)(M-18), \quad \ldots.$$

† In the same way that Pascal's theorem for a conic is duplicated into Pascal's and Brianchon's theorems. The condition for a plane curve corresponding to $m = 2n$ is class = degree.

‡ It should be noted that the foci of a curve anallagmatic with respect to j are completely determined when we know the foci lying on j. In general the foci of any curve with r^2 foci are completely determined when we know r foci no two of which lie on the same generator.

12. We may prove properties of a spherical curve anallagmatic with respect to a circle j and lying on a cone whose vertex is V by projecting it stereographically into the intersection of a sphere with a cylinder or a cone whose vertex is the centre of the sphere (*loc. cit.*, §§ 30–35). A better method, however, is to project the curve from V on to the polar plane of V with respect to the sphere (the plane of j). A circle on the sphere (not orthogonal to j) projects into a conic having double contact with j. The generator-pair through a point P on j projects into the tangent at P to j.* A circle orthogonal to j and the reciprocal pair of points project into a straight line and its pole with respect to j. Two circles orthogonal to j and each other project into two lines conjugate with respect to j.

For example:—The envelope of a chord of a conic h whose extremities are conjugate with respect to a fixed circle j is a conic touching the tangents to h and j at their points of intersection; and, reciprocally, the locus of the intersection of two tangents to h which are conjugate with respect to j is a conic passing through the points of contact with h and j of their common tangents. Hence:—If a spherical 4-ic h is anallagmatic with respect to a circle j, the envelope of a circle orthogonal to j cutting h in four points subtending a harmonic pencil at any point of the circle is a 4-ic anallagmatic with respect to j having the intersections of h and j for foci; and, reciprocally, the locus of the intersection of two circles touching h and orthogonal to j and each other is a 4-ic anallagmatic with respect to j which meets j at the four foci of h lying on j. If V lies within the sphere, we may by stereographic projection reduce these two theorems to the well known theorems:—The envelope of a quadrant chord of a sphero-conic is a sphero-conic confocal with the reciprocal of the given sphero-conic; and, reciprocally, the locus of the intersection of two orthogonal tangents to a sphero-conic is a sphero-conic whose reciprocal is confocal with the given sphero-conic.

Again:—If a plane cuspidal 3-ic touches three given straight lines at three given collinear points, the cusp lies on a fixed conic touching the three fixed lines; and, reciprocally, if the 3-ic touches three given concurrent lines at three given points, its inflexional tangent touches a fixed conic through the given points. Hence:—A spherical 6-ic has two fixed nodes A, B lying on a circle j with respect to which it is anallagmatic and

* This gives a simple proof of the theorem "Four generators of a conicoid of the same system cut all generators of the opposite system in a constant cross-ratio," and of the allied theorems.

has two cusps : if it has a fixed guiding arc,* the cusps lie on a fixed 4-ic anallagmatic with respect to j which touches the given guiding arc at two points and has A, B for foci ; if it touches at a fixed point P a fixed circle with respect to which A and B are inverse points, the circle orthogonal to j and osculating the curve in two places touches a fixed 4-ic anallagmatic with respect to j passing through A, B, and P.

Again :—A spherical 6-ic with two nodes is anallagmatic with respect to a circle j, has fixed asymptotic arcs which meet at A, B, and C, and passes through a pair of fixed points P, P' on the asymptotic arc BC. The locus of the nodes is a spherical 6-ic through A anallagmatic with respect to j whose asymptotic arcs are AB, AC, and a circle whose plane contains the line conjugate to PP'. The tangents at A to the curve and the circles through A orthogonal to j and passing through P, B, and C form a harmonic pencil.

This may be proved as before ; or, if j is imaginary, we may reduce the curve to a sphero-3-ic lying on the cone

$$(x^2+y^2+z^2)(\lambda y+\mu z)+z(ax^2+by^2+cz^2+2fyz+2gzx+2hxy)=0,$$

where the expression in the last pair of brackets breaks up into two linear factors ; and it is readily seen (by considering the intersection of the cone with the plane $z=1$) that the nodal line of this cone lies on the cone

$$(x^2+y^2+z^2)(ax+hy+gz)=x(ax^2+by^2+cz^2+2fyz+2gzx+2hxy).$$

Again :—If a fixed circle is touched by a circle cutting orthogonally in P and Q a fixed circle j through two fixed points A and B, the circles cutting j orthogonally at A and P, B and Q meet again on a fixed 4-ic anallagmatic with respect to j.

Again :—Any spherical n-ic anallagmatic with respect to j meets j at $P_1, P_2, \ldots, P_n$. If the asymptotic arcs through P_1 and P_2, P_2 and P_3, ..., P_{n-1} and P_n pass through fixed coplanar points, then the asymptotic arc through P_n and P_1 passes through two fixed points or touches two fixed circles according as n is even or odd.

13. Let a spherical curve be anallagmatic with respect to a circle j and lie on a cone whose vertex is V, and let the quantities corresponding to n, m, ... for its evolute with respect to j be denoted by n', m', ... ; then

* The guiding arc is the circle through the six points not lying on j in which the asymptotic arcs meet the curve. If the curve passes through the intersection of two asymptotic arcs (or if, as in the present case, two of the asymptotic arcs are replaced by a circle through A and B orthogonal to j), the guiding arc touches the curve (*loc. cit.*, p. 280, § 35).

n', m', ... have no longer the values obtained in § 9. The projections from V on the plane of j of circles normal to the curve and orthogonal to j are straight lines joining points on the projection of the curve to the corresponding points on the reciprocal polar of the projection with respect to j. The projection of the evolute is the envelope of these straight lines. Hence we obtain

$$n' = \iota, \qquad m' = \iota + m, \qquad \iota' = 3\iota$$

(for the projection of the evolute from V on the plane of j has no inflexions). Therefore

$$\kappa' = 3n' - 3m' + \iota' = 3(\iota - m);$$

and we see, as before, that the number of osculating circles* of the given curve which have four-point contact is $(3\iota - 3m + n)$. Two reciprocal curves have the same evolute with respect to j.

14. In the same way that we deduce properties of spherical curves from those of plane curves either by stereographic projection or by projection from any point on to the polar plane of the point, so we may deduce properties of plane curves from those of spherical curves. We may add to the examples given in the previous paper:—

The number of conics through two fixed points having double contact with a fixed conic j and touching a plane curve whose Plücker's numbers are n, m, δ, κ, τ, ι is $2(m+n)$; the number of conics through one fixed point having double contact with j and osculating the curve is

$$(6n + 2\iota) = (6m + 2\kappa);$$

and the number of conics having double contact with j and four-point contact with the curve is

$$3(2n - m + \iota) = 3(2m - n + \kappa).\dagger$$

These theorems are proved by projecting j into a circle, passing a sphere through the circle, and then projecting on to the sphere from the pole of the plane of the circle.

15. As before pointed out (*loc. cit.*, §§ 36, 37), to many of the theorems proved for the sphere correspond theorems for a conicoid. The definitions of "focus," "satellite," &c., still hold good, though, if the conicoid has

* n of these circles touch the curve at its intersections with j.

† If the fixed conic is a pair of points, the number of conics is $(m + 2n)$, $(3n + \iota)$, $(6m - 4n + 3\kappa)$ respectively; if the fixed conic is a pair of straight lines, the number of conics is $4(m + n)$, $(6n + 2\iota)$, $(6n - 4m + 3\iota)$ respectively.

real generators, we can no longer draw conclusions as to how many foci or satellites are real. Every *real* algebraic spherical curve (which is projected stereographically into a *real* algebraic plane curve) is necessarily of even degree, but this is not true of a real algebraic curve on a conicoid with real generators. As examples of properties of curves of a conicoid we may take:—

> The intersection of the osculating planes of a conicoidal 3-ic at P and Q meets the conicoid at A. If the plane APQ meets the curve again at R, the osculating plane at R also passes through A.

One of the four nodes of a conicoidal 6-ic is the 4-ple satellite of a focus; if the osculating planes at four points of the curve pass through that node, the four points are coplanar.

In conclusion I have to express my thanks to Prof. E. B. Elliott and a referee of this paper for their kind help.

AUTHORS of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. d.	s. d.	s. d.
4 pages	Free	3 6	5 0	6 0
8 "	"	4 6	6 6	8 0
12 "	"	7 0	10 6	13 0
16 "	"	8 0	12 0	15 0
32 "	"	14 0	21 0	26 0

It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1s. 6d. per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1s. 4d. per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. FRANCIS HODGSON, 89 Farringdon Street, E.C., at the following rates:—

Vol.	I.	Nos.	1–8	... Jan. 1865	to Nov. 1866	...	10s.
"	II.	"	9–20	... Nov. 1866	" 1869	...	16s.
"	III.	"	21–40	... " 1869	" 1871	...	20s.
"	IV.	"	41–66	... " 1871	" 1873	...	31s. 6d.
"	V.	"	67–75	... " 1873	" 1874	...	15s.
"	VI.	"	76–88	... " 1874	" 1875	...	21s.
"	VII.	"	89–103	... " 1875	" 1876	...	21s.
"	VIII.	"	104–123	... " 1876	" 1877	...	21s.
"	IX.	"	124–140	... " 1877	" 1878	...	21s.
"	X.	"	141–155	... " 1878	" 1879	...	18s.
"	XI.	"	156–165	... " 1879	" 1880	...	12s. 6d.
"	XII.	"	166–179	... " 1880	" 1881	...	16s.
"	XIII.	"	180–194	... " 1881	" 1882	...	18s.
"	XIV.	"	195–214	... " 1882	" 1883	...	20s.
"	XV.	"	215–232	... " 1883	" 1884	...	20s.
"	XVI.	"	233–252	... " 1884	" 1885	...	20s.
"	XVII.	"	253–279	... " 1885	" 1886	...	25s.
"	XVIII.	"	280–304	... " 1886	" 1887	...	25s.
"	XIX.	"	305–342	... " 1887	" 1888	...	30s.
"	XX.	"	343–369	... " 1888	" 1889	...	25s.
"	XXI.	"	370–398	... " 1889	" 1890	...	25s.
"	XXII.	"	399–429	... " 1890	" 1891	...	27s. 6d.
"	XXIII.	"	430–449	... " 1891	" 1892	..	20s.
"	XXIV.	"	450–474	... " 1892	" 1893	...	22s.
"	XXV.	"	475–499	... " 1893	" 1894	...	22s.
"	XXVI.	"	500–534	... " 1894	" 1895	...	30s.
"	XXVII.	"	535–574	... " 1895	" 1896	...	35s.
"	XXVIII.	"	575–611	... " 1896	" 1897	...	32s. 6d.
"	XXIX.	"	612–657	... " 1897	" 1898		Pt. I. 20s. Pt. II. 20s.
"	XXX.	"	658–678	... " 1898	to Mar. 1899	...	20s.
"	XXXI.	"	679–709	... April 1899	" Dec. 1899	...	20s.
"	XXXII.	"	710–737	... Jan. 1900	" June 1900	...	20s.
"	XXXIII.	"	738–763	... Nov. 1900	" Feb. 1901	...	20s.
"	XXXIV.	"	764–789	... Mar. 1901	" April 1902	...	20s.
"	XXXV.	"	790–819	... May 1902	" Jan. 1903	...	20s.

Cases for binding the above Volumes can also be procured from the Publisher, price 1s. each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series will be 25s. each Volume.

The price of cases for binding the Volumes of the Second Series will be 1s. 3d.

Vol. 1 of Series 2 (1904) has been published.

ɩ

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 2.—PART 3.

CONTENTS.

Issued September 17, 1904.

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. LOVE. / W. BURNSIDE.

34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

The Croft, Bromley Road, Catford, S.E.—W. BURNSIDE.

SOME EXTENSIONS TO MULTIPLE SERIES OF ABEL'S THEOREM ON THE CONTINUITY OF POWER SERIES

By T. J. I'A. BROMWICH *and* G. H. HARDY.*

[Received March 23rd, 1904.]

1.

The object of this paper is to investigate certain extensions to multiple and repeated series of the following well-known theorem due to Abel:—

If the series

$$(1) \qquad a_0+a_1+a_2+\ldots$$

is convergent, the series

$$(2) \qquad a_0+a_1x+a_2x^2+\ldots$$

is absolutely convergent for all values of x whose modulus is less than unity, and if $f(x)$ denotes the function represented by the series (2), the limit of $f(x)$ when x approaches 1 along the straight line (0, 1) is equal to the sum of the series (1).†

Notation and Terminology.

It will be found essential in dealing with these questions to lay down as definite and concise a notation and as unambiguous a terminology as is possible, since those usually employed are in some ways misleading.

Suppose that

$$s_{m_1, m_2, \ldots, m_n} = \sum_{i_1=0}^{i_1=m_1} \sum_{i_2=0}^{i_2=m_2} \ldots \sum_{i_n=0}^{i_n=m_n} a_{i_1, i_2, \ldots, i_n};$$

then we denote by

$$\sum_{(1, 2, \ldots, p)(p+1, \ldots, q) \ldots (r+1, \ldots, n)} a$$

* Mr. Hardy communicated his share of the paper on February 11th, 1904, and discovered shortly afterwards that Prof. Bromwich had at an earlier date arrived independently at the results of §§ 1–5. § 6 and §§ 12–17 are due more particularly to Mr. Hardy, and §§ 7–11 to Prof. Bromwich. Some of the earlier results (those relating to double series summed by rows or columns) were also obtained by Mr. A. Brown, to whom the subject had been suggested by Prof. Bromwich for a dissertation. As regards the latter part of the paper, each of the authors had arrived by conjecture at the other's results, but had not worked out formal proofs at the time when it was decided to unite them in one paper.

† The theorem is still true if x approaches 1 by any path (in the complex-plane) which does not touch the circle of convergence; but it is not with extensions of this kind that we shall be concerned now.

the result (if it be determinate) of making the suffixes $m_1, m_2, \ldots, m_n$ tend to infinity in groups, the group $m_{r+1}, \ldots, m_n$ being made first to tend *simultaneously* to infinity, and so on, the groups corresponding to the brackets written under the sign of summation. Thus, to take the simplest case—that of two integral parameters i_1, i_2—the expressions

$$\sum_{(1,2)} a, \quad \sum_{(1)(2)} a, \quad \sum_{(2)(1)} a$$

denote respectively the double series

$$\Sigma a_{i_1, i_2}$$

in Pringsheim's sense, and the two repeated series in which the sum is effected with respect to one parameter first. A similar notation will be used for limits. Thus,

$$\sum_{(1,2)} a = \lim_{(1,2)} s, \qquad \sum_{(1)(2)} a = \lim_{(1)(2)} s.$$

Where there is more than one bracket the operation of proceeding to the limit which corresponds to the bracket on the *right* is always to be performed first. The same notation applies to limits of functions of continuous variables. Thus, if $f(x_1, x_2)$ is a function of x_1 and x_2, both of these being positive and less than 1, $\lim\limits_{(1)(2)} f$ means $\lim\limits_{x_1=1} (\lim\limits_{x_2=1} f)$ and $\lim\limits_{(1,2)} f$ means the *double* limit $\lim\limits_{x_1=1, x_2=1} f$.

It is always to be understood that the limits of summation, unless the contrary is expressly stated, are zero and infinity, and the limiting value of every variable, which we shall always assume to be real and positive,* unless the contrary is expressly stated, is 1, and the term "double limit" will be used always as indicating that two variables (integral or continuous) are made to tend *simultaneously* to their limiting values. When there are several distinct passages to the limit the result is a *repeated* limit; thus,

$$\lim_{(1,2)(3,4)}$$

would denote a repeated limit—in this case the double limit of a double limit.

The expression $$\sum_{(1)} a$$

denotes the result of summing with respect to i_1 *only*, and so on. Also,

---

* There is, of course, no such limitation on the value of a.

if b depends on $i_1, \ldots, i_n$,

$$\underset{(1)}{\Delta} b = b_{i_1+1, i_2, \ldots, i_n} - b_{i_1, i_2, \ldots, i_n},$$

$$\underset{(1, 2)}{\Delta} b = \underset{(2, 1)}{\Delta} b = \underset{(2)}{\Delta} \underset{(1)}{\Delta} b$$

$$= b_{i_1+1, i_2+1, i_3, \ldots, i_n} - b_{i_1, i_2+1, i_3, \ldots, i_n} - b_{i_1+1, i_2, i_3, \ldots, i_n} + b_{i_1, i_2, i_3, \ldots, i_n},$$

and so on.

Finally, all this notation may be generalised to denote, not limits, but maximum and minimum limits;* thus,

$$\underset{(\overline{1})(\underline{2})}{\Sigma} a$$

denotes the maximum limit for $i_1 = \infty$ of the minimum limit of s_{i_1, i_2} for $i_2 = \infty$, and

$$\underset{(1, 2)}{\Sigma} a$$

denotes the maximum limit of s_{i_1, i_2} when i_1 and i_2 tend together to infinity. And, again, exactly the same applies to such expressions as

$$\underset{(\overline{1})(\underline{2})}{\lim} f.$$

2. *Statement of the Analogue of Abel's Theorem for the General Series.*

If the simple series Σa_i is convergent, there is certainly a constant C, such that

$$| s_i | < C$$

for all values of i. We express this by saying that such a convergent series necessarily satisfies the *condition of finitude.* The same is not true for multiple series. This being so, we cannot affirm that, if, *e.g.*,

$$\underset{(1, 2, \ldots, n)}{\Sigma} a$$

is convergent, then

$$\underset{(1, 2, \ldots, n)}{\Sigma} a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

converges for values of $x_1, x_2, \ldots, x_n$ less than 1, and it is easy to see by examples that this is not necessarily the case.†

It is therefore essential to subject our series to some condition beyond that of mere convergence. We shall assume that it *does* satisfy the "condition of finitude," that is to say, that

$$(3) \qquad | s_{m_1, m_2, \ldots, m_n} | < C$$

* Sometimes called "upper and lower limits of indetermination."

† For instance, compare § 3, end.

for all values of $m_1, m_2, \ldots, m_n$. Doubtless this condition is unnecessarily narrow, but it is simple and fulfils all requirements.

The analogue of Abel's theorem is then as follows:—*If the condition of finitude is satisfied, and*

(4) $$\sum_{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)} a$$

is convergent, then

(5) $$\Sigma\, a x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$$

is absolutely convergent for all values of $x_1, \ldots, x_n$ *whose moduli are less than* 1, *and if* $f(x_1, \ldots, x_n)$ *is the function represented by this series, then*

(6) $$\lim_{(1,2,\ldots,p)(p+1,\ldots,q)\ldots(r+1,\ldots,n)} f$$

is determinate and equal to the sum of the series (4).

We shall prove this theorem first for double series and give some illustrations in which the series Σa has different sums when summed in different ways, so that f has different limits when we proceed to the limit in different ways.* We shall then consider some further extensions of a different kind connected with double series. Finally, we shall establish the general theorem by induction. In dealing with double series we shall use i, j, x, y for i_1, i_2, x_1, x_2 in order to avoid suffixes, and we shall write $\sum_{(i)(j)}, \sum_{(j)(i)}, \sum_{(i,j)}, \lim_{(x)(y)}, \lim_{(y)(x)}, \lim_{(x,y)}$ for $\sum_{(1)(2)}, \ldots$.

3. *Double Series.*

Since $$a_{i,j} = \Delta_{(i,j)}\, s_{i-1,j-1}$$

and $$|s_{m,n}| < C,$$

it follows that

(7) $$|a_{i,j}| < 4C,$$

and hence that $$\Sigma a_{i,j} x^i y^j$$

is absolutely convergent. Let $f(x, y)$ denote its sum. Then

(8) $$f(x, y) = \Sigma s_{i,j} (1-x)(1-y) x^i y^j,$$

* This course seems best because this simple case affords the clearest illustration of the ideas on which our extensions of Abel's theorem are based, and its treatment does not involve the algebraical difficulties which occur in proving the more general theorems.

as is at once evident if we compare the coefficients and use condition (7).*

Now to say that $\sum_{(i,j)} a$ is convergent is the same as to say that there is a quantity s such that, however small be σ, we can determine M and N so that

$$|s_{m,n}-s| < \sigma,$$

if only $m \geqslant M$ and $n \geqslant N$. It is evident, moreover, that $|s| \leqslant C$.

Now, since

$$\Sigma(1-x)(1-y)x^i y^j = 1,$$

it follows that $\quad f(x, y)-s = \Sigma(s_{i,j}-s)(1-x)(1-y)x^i y^j$

and $|f(x,y)-s| \leqslant \left|\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\right| + \left|\sum_{i=0}^{M-1}\sum_{j=N}^{\infty}\right| + \left|\sum_{j=0}^{N-1}\sum_{i=M}^{\infty}\right| + \left|\sum_{i=M}^{\infty}\sum_{j=N}^{\infty}\right|.$

But

$$\left|\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\right| < 2CMN(1-x)(1-y),$$

since $x<1$, $y<1$, and $|s_{i,j}-s| < 2C$; also

$$\left|\sum_{i=0}^{M-1}\sum_{j=N}^{\infty}\right| < 2CM(1-x)\sum_{j=0} y^j(1-y) < 2CM(1-x),$$

$$\left|\sum_{j=0}^{N-1}\sum_{i=M}^{\infty}\right| < 2CN(1-y),$$

and

$$\left|\sum_{i=M}^{\infty}\sum_{j=N}^{\infty}\right| < \sigma\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j(1-x)(1-y) < \sigma.$$

Thus

$$|f(x,y)-s| < 2CMN(1-x)(1-y)+2CM(1-x)+2CN(1-y)+\sigma.$$

But when σ has been fixed M and N are fixed, and we can determine δ, ϵ, so that

$$|f(x,y)-s| < 2\sigma,$$

if $1-x<\delta$, $1-y<\epsilon$. Therefore

$$\lim_{(x,y)} f = s.$$

---

* The transformation

$$a_0+a_1x+a_2x^2 = \ldots = (1-x)(s_0+s_1x+s_2x^2+\ldots)$$

was given by Dirichlet and used as the basis of a proof of Abel's theorem identical in principle with the proof stated here of the corresponding theorem for double series, though (at any rate in the form in which he presents it) less simple than Abel's original proof. See Abel, *Œuvres*, Vol. I., p. 223; Dirichlet, *Werke*, Vol. II., p. 305; Pringsheim, *Münch. Ber.*, 1897, p. 344.

We may remark in passing that a similar proof applies to the general case when it is the convergence of the multiple series

$$\sum_{(1, 2, \ldots, n)} a$$

which is given. The real difficulties begin when *repeated* limits are introduced.

We may further remark that the necessity of some such limitation as is implied by the condition of finitude becomes apparent when we consider that, for example, the double series defined by the scheme

$$\begin{array}{ccccc} a_0+b_0, & a_1-b_0, & a_2, & a_3, & \ldots, \\ -a_0+b_1, & -a_1-b_1, & -a_2, & -a_3, & \ldots, \\ b_2, & -b_2, & 0, & 0, & \ldots, \\ b_3, & -b_3, & 0, & 0, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{array}$$

is convergent and has the sum 0 *whatever* be the quantities a, b; even if $a_\nu = b_\nu = \nu!$, in which case $\Sigma a_{i,j} x^i y^j$ is not convergent for any values of x and y except $x = 0$, $y = 0$ and $x = 1$, $y = 1$. If $a_\nu = b_\nu = 2^\nu$, the series is convergent and equal to $(1-y)/(1-2x)+(1-x)/(1-2y)$ if x and y are both $< \frac{1}{2}$, but divergent if $\frac{1}{2} \leqslant x < 1$ or $\frac{1}{2} \leqslant y < 1$.

4. *Repeated (Two-fold) Series.*

Now let us suppose that Σa is convergent when summed by *columns*, thus implying the convergence of every column, and that

$$\sum_{(i)(j)} a = s.$$

The series is of course absolutely convergent as before, in virtue of the condition of finitude. To illustrate the necessity of some such condition in this case we might suppose $a_{i,j}$ given by the scheme

$$\begin{array}{ccccc} 1, & 2, & 4, & 8, & \ldots, \\ -\frac{1}{2}, & -1, & -2, & -4, & \ldots, \\ -\frac{1}{4}, & -\frac{1}{2}, & -1, & -2, & \ldots, \\ -\frac{1}{8}, & -\frac{1}{4}, & -\frac{1}{2}, & -1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \end{array}$$

Then $\sum_{(i)(j)} a = 0$, but the power series does not converge for any value of y if $\frac{1}{2} \leqslant x < 1$.

Let
$$b_i = \sum_{(j)} a_{i,j};$$
then, since
$$\left|\sum_{j=0}^{n} a_{i,j}\right| = \left|\Delta_{(i)} s_{i-1,n}\right| < 2C,$$
$$|b_i| \leqq 2C$$
and

(9)
$$\sum_{(i)} b_i x^i$$
is absolutely convergent. Similarly,
$$\sum_{(i)} a_{i,j} x^i$$
is absolutely convergent. Further we can prove that

(10)
$$\sum_{(j)(i)} a_{i,j} x^i$$
is convergent, and its sum equal to that of (9).*

For, if we introduce the abbreviation
$$b_{i,j} = \sum_{l=0}^{j} a_{i,l} = \Delta_{(i)} s_{i-1,j},$$
then the series (10) is equal to the limit
$$\lim_{j=\infty}\left(\sum_{i=0}^{\infty} b_{i,j} x^i\right),$$
provided that this limit exists.

Now, by the condition of finitude,
$$|b_{i,j}| = |\Delta_{(i)} s_{i-1,j}| < 2C;$$
so that $|b_{i,j} - b_i| < 4C$, for all values of i, j.

Hence, for all values of j,
$$\left|\sum_{i=M}^{\infty} (b_{i,j} - b_i) x^i\right| < 4C \sum_{i=M}^{\infty} x^i = 4Cx^M/(1-x).$$

Let M be chosen so as to make $4Cx^M/(1-x)$ less than an assigned positive number σ; M being now fixed, N can be chosen so as to give
$$|b_{i,j} - b_i| < \sigma(1-x)$$

* This is a kind of converse of Weierstrass's theorem concerning series of power series.

for every value of $j \geqslant N$ and for $i = 0, 1, 2, \ldots, M-1$, since

$$\lim_{j=\infty} b_{i,j} = b_i.$$

Then $$\left| \sum_{i=0}^{M-1} (b_{i,j} - b_i) x^i \right| < \sigma(1-x) \sum_{i=0}^{M-1} x^i < \sigma,$$

and hence $$\left| \sum_{i=0}^{\infty} (b_{i,j} - b_i) x^i \right| \leqslant \left| \sum_{i=0}^{M-1} \right| + \left| \sum_{i=M}^{\infty} \right| < 2\sigma, \quad \text{if } j \geqslant N.$$

Thus*

$$\lim_{j=\infty} \sum_{i=0}^{\infty} b_{i,j} x^i = \sum_{i=0}^{\infty} b_i x^i;$$

that is to say, the series (10) converges and has the same sum as (9).

5.

Hence $$\lim_{(x)(y)} f(x, y) = \lim_{(x)(y)} \sum_{(j)} y^j \sum_{(i)} a_{i,j} x^i = \lim_{(x)} \sum_{(j)} \sum_{(i)} a_{i,j} x^i$$

(by Abel's theorem)

$$= \lim \sum_{(i)} x^i \sum_{(j)} a_{i,j} \quad \text{(by § 4)}$$

$$= \sum_{(i)(j)} a_{i,j} \quad \text{(by Abel's theorem).}$$

An exactly similar proof applies to the case in which the convergence of $\sum_{(j)(i)} a$ is given. Hence, *if the condition of finitude is satisfied, and any one of the three series* $\sum_{(i,j)} a$, $\sum_{(i)(j)} a$, $\sum_{(j)(i)} a$ *is convergent, the corresponding one of the three limits* $\lim_{(x,y)} f$, $\lim_{(x)(y)} f$, $\lim_{(y)(x)} f$ *is determinate and equal to the sum of the series.*

By similar methods we can easily establish corresponding theorems, in case the series $\sum_{(i)(j)} a_{i,j}$, $\sum_{(j)(i)} a_{i,j}$, $\sum_{(i,j)} a_{i,j}$ do not converge, but oscillate.

---

* An alternative proof of this equation can be found by writing each side as a repeated limit, in the form

$$\lim_{(j)(i)} \left(\sum_{m=0}^{i} b_{m,j} x^m \right), \quad \lim_{(i)(j)} \left(\sum_{m=0}^{i} b_{m,j} x^m \right).$$

The equality can be then obtained by using conditions given by Bromwich (*Proc. London Math. Soc.*, Ser. 2, Vol. 1, 1903, p. 184).

Thus we find

$$\sum_{(\underline{i})(\underline{j})} a_{i,j} \leqslant \lim_{(\underline{x})(\underline{y})} f(x, y) \leqslant \lim_{(\overline{x})(\overline{y})} f(x, y) \leqslant \sum_{(\overline{i})(\overline{j})} a_{i,j},$$

$$\sum_{(\underline{j})(\underline{i})} a_{i,j} \leqslant \lim_{(\underline{y})(\underline{x})} f(x, y) \leqslant \lim_{(\overline{x})(\overline{y})} f(x, y) \leqslant \sum_{(\overline{i})(\overline{j})} a_{i,j},$$

$$\sum_{\underline{(i,j)}} a_{i,j} \leqslant \lim_{\underline{(x,y)}} f(x, y) \leqslant \lim_{\overline{(x,y)}} f(x, y) \leqslant \sum_{\overline{(i,j)}} a_{i,j}.$$

These results may be summed up in the statement that *the maximum and minimum limits of $f(x, y)$, when x, y approach unity in any one of the three standard ways, are included between the maximum and minimum limits of $\Sigma a_{i,j}$, when i, j approach infinity in the same way as x, y approach unity.*

6.

Before proceeding to the general case we shall illustrate this result by some examples:—

(i.) Suppose
$$a_{i,j} = \frac{i-j}{2^{i+j}} \frac{(i+j-1)!}{i!\,j!} \qquad (i, j > 0)$$

and $a_{0,j} = -2^{-j}$ $(j > 0)$, $a_{i,0} = 2^{-i}$ $(i > 0)$, $a_{0,0} = 0$. Then, if $j > 0$,

$$\sum_{i=0}^{\infty} a_{i,j} = -2^{-j} + \frac{2^{-j}}{j!} \sum_{1}^{\infty} \frac{(i+j-1)!}{(i-1)!} 2^{-i} - \frac{2^{-j}}{(j-1)!} \sum_{1}^{\infty} \frac{(i+j-1)!}{i!} 2^{-i}$$
$$= -2^{-j} + 2^{-j-1}(1-\tfrac{1}{2})^{-j-1} - 2^{-j}\{(1-\tfrac{1}{2})^{-j} - 1\} = 0;$$

but
$$\sum_{0}^{\infty} a_{i,0} = \sum_{1}^{\infty} 2^{-i} = 1.$$

Hence
$$\sum_{(j)(i)} a = 1$$

and, as
$$a_{j,i} = -a_{i,j},$$
$$\sum_{(i)(j)} a = -1.$$

It follows by a well known theorem of Pringsheim's that the double series $\sum_{(i,j)} a$ is not convergent. Hence we infer (assuming for a moment that the condition of finitude is satisfied) that

$$\lim_{(x)(y)} f = -1, \qquad \lim_{(y)(x)} f = 1,$$

and therefore (by the same theorem) $\lim_{(x,y)} f$ is not determinate. It is interesting to note that in such a case as this we can make this last *negative* inference. In the case of Abel's theorem *no* negative inference is possible.

To verify that, as a matter of fact, the condition of finitude is satisfied, we have only to observe that, if $m = n$,

$$s_{m,n} = 0,$$

while, if $m > n$, $\quad s_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{i,j} = \sum_{i=n+1}^{m} \sum_{j=0}^{n} a_{i,j}.$

In this last expression every term is positive, and

$$s_{m,n} < \sum_{i=n+1}^{\infty} \sum_{j=0}^{n} a_{i,j};$$

but, since $s_{n,n} = 0$,

$$\sum_{i=n+1}^{\infty} \sum_{j=0}^{n} a_{i,j} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} a_{i,j} = 1.$$

Thus we may take $C = 1$.

It is easy to verify our conclusions, for

$$\Sigma a_{i,j} x^i y^j = \frac{x-y}{2-x-y} = \frac{(1-y)-(1-x)}{(1-y)+(1-x)},$$

$$\lim_{(x)(y)} f = -1, \qquad \lim_{(y)(x)} f = 1.$$

(ii.) Suppose that

$$\sin\frac{1}{1-x} = a_0+a_1x+a_2x^2+\ldots \qquad (0 < x < 1),$$

and consider the double series defined by the scheme

$$\begin{array}{ccccc} a_0+a_0, & a_1-a_0, & a_2, & a_3, & \ldots, \\ a_1-a_0, & -a_1-a_1, & -a_2, & -a_3, & \ldots, \\ a_2, & -a_2, & 0, & 0, & \ldots, \\ a_3, & -a_3, & 0, & 0, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots. \end{array}$$

Then, if $m \geqslant 2$, $n \geqslant 2$, $s_{m,n} = 0$; so that

$$\sum_{(i,j)} a_{i,j} = 0.$$

But neither repeated series is convergent, since $a_0+a_1+a_2+\ldots$ is not convergent.* In this case

$$f(x, y) = (1-x)\sin\frac{1}{1-y} + (1-y)\sin\frac{1}{1-x};$$

so that

$$\lim_{(x,y)} f = 0$$

while neither repeated limit exists.

---

* For, if it were, $\sin\frac{1}{1-x}$ would by Abel's theorem have a limit for $x = 1$, which is not the case.

It is true that we have not in this case verified the condition of finitude, and it is difficult to see exactly how this can be done, as a_ν is a complicated function of ν. But it is only necessary to observe that to remove this objection we may replace $\sin \dfrac{1}{1-x}$ by any function of x which satisfies the following conditions:—

(i.) $$f(x) = a_0+a_1x+\dots \qquad (0 < x < 1);$$

(ii.) $$|a_0+a_1+\dots+a_\nu| < C;$$

(iii.) $f(x)$ oscillates between finite limits of indetermination for $x = 1$. Such functions certainly exist.*

7. *Statement of the Theorems of Frobenius and Hölder.*

Abel's theorem gives no information as to the behaviour near $x = 1$ of the function $f(x)$, in case the series (1) is not convergent; but if the series oscillates it is quite possible that the limit

$$\lim_{x=1} f(x)$$

may be finite and determinate, in spite of the divergence of the series.† Frobenius‡ was the first to obtain a result giving information about this case; his theorem may be stated as follows:—

Let $$s_n = \sum_{j=0}^{n} a_j;$$

in case s_n approaches no definite limit as n increases to infinity, it may

---

* One may, in fact, be constructed as follows. Divide (0, 1) into the intervals

$$i_n = \left(1-\frac{1}{2^n},\ 1-\frac{1}{2^{n+1}}\right) \quad (n = 0, 1, 2, \dots).$$

Let σ be an assigned small positive quantity. Choose n_1 so that throughout i_{n_1}

$$|(1-2x)-(-1)| < \sigma.$$

Now choose p_2 so that throughout i_{n_1}

$$x^{p_2}(1+x+x^2+\dots) < \sigma,$$

n_2 so that throughout i_{n_2} $$|1-2x+2x^{p_2}-(+1)| < \sigma,$$

p_3 so that throughout i_{n_2} $$x^{p_3}(1+x+x^2+\dots) < \sigma,$$

and so on. Then it is easy to see that, if

$$f(x) = 1-2x^{p_1}+2x^{p_2}-2x^{p_3}+\dots \quad (p_1 = 1),$$

$f(x)$ differs from -1 by less than 3σ in $i_{n_1}, i_{n_3}, i_{n_5}, \dots$, and from $+1$ by less than 3σ in $i_{n_2}, i^{n}_{4}, i_{n_6}, \dots$. The numbers $p_1, p_2, p_3, \dots$ increase with very great rapidity.

† For example, let $f(x) = 1/(1+x) = 1-x+x^2-x^3+\dots$; then $\lim_{x=1} f(x)$ is equal to $\frac{1}{2}$, although $1-1+1-1+1-1+\dots$ is oscillatory. But it has been proved that if $\lim_{n=\infty} s_n = \infty$, then $\lim_{x=1} f(x) = \infty$.

‡ *Crelle's Journal*, Bd. LXXXIX., 1880, p. 262.

happen that the arithmetic mean

$$s_n^{(1)} = \frac{1}{n+1}(s_0+s_1+s_2+\ldots+s_n)$$

approaches a limit l; then the limit

$$\lim_{x=1} f(x)$$

exists and is equal to l.

It may be noticed incidentally that, if s_j does approach a definite limit l, then the arithmetic mean $s_n^{(1)}$ will approach the same limit. For an integer n can be chosen so that

$$|s_j - l| < \sigma,$$

if $j > n$; n being fixed, choose N so that

$$|s_0+s_1+s_2+\ldots+s_{n-1}-nl| < N\sigma.$$

Then

$$|s_j^{(1)}-l| = \frac{1}{j+1}\left|(s_0+s_1+\ldots+s_{n-1}-nl)+(s_n-l)+(s_{n+1}-l)+\ldots+(s_j-l)\right|$$

$$< \frac{1}{j+1}[(N\sigma)+(j-n+1)\sigma] < 2\sigma,$$

if $j > n$ and N; that is, $\lim_{j=\infty} s_j^{(1)} = l.$

A similar method can be used to prove that if s_n tends to infinity with n, then the same is true of $s_n^{(1)}$.

The theorem of Frobenius was extended further by Hölder,* so as to cover cases in which the first arithmetic mean has no definite limit.

Hölder writes

$$
\begin{aligned}
s_n^{(1)} &= \frac{1}{n+1}(s_0+s_1+s_2+\ldots+s_n),\\
s_n^{(2)} &= \frac{1}{n+1}(s_0^{(1)}+s_1^{(1)}+s_2^{(1)}+\ldots+s_n^{(1)}), \qquad (11)\\
\ldots &\quad \ldots \quad \ldots \quad \ldots \quad \ldots,\\
s_n^{(k)} &= \frac{1}{n+1}(s_0^{(k-1)}+s_1^{(k-1)}+s_2^{(k-1)}+\ldots+s_n^{(k-1)}).
\end{aligned}
$$

The extended theorem is then

$$\varliminf_{(n)} s_n^{(k)} \leqslant \varliminf_{(x)} f(x) \leqslant \varlimsup_{(x)} f(x) \leqslant \varlimsup_{(n)} s_n^{(k)},$$

provided that $|s_n^{(k)}| < C$ for all values of n.

---

* *Math. Annalen*, Bd. xx., 1882, p. 535.

8. *Extension of Frobenius's Theorem to Double Series.*

Let us write

(12) $$s_{m,n} = \sum_{i,j=0} a_{i,j}, \quad s^{(1)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s_{i,j};$$

so that $s^{(1)}_{m,n}$ is an arithmetic mean amongst the sums s_{mn}. Then the theorem is:—

If $$\lim_{(m,n)} s^{(1)}_{m,n} = l,$$

then also $$\lim_{(x,y)} f(x, y) = l,$$

provided that

(13) $$| s^{(1)}_{m,n} | < C$$

for all values of m, n (the present form of the condition of finitude).

In virtue of equations (12), we have

$$\Delta_{(i,j)} [ij\, s^{(1)}_{i-1,j-1}] = s_{i,j}, \quad \Delta_{(i,j)} [s_{i-1,j-1}] = a_{i,j}.$$

Hence, using (13), we deduce

(14) $$| s_{i\,j} | < C[(i+1)(j+1)+i(j+1)+(i+1)j+ij] < 4C(i+1)(j+1)$$

and

(15) $$| a_{i,j} | < 16C(i+1)(j+1).$$

It follows at once, from (13), (14), and (15), that the three series

$$\Sigma(i+1)(j+1)s^{(1)}_{i,j}x^i y^j, \quad \Sigma s_{i,j}x^i y^j, \quad \Sigma a_{i,j}x^i y^j$$

are all absolutely convergent, since their terms are less numerically than the corresponding terms in the series for

$$16C(1-x)^{-2}(1-y)^{-2}.$$

Further we find by direct multiplication that

$$(1-x)(1-y)\Sigma(i+1)(j+1)s^{(1)}_{i,j}x^i y^j = \Sigma s_{i,j}x^i y^j.$$

Thus, using (8), it is clear that

(16) $$f(x, y) = (1-x)^2(1-y)^2\Sigma(i+1)(j+1)s^{(1)}_{i,j}x^i y^j.$$

But, since the arithmetic means have the limiting value l, an integer N can be found such that

(17) $$| s^{(1)}_{i,j} - l | < \sigma, \text{ for } i, j \geqslant N,$$

however small the positive number σ may be; further, from (13), it follows that

$$| l | \leqslant C, \quad | s^{(1)}_{i,j} - l | < 2C, \text{ for all values of } i, j.$$

Thus, since $\qquad (1-x)^2(1-y)^2\Sigma(i+1)(j+1)x^i y^j = 1,$

it follows, from (16), that

$$\begin{aligned} f(x, y)-l &= (1-x)^2(1-y)^2\Sigma(i+1)(j+1)(s_{i,j}^{(1)}-l)x^i y^j \\ &= (1-x)^2(1-y)^2\left[\sum_{i,j=0}^{N-1} + \sum_{i=0}^{N-1}\sum_{j=N}^{\infty} + \sum_{j=0}^{N-1}\sum_{i=N}^{\infty} + \sum_{i,j=N}^{\infty}\right]. \end{aligned}$$

But, from (17), it is evident that

$$\left|\sum_{i,j=N}^{\infty}\right| < \sigma\sum_{i,j=N}^{\infty}(i+1)(j+1)x^i y^j < \sigma(1-x)^{-2}(1-y)^{-2};$$

also $$\left|\sum_{i,j=0}^{N-1}\right| < 2C\sum_{i,j=0}^{N-1}(i+1)(j+1) = 2C[\tfrac{1}{2}N(N+1)]^2,$$

$$\left|\sum_{i=0}^{N-1}\sum_{j=N}^{\infty}\right| < 2C\sum_{i=0}^{N-1}(i+1)\sum_{j=N}^{\infty}(j+1)y^j < N(N+1)C(1-y)^{-2}.$$

$$\left|\sum_{j=0}^{N-1}\sum_{i=N}^{\infty}\right| < 2C\sum_{j=0}^{N-1}(j+1)\sum_{i=N}^{\infty}(i+1)x^i < N(N+1)C(1-x)^{-2}.$$

Combining these four inequalities, we obtain

(18) $$\begin{aligned} &|f(x,y)-l| \\ &\quad < \sigma+N(N+1)C[(1-x)^2+(1-y)^2+\tfrac{1}{2}N(N+1)(1-x)^2(1-y)^2]. \end{aligned}$$

Now choose δ so that

$$N(N+1)C\delta^2[2+\tfrac{1}{2}N(N+1)\delta^2] < \sigma,$$

which is possible, since N is now fixed.* Then plainly

$$N(N+1)C[(1-x)^2+(1-y)^2+\tfrac{1}{2}N(N+1)(1-x)^2(1-y)^2] < \sigma,$$

if $1-x < \delta$, $1-y < \delta$; and so (18) leads to the result

$$|f(x, y)-l| < 2\sigma,$$

if $1-x < \delta$, $1-y < \delta$; that is,

(19) $$\lim_{(x,y)} f(x, y) = l,$$

which is the analogue of Frobenius's theorem.

It is easy to prove, by a similar method, that, in case $s_{i,j}^{(1)}$ does not approach a definite limit, but oscillates between a maximum limit and a minimum limit, then

$$\varliminf_{(i,j)} s_{i,j}^{(1)} \leqslant \varliminf_{(x,y)} f(x, y) \leqslant \varlimsup_{(x,y)} f(x, y) \leqslant \varlimsup_{(i,j)} s_{i,j}^{(1)}.$$

Before considering the case of *repeated* limits of the double series, we shall give an example of the result contained in equation (19).

---

* One way of doing it is to take for δ the smaller of the two values $[\sigma/4N(N+1)C]^{\frac{1}{2}}$, $[\sigma/N^2(N+1)^2C]^{\frac{1}{4}}$; the smaller will usually be the first.

9. *Lord Kelvin's Series.*

In Lord Kelvin's discussion of the electrical force between two equal conducting spheres in contact,* he employs the double series given by

$$a_{i,j} = (-1)^{i+j} ij/(i+j)^2 \quad (i, j = 1, 2, 3, \ldots),$$

the scheme for which is

$$\begin{array}{cccccc}
+\frac{1.1}{2^2}, & -\frac{2.1}{3^2}, & +\frac{3.1}{4^2}, & -\frac{4.1}{5^2}, & +\ldots, \\
-\frac{1.2}{3^2}, & +\frac{2.2}{4^2}, & -\frac{3.2}{5^2}, & +\frac{4.2}{6^2}, & -\ldots, \\
+\frac{1.3}{4^2}, & -\frac{2.3}{5^2}, & +\frac{3.3}{6^2}, & -\frac{4.3}{7^2}, & +\ldots, \\
-\frac{1.4}{5^2}, & +\frac{2.4}{6^2}, & -\frac{3.4}{7^2}, & +\frac{4.4}{8^2}, & -\ldots, \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots .
\end{array}$$

He shows that† $\sum\limits_{(i)(j)} a_{i,j} = \sum\limits_{(j)(i)} a_{i,j} = \frac{1}{8}(\log 2 - \frac{1}{2}) = l,$

say, the method employed being, essentially, the same as that used below.

Before proceeding to the general discussion, we shall evaluate $f(x, x)$; now here $|a_{ij}| \leqslant \frac{1}{4}$, so that the series for $f(x, y)$ is absolutely convergent. Thus, we may write

$$f(x, x) = \sum_{n=2}^{\infty} x^n \left(\sum_{i=1}^{n-1} a_{i,\,n-i}\right).$$

But $\displaystyle\sum_{i=1}^{n-1} a_{i,\,n-i} = (-1)^n \sum_{i=1}^{n-1} i\,(n-i)/n^2 = (-1)^n \tfrac{1}{6}(n-1/n),$

and thus

$$f(x, x) = \tfrac{1}{6} \sum_{n=2}^{\infty} (n-1/n)(-x)^n = \tfrac{1}{6} \sum_{n=1}^{\infty} (n-1/n)(-x)^n$$
$$= \tfrac{1}{6}\left[\log(1+x) - x/(1+x)\right].$$

From this equation it is plain that

$$\lim_{x=1} f(x, x) = \tfrac{1}{6}(\log 2 - \tfrac{1}{2}) = l,$$

---

* *Phil. Mag.*, April and August, 1853; *Reprint of Electrical Papers*, No. VI., Art. 140.

† It is of some interest to observe that it is the *repeated* summation which gives the correct expression for *the force between the spheres*. But this is NOT *the force between the two sets of images*; in fact, the latter force can only be regarded as lim s_{ij}, where i, j approach infinity in such a way that i/j tends to the limit unity; but, as will be seen below, lim s_{ij} is then *not determinate*.

a result which has sometimes been used to evaluate the sum of Kelvin's series.*

Next, to find the general value of $f(x, y)$, we write

$$f(x, y) = \sum_{n=2}^{\infty} \left(\sum_{i=1}^{n-1} a_{i,\, n-i} x^i y^{n-i} \right);$$

but

$$\sum_{i=1}^{n-1} a_{i,\, n-i} x^i y^{n-i} = (-1)^n \frac{1}{n^2} \frac{\partial^2}{\partial x \partial y} \left(\sum_{i=0}^{n} x^i y^{n-i} \right) = (-1)^n \frac{1}{n^2} \frac{\partial^2}{\partial x \partial y} \left(\frac{x^{n+1} - y^{n+1}}{x-y} \right)$$

$$= (-1)^n \frac{1}{n^2} \left[(n+1) \frac{x^n + y^n}{(x-y)^2} - 2 \frac{x^{n+1} - y^{n+1}}{(x-y)^3} \right].$$

It will be observed that this expression is identically zero for $n = 1$, and so the summation may be extended to include $n = 1$; then we have

$$(x-y)^3 f(x, y) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2} [(n+1)(x-y)(x^n + y^n) - 2(x^{n+1} - y^{n+1})]$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} [(x+y)(x^n - y^n) - n(x-y)(x^n + y^n)].$$

If we introduce the function $\phi(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^n / n^2$, it is clear that

$$\phi'(x) = \sum_{n=1}^{\infty} (-1)^{n-1} x^{n-1}/n = \frac{1}{x} \log(1+x);$$

and then

$$(x-y)^3 f(x, y) = (x+y)[\phi(x) - \phi(y)] - (x-y)[x\phi'(x) + y\phi'(y)].$$

If we write, for the moment,

$$\xi = \tfrac{1}{2}(x+y), \qquad \eta = \tfrac{1}{2}(x-y),$$

it will be found (after some reductions which are tedious, but not difficult) that

$$f(x, y) = -\tfrac{1}{6}\xi\phi'''(\xi) - \tfrac{1}{2}\phi''(\xi) + \tfrac{1}{24}\eta R$$

where

$$|R| \leqslant (\tfrac{5}{2}|\xi| + 2|\eta|)\lambda < \tfrac{9}{2}\lambda,$$

λ being the greatest value of $|\phi^{\text{iv}}(\zeta)|$ when ζ takes all values from x to y, inclusive.

Thus

$$\lim_{(x,\, y)} f(x, y) = \lim_{\xi=1,\, \eta=0} [-\tfrac{1}{6}\xi\phi'''(\xi) - \tfrac{1}{2}\phi''(\xi) + \tfrac{1}{24}\eta R]$$

$$= -\tfrac{1}{6} \lim_{\xi=1} [\phi'''(\xi) + 3\phi''(\xi)]$$

$$= -\tfrac{1}{6}[(2\log 2 - \tfrac{5}{4}) + 3(\tfrac{1}{2} - \log 2)] = \tfrac{1}{6}(\log 2 - \tfrac{1}{4}) = l;$$

and it is clear that l is also the value of the two repeated limits

$$\lim_{(x)(y)} f(x, y) \qquad \text{and} \qquad \lim_{(y)(x)} f(x, y).$$

---

* For example, by Prof. Tarleton, in his book on *Attractions* (Ex. 9, p. 279), where the result is obtained by processes which can hardly be justified.

Next we consider the value of $s_{m,n}$; and, to find this, use the theorem

$$(i+j)^{-2} = \int_0^\infty e^{-(i+j)t}\,t\,dt\,;$$

so that

$$a_{i,j} = (-1)^{i+j}\int_0^\infty ij\,e^{-(i+j)t}\,t\,dt.$$

Hence

$$s_{m,n} = \int_0^\infty t\,dt\left[\sum_{i,j=0}^{m,n}(-1)^{i+j}ij\,e^{-(i+j)t}\right] = \int_0^\infty \phi(m,t)\,\phi(n,t)\,\frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4},$$

where

$$\phi(m,t) = 1+(-1)^{m-1}\left\{(m+1)e^{-mt}+me^{-(m+1)t}\right\}.$$

Now

$$\int_0^\infty \frac{me^{-(m+3)t}\,t\,dt}{(1+e^{-t})^4} < \int_0^\infty \frac{me^{-(m+2)t}\,t\,dt}{(1+e^{-t})^4} < \int_0^\infty me^{-mt}\,t\,dt = \frac{1}{m},$$

and accordingly

$$\lim_{m=\infty}\int_0^\infty \frac{me^{-(m+3)t}\,t\,dt}{(1+e^{-t})^4} = 0 = \lim_{m=\infty}\int_0^\infty \frac{(m+1)e^{-(m+2)t}\,t\,dt}{(1+e^{-t})^4}.$$

Similarly

$$\lim_{m=\infty}\int_0^\infty \frac{m(n+1)e^{-(m+n+3)t}\,t\,dt}{(1+e^{-t})^4} = 0,$$

and so on; and hence

$$\lim_{(n)(m)} s_{m,n} = \int_0^\infty \frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4} = \tfrac{1}{6}(\log 2-\tfrac{1}{4}) = l,$$

the value of the integral being obtained by direct integration.* In the same way,

$$\lim_{(m)(n)} s_{m,n} = l.$$

We have thus obtained an illustration of part of the theorem given in § 5; for we have proved directly that

$$\sum_{(i)(j)} a_{i,j} = \lim_{(x)(y)} f, \qquad \sum_{(j)(i)} a_{i,j} = \lim_{(y)(x)} f.$$

However, the double series $\sum_{(i,j)} a$ is not convergent, in spite of the fact

---

* The indefinite integral is

$$\tfrac{1}{6}\left[\frac{e^t}{(1+e^t)^2} - \frac{t(1+3e^t)}{(1+e^t)^3} - \log(1+e^{-t})\right].$$

This is the method employed by Kelvin, *loc. cit.*

that $\lim\limits_{(x,y)} f$ is perfectly determinate. For

$$\lim_{m=\infty}\int_0^\infty \frac{m^2 e^{-mt}\,t\,dt}{(1+e^{-t})^4} = \lim_{m=\infty}\int_0^\infty \frac{e^{-x}x\,dx}{(1+e^{-x/m})^4} = \tfrac{1}{16}\int_0^\infty e^{-x}x\,dx = \tfrac{1}{16},^*$$

from which it easily follows that

$$\lim_{m=\infty} s_{m,m+1} = l-\tfrac{1}{16}, \qquad \lim_{m=\infty} s_{m,m} = l+\tfrac{1}{16}.$$

It is not difficult to prove that these are the general values of

$$\underline{\lim}_{(m,n)}\, s_{m,n} \quad \text{and} \quad \overline{\lim}_{(m,n)}\, s_{m,n}.$$

If m, n tend to infinity in such a way that $\lim(m/n)=1$, $s_{m,n}$ oscillates between these values; if in such a way that $\lim(m/n)=0$ or ∞, $s_{m,n}$ tends to the determinate limit l.

It will be seen that, in agreement with § 5,

$$\underline{\Sigma}_{(i,j)}\, a_{i,j} < \lim_{(x,y)} f < \overline{\Sigma}_{(i,j)}\, a_{i,j}.$$

Next, if we form the arithmetic mean of $s_{m,n}$, it will be found that

$$s^{(1)}_{m,n} = \int_0^\infty \psi(m,t)\,\psi(n,t)\,\frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4},$$

where $\psi(m,t) = 1+\dfrac{(-)^{m-1}me^{-(m+1)t}}{m+1}+\dfrac{2}{m+1}\,\dfrac{e^{-t}+(-)^{m-1}e^{-(m+1)t}}{1+e^{-t}}.$

This gives at once

$$\lim_{(m,n)} s^{(1)}_{m,n} = \int_0^\infty \frac{e^{-2t}\,t\,dt}{(1+e^{-t})^4} = l = \lim_{(x,y)} f(x,y);$$

and, to verify the condition of finitude, we observe that, since $|\psi(m,t)|<4$,

$$|s^{(1)}_{m,n}| < 16\int_0^\infty e^{-2t}\,t\,dt \text{ for all values of } m, n,$$

or

$$|s^{(1)}_{m,n}| < 4.$$

Thus the equation $$\lim_{(m,n)} s^{(1)}_{m,n} = \lim_{(x,y)} f(x,y)$$

is in complete agreement with the theorem proved in § 8.

From the preceding work it is clear that there is no justification for *assuming* the equation

$$\Sigma_{(i)(j)}\, a_{i,j} = \Sigma_{(j)(i)}\, a_{i,j} = \lim_{x=1} f(x,x),$$

* It is easy to see that the conditions given by Bromwich (*l.c.*, p. 201) for this inversion of limits are satisfied.

until we have proved (i.) that the *repeated* sums $\sum_{(i)(j)} a_{i,j}$, $\sum_{(j)(i)} a_{i,j}$ are convergent; and (ii.) that the *double* limit $\lim_{(m,n)} s^{(1)}_{m,n}$ is determinate, in addition to verifying the condition of finitude.

It follows that this method of evaluating the repeated sums is really far more complicated than Kelvin's direct method of summation; although, superficially, the former method appears to be the easier.

10. *Extension to Repeated (Two-fold) Series of the Theorems of Frobenius and Hölder.*

Returning to the notation of § 4, suppose that the limit

$$\lim_{j=\infty} b_{i,j}$$

does not exist; it may then happen that the arithmetic means of $b_{i,j}$, namely,

$$b^{(1)}_{i,j} = \frac{1}{j+1} \sum_{n=0}^{j} b_{i,n},$$

approach a limit $b^{(1)}_i$; so that

$$\lim_{j=\infty} b^{(1)}_{i,j} = b^{(1)}.$$

Suppose further that the condition of finitude is satisfied in the form

$$|\, b^{(1)}_{i,j} \,| < C, \quad \text{for all values of } i, j;$$

it follows that the two series

$$\sum_{(i)} b^{(1)}_{i,j} x^i, \qquad \sum_{(i)} b^{(1)}_i x^i$$

are absolutely convergent. The same is true of the series

$$\sum a_{i,j} x^i,$$

since $$b_{i,j} = \Delta_{(j)} [j\, b^{(1)}_{i,j-1}], \qquad a_{i,j} = \Delta_{(j)} [b_{i,j-1}];$$

so that $$|\, b_{i,j} \,| < 2C(j+1), \qquad |\, a_{i,j} \,| < 4C(j+1).$$

Now write

$$X_j = \sum_{l=0}^{j} \sum_{(i)} a_{i,l} x^i = \sum_{(i)} b_{i,j} x^i$$

and $$X^{(1)}_j = \frac{1}{j+1} (X_0 + X_1 + X_2 + \ldots + X_j).$$

Then plainly

(20) $$X_j^{(1)} = \sum_{(i)} b_{i,j}^{(1)} x^i.$$

But, by the process adopted in proving the last equation of § 4, it follows that*

$$\lim_{j=\infty} \sum_{(i)} b_{i,j}^{(1)} x^i = \sum_{(i)} b_i^{(1)} x^i,$$

and so, from (20), we find

(21) $$\lim_{j=\infty} X_j^{(1)} = \sum_{(i)} b_i^{(1)} x^i.$$

Now it has been proved that

$$|a_{i,j}| < 4C(j+1),$$

and consequently $\Sigma a_{i,j} x^i y^j$ is absolutely convergent, its terms being less numerically than those in the expansion of $4C(1-x)^{-1}(1-y)^{-2}$. Thus

$$f(x, y) = \sum_{(j)} y^j \sum_{(i)} a_{i,j} x^i.$$

Frobenius's theorem can be applied to this series: and, in virtue of equation (21), it follows that

$$\lim_{(y)} f(x, y) = \lim_{j=\infty} X_j^{(1)} = \sum_{(i)} b_i^{(1)} x^i.$$

If now either the series $\sum_{(i)} b_i^{(1)}$ *converges to a sum* l, *or if the arithmetic mean process applied to* $b_i^{(1)}$ *gives a definite limit* l, *then*

$$\lim_{(x)(y)} f(x, y) = \lim_{(x)} \sum_{(i)} b_i^{(1)} x^i = l,$$

a result which follows at once from Abel's (or Frobenius's) theorem.

Obviously a similar method can be used to find the limit

$$\lim_{(y)(x)} f(x, y),$$

the necessary modifications being made in the hypotheses.

As an illustration, take the series given by

$$a_{i,j} = (-1)^{i+j},$$

* In § 4, the condition of finitude was stated in a slightly different form; but a glance at the proof will show that $|b_{i,j}^{(1)}| < C$ is sufficient for the truth of the conclusion.

which has the scheme

$$\begin{array}{ccccc} +1, & -1, & +1, & -1, & \ldots, \\ -1, & +1, & -1, & +1, & \ldots, \\ +1, & -1, & +1, & -1, & \ldots, \\ \ldots & \ldots & \ldots & \ldots & \ldots . \end{array}$$

In this case $b_{i,j} = 0$, if j is odd; and $b_{i,j} = (-1)^i$, if j is even.

Hence $b_i^{(1)} = \lim\limits_{j=\infty} b_{i,j}^{(1)} = \frac{1}{2}(-1)^i$, and $| b_{i,j}^{(1)} | < 1$ for all values of i, j. Thus

$$\lim_{(y)} f(x, y) = \sum_{(i)} \tfrac{1}{2}(-1)^i x^i.$$

The series $\sum\limits_{(i)} \frac{1}{2}(-1)^i$ does not converge, but the arithmetic mean process leads to the limit $\frac{1}{4}$; so that

$$\lim_{(x)(y)} f(x, y) = \tfrac{1}{4},$$

which may be immediately verified, since $f(x, y) = (1+x)^{-1}(1+y)^{-1}$. In this case, as a matter of fact, the theorem of § 8 can be applied; for $s_{i,j} = 1$, if both i and j are even, while $s_{i,j} = 0$ in every other case. Thus

$$\lim_{(i,j)} s_{i,j}^{(1)} = \tfrac{1}{4},$$

and so

$$\lim_{(x,y)} f(x, y) = \tfrac{1}{4}.$$

It is clear that the method used in this paragraph is capable of immediate extension to any case in which a *finite* number* of arithmetic means must be taken in order to obtain a limit from each column of the scheme. A corresponding change must be made in the condition of finitude. Then, if the limits so found from the columns either form a convergent series with the sum l, or lead to a limit l after a finite number of arithmetic means, the equation

$$\lim_{(x)(y)} f(x, y) = l$$

is true.

A simple example which we do not pause to work out in detail is given by

$$a_{i,j} = (-)^{i+j}(i+1)^p (j+1)^q,$$

* This number may vary with i, so long as it has a finite maximum. This is clear, in consequence of a theorem proved in § 7, according to which, if a limit is obtained from an arithmetic mean of any order, the *same* limit will belong to all the subsequent arithmetic means.

or, more generally,

$$a_{i,j} = (i+1)^p (j+1)^q \exp\{(i\theta+j\phi)\sqrt{(-1)}\}.$$

11. *Extension of Hölder's Theorems to Double Series : Double Limit.*

Continuing the notation of equation (12), let us write

$$
\begin{aligned}
&s_{m,n} = \sum_{i,j=0}^{m,n} a_{i,j},\\
&s^{(1)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s_{i,j},\\
(22)\qquad &s^{(2)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s^{(1)}_{i,j},\\
&\ldots \quad \ldots \quad \ldots \quad \ldots,\\
&s^{(k)}_{m,n} = \frac{1}{(m+1)(n+1)} \sum_{i,j=0}^{m,n} s^{(k-1)}_{i,j}.
\end{aligned}
$$

Suppose that the condition of finitude

$$|s^{(k)}_{i,j}| < C$$

is verified for all values of i, j; then, by a process analogous to that used in (14) and (15), we deduce

$$
\begin{aligned}
&|a_{i,j}| < 4^{k+1}(i+1)^k(j+1)^k C,\\
(23)\qquad &|s_{i,j}| < 4^k(i+1)^k(j+1)^k C,\\
&|s^{(k-r)}_{i,j}| < 4^r(i+1)^r(j+1)^r C \quad (r = 0, 1, 2, \ldots, k-1).
\end{aligned}
$$

From (23) it is clear that each of the series

$$\Sigma a_{i,j}x^i y^j, \quad \Sigma s_{i,j}x^i y^j, \quad \Sigma s^{(r)}_{i,j}x^i y^j \quad (r = 1, 2, \ldots, k)$$

is absolutely convergent; since their terms are numerically less than the corresponding terms in $4^{k+1}(k!)^2 C(1-x)^{-(k+1)}(1-y)^{-(k+1)}$.

We prove next the following preliminary lemma :—

Assuming the truth of the equation

$$(24)\qquad \lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s^{(r)}_{i,j} x^i y^j = l,$$

where ϕ is a polynomial of the form

$$\phi(i,j) = \frac{i^p}{p!}\frac{j^q}{q!} + \text{ terms of lower degree,}$$

then also

$$\lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s_{i,j}^{(r-1)} x^i y^j = l, \tag{25}$$

provided that (24) *is valid for all integers* p, q.

To prove the lemma, we use the identity

$$s_{i,j}^{(r-1)} = \Delta_{(i,j)} [ij\, s_{i-1,\,j-1}^{(r)}],$$

which gives

$$\begin{aligned} \sum_{(i,j)} \phi(i,j)\, s_{i,j}^{(r-1)} x^i y^j = {} & (1-x)(1-y) \sum_{(i,j)} (i+1)(j+1)\, \phi(i,j)\, s_{i,j}^{(r)} x^i y^j \\ & -x(1-y) \sum_{(i,j)} (i+1)(j+1) [\Delta_{(i)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j \\ & -y(1-x) \sum_{(i,j)} (i+1)(j+1) [\Delta_{(j)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j \\ & +xy \sum_{(i,j)} (i+1)(j+1) [\Delta_{(i,j)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j. \end{aligned} \tag{26}$$

But the polynomials appearing in these series are of the forms

$$(i+1)(j+1)\phi(i,j) = (p+1)(q+1) \frac{i^{p+1}}{(p+1)!} \frac{j^{q+1}}{(q+1)!} + \text{lower terms},$$

$$(i+1)(j+1)[\Delta_{(i)} \phi(i,j)] = p(q+1) \frac{i^p}{p!} \frac{j^{q+1}}{(q+1)!} + \cdots,$$

$$(i+1)(j+1)[\Delta_{(j)} \phi(i,j)] = (p+1)q \frac{i^{p+1}}{(p+1)!} \frac{j^q}{q!} + \cdots,$$

$$(i+1)(j+1)[\Delta_{(i,j)} \phi(i,j)] = pq \frac{i^p}{p!} \frac{j^q}{q!} + \cdots.$$

Thus, in virtue of (24), we find

$$\lim_{(x,y)} (1-x)^{p+2}(1-y)^{q+2} \sum_{(i,j)} (i+1)(j+1)\phi(i,j)\, s_{i,j}^{(r)} x^i y^j = (p+1)(q+1)\, l,$$

$$\lim_{(x,y)} x(1-x)^{p+1}(1-y)^{q+2} \sum_{(i,j)} (i+1)(j+1)[\Delta_{(i)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j = p(q+1)\, l$$

$$\lim_{(x,y)} (1-x)^{p+2} y(1-y)^{q+1} \sum_{(i,j)} (i+1)(j+1)[\Delta_{(j)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j = (p+1)q l,$$

$$\lim_{(x,y)} x(1-x)^{p+1} y(1-y)^{q+1} \sum_{(i,j)} (i+1)(j+1)[\Delta_{(i,j)} \phi(i,j)]\, s_{i,j}^{(r)} x^i y^j = pql.$$

Combining the last four equations with equation (26), we see that

$$\begin{aligned} & \lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j)\, s_{i,j}^{(r-1)} x^i y^j \\ & \qquad = [(p+1)(q+1) - p(q+1) - (p+1)q + pq]\, l = l, \end{aligned}$$

and this is equation (25). Thus the lemma is proved.

It is now clear that, *if the equation*

(27) $$\lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j) s_{i,j}^{(k)} x^i y^j = l$$

is true for all integers p, q and for any particular integer k, then also the equation

(28) $$\lim_{(x,y)} (1-x)^{p+1}(1-y)^{q+1} \sum_{(i,j)} \phi(i,j) s_{i,j} x^i y^j = l$$

is true.

We shall now establish the truth of (27), on the hypothesis that

$$\lim_{(i,j)} s_{i,j}^{(k)} = l.$$

Let us write for brevity

$$\psi(i,j) = [(i+1)(i+2)\ldots(i+p)(j+1)(j+2)\ldots(j+q)]/p!\,q!,$$

so that $$\lim_{(i,j)} (\phi/\psi) = 1.$$

An integer N can now be found, corresponding to any assigned positive number σ, such that

$$|(\phi/\psi) s_{i,j}^{(k)} - l| < \sigma, \quad \text{if} \quad i, j \geqslant N.$$

Further, a number g can be found such that

$$|\phi/\psi| < g, \quad \text{for all values of } i, j;$$

and so, using the condition of finitude,

$$|\phi s_{i,j}^{(k)}| < gC\psi, \quad \text{for all values of } i, j,$$

and $$|l| \leqslant C;$$

so that $$|\phi s_{i,j}^{(k)} - l\psi| < (g+1)C\psi.$$

Now $$\sum_{(i,j)} (\phi s^k - l\psi) x^i y^j = \sum_{i,j=0}^{N-1} + \sum_{i=0}^{N-1} \sum_{j=N}^{\infty} + \sum_{j=0}^{N-1} \sum_{i=N}^{\infty} + \sum_{i,j=N}^{\infty}$$

and $$\left|\sum_{i,j=0}^{N-1}\right| < (g+1)C \sum_{i,j=0}^{N-1} \psi < (g+1)C \frac{(N+p)^{p+1}(N+q)^{q+1}}{(p+1)!\,(q+1)!},$$

$$\left|\sum_{i=0}^{N-1} \sum_{j=N}^{\infty}\right| < (g+1)C \sum_{i=0}^{N-1} \sum_{j=N}^{\infty} \psi y^i < (g+1)C \frac{(N+p)^{p+1}}{(p+1)!}(1-y)^{-(q+1)},$$

$$\left|\sum_{i=N}^{\infty} \sum_{j=0}^{N-1}\right| < (g+1)C \sum_{j=0}^{N-1} \sum_{i=N}^{\infty} \psi x^i < (g+1)C \frac{(N+q)^{q+1}}{(q+1)!}(1-x)^{-(p+1)},$$

$$\left|\sum_{i,j=N}^{\infty}\right| < \sigma \sum_{i,j=N}^{\infty} \psi x^i y^j < \sigma(1-x)^{-(p+1)}(1-y)^{-(q+1)}.$$

Hence we deduce

$$|(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}(\phi s_{i,j}^{(k)}-l\psi)x^iy^j|$$
$$<\sigma+(q+1)C\left[\frac{(N+p)^{p+1}(N+q)^{q+1}}{(p+1)!\,(q+1)!}(1-x)^{p+1}(1-y)^{q+1}\right.$$
$$\left.+\frac{(N+p)^{p+1}}{(p+1)!}(1-x)^{p+1}+\frac{(N+q)^{q+1}}{(q+1)!}(1-y)^{q+1}\right],$$

and we can choose δ so that the right-hand side of this inequality is less than 2σ, provided that $1-x$, $1-y$ are each less than δ. Hence

$$\lim_{(x,y)}(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}(\phi s_{i,j}^{(k)}-\psi l)x^iy^j=0.$$

But
$$(1-x)^{p+1}(1-y)^{q+1}\sum_{(i,j)}\psi l x^iy^j=l,$$

and equations (27), (28) follow at once.

If we now take in (28) the special values*

$$\phi(i,j)=1,\qquad p=0,\qquad q=0,$$

it will be seen that

$$\lim_{(x,y)}(1-x)(1-y)\sum_{(i,j)}s_{i,j}x^iy^j=l,$$

or, using equation (8),
$$\lim_{(x,y)}f(x,y)=l.$$

Thus the following theorem has been established:—

If, for all values of i, j, $|s_{i,j}^{(k)}|<C$, and if

$$\lim_{(i,j)}s_{i,j}^{(k)}=l,$$

then also
$$\lim_{(x,y)}f(x,y)=l.$$

This is the general extension of Hölder's theorem to double series; the method can be easily modified so as to include the possibility that $s_{i,j}^{(k)}$ may oscillate; the result is then

$$\varliminf_{(i,j)}s_{i,j}^{(k)}\leqslant\varliminf_{(x,y)}f(x,y)\leqslant\varlimsup_{(x,y)}f(x,y)\leqslant\varlimsup_{(i,j)}s_{i,j}^{(k)}.$$

12. *The General Theorem.*

We proceed now to the proof of the general theorem stated in § 2. It has been already pointed out that the argument of § 3 applies to the

* This appears to be the only case of practical importance, but the introduction of this specialization earlier does not materially simplify the work.

general case when it is the convergence of the multiple series proper

$$\sum_{(1,2,\ldots,n)} a$$

which is given. To prove the theorem in its most general form it is convenient to proceed by induction. We shall adopt the following contracted notation. We denote the *groups* of suffixes $(i_1, i_2, \ldots, i_p)$, $(i_{p+1}, \ldots, i_q)$, ..., $(i_{r+1}, \ldots, i_n)$ by (α), (β), ..., (μ); so that the series summed in the manner explained at the top of p. 162 will be written as

$$\sum_{(\alpha)(\beta)\ldots(\mu)} a. \tag{29}$$

Further, by $\sum_{\alpha=0}^{I} a$, we denote the sum in which i_1 ranges from 0 to I_1, i_2 from 0 to I_2, ..., i_p from 0 to I_p, and by $x^{(\alpha)}$ we denote $x_1^{i_1} x_2^{i_2} \ldots x_p^{i_p}$.

Let us then assume (i.) that the condition of finitude is satisfied, (ii.) that the series (29) is convergent, and (iii.) that the theorem holds in its most general form for any number of indices less than n. Let

$$s_\alpha = \sum_{(\beta)\ldots(\mu)} a. \tag{30}$$

Then, since $$\sum_{\beta,\ldots,\mu=0}^{m} a = \Delta_{(\alpha)} s_{i_1-1,\ldots,i_p-1,m_{p+1},\ldots,m_n},$$

it follows, from the condition of finitude, that

$$|s_\alpha| < 2^p C \tag{31}$$

and that

$$\sum_{(\alpha)} s_\alpha x^{(\alpha)} \tag{32}$$

is absolutely convergent. And, since

$$|a_{i_1,i_2,\ldots,i_n}| = |\Delta_{(1,2,\ldots,n)} s_{i_1-1,\ldots,i_n-1}| < 2^n C,$$

the series

$$\sum_{(\alpha)} a x^{(\alpha)} \tag{33}$$

is also absolutely convergent. We shall prove further that

$$\sum_{(\beta)\ldots(\mu)} \sum_{(\alpha)} a x^{(\alpha)} \tag{34}$$

is convergent and equal to (32).

13.

Our first step will be to prove that

$$(35) \qquad \sum_{(\mu)} \sum_{(a)} a x^{(a)}$$

is convergent and equal to

$$(36) \qquad \sum_{(a)} x^{(a)} \sum_{(\mu)} a,$$

which is convergent for the same reasons as (32) and (33).

Let

$$\sum_{\mu=0}^{m} a = b_{a,m}$$

and

$$\sum_{\mu=0}^{\infty} a = \lim_{(m)} b_{a,m} = b_a \text{ *}$$

(m of course being a *group* of suffixes). We have to prove that

$$\lim_{(m)} \sum_{(a)} (b_{a,m} - b_a) x^{(a)} = 0.$$

Now

$$\left| \left(\sum_{a=0}^{\infty} - \sum_{a=0}^{I-1} \right) (b_{a,m} - b_a) x^a \right|$$

$$< 2^{r+1} C \frac{1-(1-x_1^{I_1})(1-x_2^{I_2})\dots(1-x_p^{I_p})}{(1-x_1)(1-x_2)\dots(1-x_p)} < 2^{r+1} C \frac{x_1^{I_1}+\dots+x_p^{I_p}}{(1-x_1)\dots(1-x_p)},$$

since

$$| b_{a,m} - b_a | < 2^{r+1} C.$$

We can choose I so that this is $< \sigma$. Then, I being fixed, we can choose M so that $| b_{a,m} - b_a | < \sigma / I_1 I_2 \dots I_p$ for all values of $(m) > M$, and all values of $(a) \leqslant I$; thus

$$\left| \sum_{(a)}^{I-1} (b_{a,m} - b_a) x^{(a)} \right| < \sigma \quad \text{and} \quad \left| \sum_{(a)} (b_{a,m} - b_a) x^{(a)} \right| < 2\sigma.$$

Hence (35) is convergent and equal to (36).

14.

This argument can now be repeated. Suppose that (λ) is the group of suffixes immediately preceding (μ). We have to show that

$$(37) \qquad \sum_{(\lambda)(\mu)} \sum_{(a)} a x^{(a)}$$

---

* The existence of this limit is, of course, implied in our data.

is convergent and equal to

(38) $$\sum_{(\alpha)} x^{(\alpha)} \sum_{(\lambda)(\mu)} a,$$

which is convergent for the same reasons as the series (30), (33), and (36). To prove this we have only to observe that (37) may (after § 13) be written in the form

$$\sum_{(\lambda)} \sum_{(\alpha)} x^{(\alpha)} \sum_{(\mu)} a$$

and that a repetition of the preceding argument with $\sum_{(\mu)} a$ in place of a proves that this is convergent and equal to (38).

By repeating this line of argument as often as may be necessary we conclude finally that (34) is convergent and equal to (32).

15.

We are now in a position to prove the theorem. For

$$\lim_{(\beta)\ldots(\mu)} f = \sum_{(\beta)\ldots(\mu)} \sum_{(\alpha)} a x^{(\alpha)}$$

(since the theorem holds for any number of indices less than n) and therefore is equal to $\sum_{(\alpha)} x^{(\alpha)} \sum_{(\beta)\ldots(\mu)} a$ (by §§ 13, 14). Hence, by a further application of the theorem for p indices,

$$\lim_{(\alpha)(\beta)\ldots(\mu)} f = \lim_{(\alpha)} \sum_{(\alpha)} x^{(\alpha)} \sum_{(\beta)\ldots(\mu)} a = \sum_{(\alpha)(\beta)\ldots(\mu)} a.$$

The theorem is therefore true for n indices if it is true for any number less than n; and therefore it is true generally.

16. *Multiplication of Series.*

It is well known that from Abel's theorem we can at once deduce that, *if the three series*

$$\sum a_i, \quad \sum b_i, \quad \sum c_i,$$

where $$c_i = \sum_{(k+l=i)} a_k b_l,$$

are convergent, the third series is the product of the other two. We have in fact only to make the first two series absolutely convergent by introducing a factor x^i in each term, to multiply the resulting power series, and to proceed to the limit.

By an exactly similar process we deduce from the theorem proved in § 15 that, *if the three series*

$$\sum a_{i_1, i_2, \ldots, i_n}, \quad \sum b_{i_1, i_2, \ldots, i_n}, \quad \textit{and} \quad \sum c_{i_1, i_2, \ldots, i_n},$$

where $$c_{i_1, i_2, \ldots, i_n} = \sum_{(k_1+l_1 = i_1, \ldots, k_n+l_n = i_n)} a_{k_1, \ldots, k_n} b_{l_1, \ldots, l_n},$$

satisfy the condition of finitude and are convergent when summed in the same way (e.g., in the way specified by $\sum_{(1, 2, \ldots, p)(p+1, \ldots, q)\ldots(r+1, \ldots, n)}$*), then the third series is the product of the first two.*

Of course similar theorems can be proved for the product of any number of series.

17. *Mean Value Theorems for the General Series.*

It is easy to prove by the method of § 11 that, if $s^{(k)}_{i_1, \ldots, i_n}$ is the k-th arithmetic mean of $s_{i_1, \ldots, i_n}$, and $| s^{(k)} | < C$ for all suffixes, and

$$\lim_{(1, 2, \ldots, n)} s^{(k)} = s,$$

then
$$\lim_{(1, 2, \ldots, n)} f = s.$$

The form of the arithmetic mean theorem corresponding to the general theorem of §§ 11–15 is as follows:—

Let Σ' denote that a series is "summed" by taking any finite number of arithmetic means. *Suppose that*

$$\sum_{(\alpha)}{}' \sum_{(\beta)}{}' \ldots \sum_{(\mu)}{}' a$$

is determinate and equal to s, *and that a number* C *can be assigned such that the various quantities which we pass through before we arrive at* s *are all less than* C; *then*
$$\lim_{(\alpha)(\beta)\ldots(\mu)} f = s.$$

NOTE IN ADDITION TO A FORMER PAPER ON CONDITIONALLY CONVERGENT MULTIPLE SERIES

By G. H. HARDY.

[Received March 23rd, 1904.—Read April 14th, 1904.]

IN a paper which appeared recently in these *Proceedings** I proved the convergence of a general class of n-ple series, of which

$$\Sigma \frac{\begin{matrix}\cos\\ \sin\end{matrix}(i_1\theta_1+i_2\theta_2+\ldots+i_n\theta_n)}{(i_1a_1+i_2a_2+\ldots+i_na_n)^\rho}$$

is typical. Here $a_1, a_2, \ldots, a_n, \rho$ are all real and positive, and no one of $\theta_1, \ldots, \theta_n$ is a multiple of 2π. In that paper I was concerned entirely with *proper multiple series*; series of the type which, according to the notation developed by Prof. Bromwich and myself in the preceding paper, would be denoted by $\sum_{(1, 2, \ldots, n)}$.

I wish in this note to point out that all these series are convergent also when summed according to the type $\sum_{(1, 2, \ldots, p)(p+1, \ldots, q)\ldots(r+1, \ldots, n)}$ or $\sum_{(\alpha)(\beta)\ldots(\mu)}$. This follows at once from the following lemma, which is an obvious extension of a lemma proved by Pringsheim for double series.

LEMMA.—*The quantity* $\overline{\lim}_{(1, 2, \ldots, n)} s_{i_1, i_2, \ldots, i_n}$

is not increased, and the quantity

$$\underline{\lim}_{(1, 2, \ldots, n)} s_{i_1, i_2, \ldots, i_n}$$

is not decreased, by replacing the single bracket $(1, 2, \ldots, n)$ *by any system of brackets* $(\alpha)(\beta)\ldots(\mu)$.

To prove this it is evidently enough to prove that

$$\overline{\lim}_{(1, 2, \ldots, n)} s \geqslant \overline{\lim}_{(1, 2, \ldots, p)} \overline{\lim}_{(p+1, \ldots, n)} s \quad \text{or} \quad \geqslant \overline{\lim}_{(\alpha)} \overline{\lim}_{(\beta)} s,$$

say. Denote the quantity on the left by L; then, however small be σ, we can determine I so that if $i > I$ then $s < L+\sigma$.

* "On the Convergence of Certain Multiple Series," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 124.

Making (β) tend to infinity, we deduce $\lim_{(\beta)} s \leqslant L+\sigma$ for $(\alpha) > I$, and so $\lim_{(\alpha)(\beta)} s \leqslant L$.

The lemma is therefore proved.

Now let a, u* be two systems of quantities satisfying the conditions of § 4 of my former paper. I proved there that $\sum_{(1,\ldots,n)} au$ is convergent, and the same argument shows that $\sum_{(\beta)} au$ is convergent. Now

$$\sum_{(\overline{1, 2, \ldots, n})} au = \sum_{(\underline{1, 2, \ldots, n})} au = S,$$

say. Hence, by the lemma, $\sum_{(\alpha)(\beta)} au \leqslant S$ and also $\sum_{(\alpha)(\beta)} au \geqslant S$. That is to say, $\sum_{(\alpha)(\beta)} au$ is convergent and $= S$. Similarly, we can show (by repeating this argument a finite number of times) that, if we divide the indices into any number of groups $(\alpha)(\beta)\ldots(\mu)$, the resulting series is convergent. The most interesting special case of this theorem is that the series

$$\sum_1^\infty \sum_1^\infty \ldots \sum_1^\infty \frac{\begin{matrix}\cos\\ \sin\end{matrix}(i_1\theta_1 + \ldots + i_n\theta_n)}{(i_1 a_1 + \ldots + i_n a_n)^\rho}$$

is convergent when the summations are carried out *successively*.

---

* I write a for what was α in the former paper.

THE APPLICATION OF BASIC NUMBERS TO BESSEL'S AND LEGENDRE'S FUNCTIONS

By Rev. F. H. JACKSON.

[Received in revised form* June 2nd, 1904.]

CONTENTS.

PART I.

1.

Let $[n]$ denote $\dfrac{p^n-1}{p-1}$; then, since $[1] = 1$,

$$[2] = 1+p, \qquad [3] = 1+p+p^2,$$

and $$[-1] = -p^{-1}, \qquad [-2] = -p^{-1}-p^{-2}, \qquad \ldots.$$

The number $[n]$ is analogous to the natural number n. Various functions analogous to the functions x^n, $\exp(x)$, $\Gamma(x)$, $J_n(x)$, $P_n(x)$ may be formed, in which these numbers $[1]$, $[2]$, ... occupy the place taken by the natural numbers in the ordinary functions of analysis. Such generalized functions may be obtained as solutions of differential equations analogous to the equations satisfied by the simpler functions. In any number $[n]$ a base p is implied, and, if the base $p = 1$, the number reduces, in the limit, to the natural number n. It may be convenient to call $[n]$ the *basic num-*

* Four papers, two communicated in March and two in April, have been condensed into a single paper.

ber n, and the functions formed in this manner *basic functions*. In this paper the application of the numbers will be directed towards obtaining the following extension of Neumann's addition theorem for the function J_0.

$$\mathrm{J}_0(be^{i\beta}, ce^{i\gamma}) = J_{[0]}(b)\,\mathfrak{J}_{[0]}\left(\frac{c}{p}\right) - 2p\cos a\, J_{[1]}(b)\,\mathfrak{J}_{[1]}\left(\frac{c}{p}\right)$$

$$+2p^4\cos 2a\, J_{[2]}(b)\,\mathfrak{J}_{[2]}\left(\frac{c}{p}\right) - \ldots + (p-1)\,\phi(bc\beta\gamma)$$

$$(\beta-\gamma = a).$$

This is only one of an infinite number of independent addition theorems, all of which reduce, when $p = 1$, to Neumann's addition theorem for $J_0(R)$. The form of the general series which contains the above and a similar series symmetrical in b and c as particular cases will be indicated. The second part of the paper treats of the series corresponding to the various transformations of Legendre's functions, especially the expansions of $P_n(\cos\theta)$. Incidentally it is shown that

$$(y-x)^{-1} = \Sigma\,[2n+1]\,P_{[n]}(x)\,Q_{[n]}(y),$$

and the summation of a case of $F([a][\beta][\gamma][\delta][\epsilon])$ is effected.

2.

The results numbered (1)–(7) in this article are such as will be required in subsequent work. The theorems corresponding to the binomial and exponential theorems are well known in other forms, but it seems convenient to collect them together and express them in the form most useful for reference. If n be a positive integer, we take

$$(1-x)_n = (1-x)(1-px)(1-p^2x)\ldots(1-p^{n-1}x).$$

In general $$(1-x)_n = \prod_{r=0}^{\infty}\frac{(1-p^{n-r-1}x)}{(1-p^{-r-1}x)} \quad (p > 1). \tag{α}$$

If, however, $p < 1$, the proper infinite product expression for $(1-x)_n$ is

$$(1-x)_n = \mathop{\mathrm{L}}_{\kappa=\infty}\frac{(1-x)(1-px)\ldots(1-p^{\kappa}x)}{(1-p^nx)(1-p^{n+1}x)\ldots(1-p^{n+\kappa}x)}. \tag{β}$$

The expansion of these products in infinite series has been considered by many writers, and we may express the analogue of the binomial theorem as

$$(1-x)_n = 1-\frac{[n]}{[1]}x+p\frac{[n][n-1]}{[2]!}x^2-\ldots$$

$$+(-1)^r p^{\frac{1}{2}r(r-1)}\frac{[n][n-1]\ldots[n-r+1]}{[r]!}x^r+\ldots. \tag{1}$$

The series is convergent if $p > 1$ and $x < p$, if $p < 1$ and $x < p^{-n}$, if $p = 1$ and $x < 1$.

From (α) and (β) we obtain without difficulty

$$\frac{(1-x)_{-n}}{(1-p^{-n-1}x)} = (1-x)_{-n-1}, \qquad \frac{(1-x)_{-n}}{(1-p^{-n-1}x)(1-p^{-n-2}x)} = (1-x)_{-n-2}, \tag{2}$$

which will be required in subsequent work.

The Function E_p:—If in series (1) we replace x by $\lambda(1-p)/p^n$, and make n infinite, we obtain

$$\mathop{\mathrm{L}}_{n=\infty} \left(1+\frac{\lambda(p-1)}{p^n}\right)_n = 1+\frac{\lambda}{[1]!}+\frac{\lambda^2}{[2]!}+\dots. \tag{γ}$$

The series is absolutely convergent if $p > 1$, and is convergent for all other values of p, subject to obvious limitations of λ. The infinite product is convergent in form (α) when $p > 1$, and in form (β) when $p < 1$. We therefore write the function (γ) as $E_p(\lambda)$, the analogue of the exponential function.

If we invert the base p, the number $[1]$ is unchanged, but the basic number $[r]$ is transformed into $p^{1-r}[r]$, and the function $E_p(x)$ becomes

$$E_{p^{-1}}(x) = 1+\frac{x}{[1]}+p\frac{x^2}{[2]!}+\dots+p^{\frac{1}{2}r(r-1)}\frac{x^r}{[r]!}+\dots.$$

The following properties may be obtained without difficulty:—

$$E_p(x)\,E_{p^{-1}}(y) = 1+\frac{(x+y)}{[1]}+\frac{(x+y)(x+py)}{[2]!}+\frac{(x+y)(x+py)(x+p^2y)}{[3]!}+\dots, \tag{3}$$

$$E_p(a)\,E_{p^{-1}}(-a) = 1 = E_p(-a)\,E_{p^{-1}}(a), \tag{4}$$

which is the analogue of $\exp(-a) = 1/\exp(a)$,

$$E_{p^2}\left(\frac{\lambda}{[2]}\right) = 1+\frac{\lambda}{[2]}+\frac{\lambda^2}{[2][4]}+\dots+\frac{\lambda^r}{\{2r\}!}+\dots. \tag{5}$$

The use of the symbol $\{2r\}!$ to denote $[2][4]\dots[2r]$ will be convenient in lengthy expressions.

It is well known that

$$\begin{aligned} F([a][\beta][\gamma]p^{\gamma-a-\beta}) &= 1+\frac{[a][\beta]}{[1][\gamma]}p^{\gamma-a-\beta}+\dots \\ &= \prod_{n=0}^{n=\infty}\frac{[\gamma-a+n][\gamma-\beta+n]}{[\gamma-a-\beta+n][\gamma+n]}. \end{aligned} \tag{6}$$

If we invert the base p, the series becomes $F([\alpha][\beta][\gamma]p)$, while the form of the infinite product remains unchanged. Both series are, however, required to fully represent the infinite product, because, while the product is absolutely convergent for all values of the base p, the series $F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta})$ is absolutely convergent if $p > 1$ and the series $F([\alpha][\beta][\gamma]p)$ is absolutely convergent if $p < 1$. Both series are, however, convergent for all other values of p including unity, subject to the condition $\gamma-\alpha-\beta > 1$.

The infinite product, when expressed in terms of the generalized gamma function, takes the form

$$\frac{1}{p^{\alpha\beta}}\frac{\Gamma_p([\gamma-\alpha-\beta])\,\Gamma_p([\gamma])}{\Gamma_p([\gamma-\alpha])\,\Gamma_p([\gamma-\beta])},$$

and this is equal to $F([\alpha][\beta][\gamma]p)$ or $F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta})$, subject to the conditions stated above. The function Γ_p is defined, p. 361, Ser. 2, Vol. 1. Some detailed properties of this function and its derivatives will be found in a paper in *Proc. R.S. London*, Vol. LXXIV. (1904).

3. *The Functions $J_{[n]}$ and $\mathfrak{J}_{[n]}$.*

We define $J_{[n]}(\lambda, x)$ as $\displaystyle\sum_{r=0}^{r=\infty}(-1)^r\frac{\lambda^{n+2r}x^{[n+2r]}}{[n+r]!\,[r]!\,(2)_n(2)_{n+r}}$,

in which $[n+r]! = \Gamma_p([n+r+1])$ and $(2)_n$ satisfies the relation

$$(2)_n\Gamma_p([n+1]) = [2]^n\,\Gamma_{p^2}([n+1]).$$

The function $(2)_n$ reduces to 2^n if the base $p = 1$. It will be convenient to denote $(2)_n\Gamma_p([n+1])$ by $\{2n\}!$, a function which has the following difference equation :—

$$\{2n\}! = [2n]\,\{2n-2\}!.$$

The complete solution of the differential equation satisfied by $J_{[n]}$ is given in *Proc. R.S. Edin.*, Session 1903–1904. If we invert the base p in the function

$$J_{[n]}(\lambda) = \Sigma(-1)^r\frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!},$$

we obtain the series

$$p^{n^2}\Sigma(-1)^r\frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}p^{2r(n+r)}$$

and denote this by $p^{n^2}\mathfrak{J}_{[n]}(\lambda)$.

It has been shown in result (5) that

$$E_{p^2}(x) = \Sigma \frac{x^r [2]^r}{\{2r\}!};$$

therefore $$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = 1 + \frac{\lambda t}{\{2\}!} + \frac{\lambda^2 t^2}{\{4\}!} + \ldots$$

and $$E_{p^2}\left(-\frac{\lambda t}{[2]}\right) = 1 - \frac{\lambda t^{-1}}{\{2\}!} + \frac{\lambda^2 t^{-2}}{\{4\}!} - \ldots .$$

Taking the product of these, we obtain, on arranging according to powers of t,

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]t}\right) = J_{[0]}(\lambda) + tJ_{[1]}(\lambda) + t^2 J_{[2]}(\lambda) + \ldots$$
$$- t^{-1} J_{[1]}(\lambda) + t^{-2} J_{[2]}(\lambda) - \ldots,$$

which, since* $$J_{[n]} = (-1)^n J_{[-n]},$$

may be written $$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda t}{[2]}\right) = \sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\lambda).$$

Similarly we can obtain

$$E_{p^{-2}}\left(\frac{\lambda t}{[2]}\right) E_{p^{-2}}\left(-\frac{\lambda t}{[2]}\right) = \sum_{n=-\infty}^{+\infty} p^{n^2} t^n \mathfrak{J}_{[n]}\left(\frac{\lambda}{p}\right).$$

Both of these are analogous to

$$\exp\left\{\frac{\lambda}{2}\left(t - \frac{1}{t}\right)\right\} = \sum_{-\infty}^{+\infty} t^n J_n(\lambda).$$

Writing at length

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]t}\right) = J_{[0]}(\lambda) + tJ_{[1]}(\lambda) + \ldots + t^r J_{[r]}(\lambda) + \ldots$$
$$- \frac{1}{t} J_{[1]}(\lambda) + \frac{1}{t^2} J_{[2]}(\lambda) - \ldots, \quad (8)$$

$$E_{p^{-2}}\left(-\frac{\lambda t}{[2]}\right) E_{p^{-2}}\left(\frac{\lambda}{[2]t}\right) = \mathfrak{J}_{[0]}\left(\frac{\lambda}{p}\right) - pt\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right) + \ldots$$
$$+ (-)^r p^{r^2} t^r \mathfrak{J}_{[r]}\left(\frac{\lambda}{p}\right) - \ldots + \frac{p}{t}\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right) + \frac{p^4}{t^2}\mathfrak{J}_{[2]}\left(\frac{\lambda}{p}\right) + \ldots, \quad (9)$$

we see that the product of the left-hand sides of (8) and (9) is unity, since $E_p(a)E_{p^{-1}}(-a) = 1$, as was shown in result (14). This gives rise to various interesting results, by equating to zero the coefficients of various

* *Trans. R.S. Edin.*, Vol. XLI., Part 1, 1904, pp. 112, 113.

powers of t in the product of the series on the right-hand sides of the above expressions. By considering the terms independent of t we obtain

$$1 = J_{[0]}(\lambda)\,\mathfrak{J}_{[0]}\left(\frac{\lambda}{p}\right)+2p\,J_{[1]}(\lambda)\,\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right)+\ldots+2p^{r^2}J_{[r]}(\lambda)\,\mathfrak{J}_{[r]}\left(\frac{\lambda}{p}\right)+\ldots.$$

In a paper (*Trans. R.S. Edin.*, Vol. XLI.) a result

$$1 = J_{[0]}(\lambda)\,\mathfrak{J}_{[0]}(\lambda)+\frac{[4]}{[2]}J_{[1]}(\lambda)\,\mathfrak{J}_{[1]}(\lambda)+\ldots+p^{r(r-1)}\frac{[4r]}{[2r]}J_{[r]}(\lambda)\,\mathfrak{J}_{[r]}(\lambda) \quad (9\text{A})$$

is obtained, and subsequently the general form including an infinite number of theorems similar to the above is given. This is of interest as showing that an infinite number of forms of the general addition theorem exist.

4.

By (8), we have

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = \frac{1}{E_{p^2}\left(-\frac{\lambda}{[2]\,t}\right)}\left\{J_{[0]}+tJ_{[1]}+t^2J_{[2]}+\ldots-\frac{1}{t}J_{[1]}+\frac{1}{t^2}J_{[2]}-\ldots\right\},$$

but
$$E_{p^2}\left(-\frac{\lambda}{[2]\,t}\right)E_{p^{-2}}\left(\frac{\lambda}{[2]\,t}\right) = 1;$$

therefore
$$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = E_{p^{-2}}\left(\frac{\lambda}{[2]\,t}\right)\left\{J_{[0]}+tJ_{[1]}+\ldots-\frac{1}{t}J_{[1]}+\ldots\right\}. \quad (10)$$

Expanding the exponential basic functions, and equating coefficients of the various powers of t, from the terms independent of t, we get

$$1 = J_{[0]}(\lambda)+\frac{\lambda}{[2]}J_{[1]}(\lambda)+p^2\frac{\lambda^2}{[2][4]}J_{[2]}(\lambda)+\ldots+p^{r(r-1)}\frac{\lambda^r}{\{2r\}!}J_{[r]}(\lambda)+\ldots.$$

This theorem is a particular case of a more general theorem for $J_{[n]}(\lambda, x)$. (*Trans. R.S. Edin.*, Vol. XLI., Part 1, Art. 8, p. 25.) By equating the coefficients of t^n in (10), we obtain

$$\frac{\lambda^n}{\{2n\}!} = J_{[n]}+\frac{\lambda}{\{2\}!}J_{[n+1]}+p^2\frac{\lambda^2}{\{4\}!}J_{[n+2]}+\ldots+p^{r(r-1)}\frac{\lambda^r}{\{2r\}!}J_{[n+r]}. \quad (11)$$

Equate the coefficients of t^{-n}; then we obtain

$$p^{n(n-1)}\frac{\lambda^n}{\{2n\}!}J_{[0]}+p^{(n+1)n}\frac{\lambda^{n+1}}{\{2n+2\}!}J_{[1]}+\ldots \text{ ad inf.}$$

$$= p^{(n-1)(n-2)}\frac{\lambda^{n-1}}{\{2n-2\}!}J_{[1]}-p^{(n-2)(n-3)}\frac{\lambda^{n-2}}{\{2n-4\}!}J_{[2]}+\ldots. \quad (12)$$

Putting n in succession as 1, 2, 3, ..., we obtain from the latter formula

$$J_{[1]} = \frac{\lambda}{[2]} J_{[0]} + p^2 \frac{\lambda^2}{[2][4]} J_{[1]} + p^6 \frac{\lambda^3}{[2][4][6]} J_{[2]} + \dots$$

and $$\frac{\lambda}{[2]} J_{[1]} - J_{[2]} = p^2 \frac{\lambda^2}{[2][4]} J_{[0]} + p^6 \frac{\lambda^3}{[2][4][6]} J_{[1]} + \dots,$$

and so on. From the two expressions just given, we find

$$J_{[2]} = \frac{\lambda^2}{[2][4]} J_{[0]} + p^2 \frac{\lambda^3}{[2][4][6]} \frac{[4]}{[2]} J_{[1]}(\lambda) + \dots,$$

and, finally,

$$J_{[n]}(\lambda) = \frac{\lambda^n}{[n]!\,(2)_n} J_{[0]}(\lambda) + p^2 \frac{\lambda^{n+1}}{[n+1]!\,(2)_{n+1}} J_{[1]}(\lambda) + \dots$$
$$+ p^{r(r-1)} \frac{\lambda^{n+r-1}}{[n+r-1]!\,(2)_{r-1}} J_{[r-1]}(\lambda) + \dots. \quad (13)$$

5.

In this article it will be shown that

$$J_{[n]}(\kappa\lambda) = \kappa^n J_{[n]}(\lambda) - \kappa^{n+1} \frac{\lambda}{[2]} \left(\kappa - \frac{1}{\kappa}\right) J_{[n+1]}(\lambda)$$
$$+ \kappa^{n+2} \frac{\lambda^2}{[2][4]} \left(\kappa - \frac{1}{\kappa}\right)\left(\kappa - \frac{p^2}{\kappa}\right) J_{[n+2]}(\lambda) - \dots, \quad (14)$$

the coefficient of $J_{[n+r]}(\lambda)$ being

$$(-1)^r \kappa^{n+r} \frac{\lambda^r}{[2][4]\dots[2r]} \left(\kappa - \frac{1}{\kappa}\right)\left(\kappa - \frac{p^2}{\kappa}\right)\dots\left(\kappa - \frac{p^{2r-2}}{\kappa}\right).$$

When $p = 1$ this theorem reduces to

$$J_n(\kappa\lambda) = \kappa^n J_n(\lambda) - \lambda\kappa^{n+1}\left(\frac{\mu}{2}\right) J_{n+1}(\lambda) + \lambda^2 \frac{\kappa^{n+2}}{2!}\left(\frac{\mu}{2}\right)^2 J_{n+2}(\lambda) - \dots,$$

in which $\mu = \kappa - \kappa^{-1}$: see Todhunter, *Functions of Laplace, Lamé, and Bessel*, p. 326. A theorem which may be derived from the above is

$$p^{-mn} J_{[n]}(p^m\lambda) = J_{[n]}(\lambda) - \lambda \frac{[2m]}{[2]} (p-1) J_{[n+1]}(\lambda)$$
$$+ \lambda^2 \frac{[2m][2m-2]}{[2][4]} (p-1)^2 p^2 J_{[n+2]}(\lambda) - \dots. \quad (15)$$

As before, we have

$$E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\kappa\lambda}{[2]t}\right)$$

$$= J_{[0]}(\kappa\lambda)+tJ_{[1]}(\kappa\lambda)+\ldots+t^r J_{[r]}(\kappa\lambda)+\ldots-t^{-1}J_{[1]}(\kappa\lambda)+t^{-2}J_{[2]}(\kappa\lambda)-\ldots,$$

$$E_{p^2}\left(\frac{\lambda\kappa t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]\kappa t}\right)$$

$$= J_{[0]}(\lambda)+\kappa t J_{[1]}(\lambda)+\ldots+\kappa^r t^r J_{[r]}(\lambda)+\ldots$$

$$-\kappa^{-1}t^{-1}J_{[1]}(\lambda)+\ldots+(-1)^r\kappa^{-r}t^{-r}J_{[r]}(\lambda)-\ldots.$$

Now, by means of results (3) and (4), we can show that

$$\frac{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\kappa\lambda}{[2]t}\right)}{E_{p^2}\left(\frac{\kappa\lambda t}{[2]}\right) E_{p^2}\left(-\frac{\lambda}{[2]\kappa t}\right)} = \frac{E_{p^2}\left(-\frac{\kappa\lambda}{[2]t}\right)}{E_{p^2}\left(-\frac{\lambda}{[2]\kappa t}\right)} = E_{p^2}\left(-\frac{\kappa\lambda}{[2]t}\right) E_{p^{-2}}\left(\frac{\lambda}{[2]\kappa t}\right)$$

$$= 1-\frac{\lambda}{t}\frac{\left(\kappa-\frac{1}{\kappa}\right)}{[2]}+\frac{\lambda^2}{t^2}\frac{\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)}{[2][4]}-\ldots; \tag{16}$$

so that

$$J_{[0]}(\kappa\lambda)+tJ_{[1]}(\kappa\lambda)+\ldots \text{ ad inf.}-t^{-1}J_{[1]}(\kappa\lambda)+t^{-2}J_{[2]}(\kappa\lambda)-\ldots \text{ ad inf.}$$

$$=\left\{1-\frac{\lambda}{t}\frac{\left(\kappa-\frac{1}{\kappa}\right)}{[2]}+\ldots\right\}$$

$$\times\left\{J_{[0]}(\lambda)+\kappa t J_{[1]}(\lambda)+\ldots-\kappa^{-1}t^{-1}J_{[1]}(\lambda)+\kappa^{-2}t^{-2}J_{[2]}(\lambda)-\ldots\right\}.$$

Equating the coefficients of t^n in these products, we obtain

$$J_{[n]}(\kappa\lambda) = \kappa^n J_{[n]}(\lambda)-\kappa^{n+1}\frac{\lambda}{[2]}\left(\kappa-\frac{1}{\kappa}\right)J_{[n+1]}(\lambda)$$

$$+\kappa^{n+2}\frac{\lambda^2}{[2][4]}\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)J_{[n+2]}(\lambda)-\ldots$$

In a similar manner we can obtain

$$\kappa^n J_{[n]}(\lambda) = J_{[n]}(\kappa\lambda)+\frac{\lambda}{[2]}\left(\kappa-\frac{1}{\kappa}\right)J_{[n+1]}(\kappa\lambda)$$

$$+\frac{\lambda^2}{[2][4]}\left(\kappa-\frac{1}{\kappa}\right)\left(p^2\kappa-\frac{1}{\kappa}\right)J_{[n+2]}(\lambda)+\ldots. \tag{17}$$

Interesting particular cases of these are ($\kappa = \sqrt{2}$)

$$J_{[n]}(\sqrt{2}\lambda) = 2^{\frac{1}{2}n}\left\{J_{[n]}(\lambda) - \frac{\lambda}{[2]} J_{[n+1]}(\lambda) + \frac{\lambda^2}{[2][4]}(2-p^2)J_{[n+2]}(\lambda) - \dots\right\} \tag{18}$$

and ($\kappa = p^m$)

$$J_{[n]}(p^m\lambda) = p^{mn}\left\{J_{[n]}(\lambda) - \lambda\frac{[2m]}{[2]}(p-1)J_{[n+1]}(\lambda) + \dots \right.$$
$$\left. + (-1)^r\lambda^r\frac{[2m]\dots[2m-2r+2]}{[2][4]\dots[2r]}p^{r(r-1)}(p-1)^r J_{[n+r]}(\lambda) + \dots\right\}, \tag{19}$$

$$p^{mn}J_{[n]}(\lambda) = J_{[n]}(p^m\lambda) + \lambda\frac{[2m]}{[2]}(p-1)p^{-m}J_{[n+1]}(p^m\lambda)$$
$$+\lambda^2\frac{[2m][2m-2]}{[2][4]}(p-1)^2p^{-2m}J_{[n+2]}(p^m\lambda) - \dots. \tag{20}$$

The equations (8) and (9) of Art. 3 easily lead to various theorems respecting the functions $J_{[n]}$ and $\mathfrak{J}_{[n]}$ when n is a positive integer. The theorems obtained as above in this way are, however, valid when n is not restricted to positive integral values; this is evident at once if we compare the coefficients of the powers of λ in the series on the right side of (14) with the coefficients of like powers of λ in $J_{[n]}(\kappa\lambda)$. We have from the right side the following terms involving λ^{n+2r}:—

$$(-1)^r\left[\frac{\kappa^n\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!} + \frac{\kappa^{n+1}\left(\kappa-\frac{1}{\kappa}\right)\lambda^{n+2r}}{\{2\}!\,\{2n+2r\}!\,\{2r-2\}!}\right.$$
$$\left. + \frac{\kappa^{n+2}\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)\lambda^{n+2r}}{\{4\}!\,\{2n+2r\}!\,\{2r-4\}!} + \dots\right],$$

which may be expressed as

$$(-1)^r\frac{\kappa^n\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}\left\{1 + \frac{[2r]}{[2]}(\kappa^2-1) + \frac{[2r][2r-2]}{[2][4]}(\kappa^2-1)(\kappa^2-p^2) + \dots\right\}$$
$$\equiv \frac{\kappa^{n+2r}\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}.$$

This holds for all values of n, subject to the interpretation of $\{2n+2r\}!$ as

$$\{2n+2r\}! = (2)_{n+r}\Gamma_p([n+r+1]) = [2]^{n+r}\Gamma_{p^2}([n+r+1]).$$

6.

We have so far introduced two generalized forms of Bessel's function. The function E_p gives us also a third form, which we denote $\mathrm{J}_{[n]}$, as

distinguished from $J_{[n]}$ and $\mathfrak{J}_{[n]}$. $\mathrm{J}_{[n]}$ is perhaps worth noticing on account of its connexion with an expression analogous to $\cos(x\cos\phi)$.

Since $$E_{p^2}\left(\frac{\lambda t}{[2]}\right) = 1+\frac{\lambda t}{[2]}+\frac{\lambda^2 t^2}{[2][4]}+\ldots+\frac{\lambda^r t^r}{\{2r\}!}+\ldots$$

and $$E_{p^{-2}}\left(-\frac{\lambda}{[2]t}\right) = 1-\frac{\lambda t^{-1}}{[2]}+p^2\frac{\lambda^2 t^{-2}}{[2][4]}-\ldots+(1)^r p^{r(r-1)}\frac{\lambda^r t^{-r}}{\{2r\}!}+\ldots,$$

forming the product of these, we obtain, by (3),

$$E_{p^2}\left(\frac{\lambda t}{[2]}\right)E_{1/p^2}\left(-\frac{\lambda}{[2]t}\right) = 1+\frac{\lambda}{[2]}\left(t-\frac{1}{t}\right)+\frac{\lambda^2}{[2][4]}\left(t-\frac{1}{t}\right)\left(t-\frac{p^2}{t}\right)$$
$$+\frac{\lambda^3}{[2][4][6]}\left(t-\frac{1}{t}\right)\left(t-\frac{p^2}{t}\right)\left(t-\frac{p^4}{t}\right)+\ldots. \quad (21)$$

If, however, the product be formed in a series of ascending and descending powers of t, the expression for the product is

$$\mathrm{J}_{[0]}(\lambda)+t\mathrm{J}_{[1]}(\lambda)+t^2\mathrm{J}_{[2]}(\lambda)+\ldots-pt^{-1}\mathfrak{J}_{[1]}\left(\frac{\lambda}{p}\right)+p^4t^{-2}\mathfrak{J}_{[2]}\left(\frac{\lambda}{p}\right)-\ldots, \quad (22)$$

in which $$\mathrm{J}_{[n]}(\lambda) = \Sigma(-1)^r p^{r(r-1)}\frac{\lambda^{n+2r}}{\{2n+2r\}!\,\{2r\}!}.$$

If $p=1$ and $t=e^{i\phi}$, the series (21) becomes $e^{i\lambda\sin\phi}$.

Since the expansion in ascending and descending powers of t is a Laurent series, we have

$$\mathrm{J}_{[n]}(\lambda) = \frac{1}{2\pi i}\int_C E_{p^2}\left(\frac{\lambda t}{[2]}\right)E_{p^{-2}}\left(-\frac{\lambda}{[2]t}\right)\frac{dt}{t^{n+1}},$$
$$p^{n^2}\mathfrak{J}_{[n]}(\lambda) = \frac{1}{2\pi i}\int_C E_{p^2}\left(\frac{\lambda t}{[2]}\right)E_{p^{-2}}\left(-\frac{\lambda}{[2]t}\right)\frac{dt}{t^{1-n}}.$$

Taking C to be a circle of unit radius, so that $t=e^{i\theta}$,

$$\mathrm{J}_{[n]}(\lambda) = \frac{1}{2\pi}\int_0^{2\pi}\left(1+\frac{\lambda}{[2]}(e^{i\theta}-e^{-i\theta})\right.$$
$$\left.+\frac{\lambda^2}{[2][4]}(e^{i\theta}-e^{-i\theta})(e^{i\theta}-p^2e^{-i\theta})+\ldots\right)e^{-ni\theta}d\theta, \quad (23)$$

subject to the uniform convergence of the series. Also

$$p^{n^2}\mathfrak{J}_{[n]}(\lambda) = \frac{1}{2\pi}\int_0^{2\pi}E_{p^2}\left(\frac{\lambda e^{i\theta}}{[2]}\right)E_{p^{-2}}\left(-\frac{\lambda e^{-i\theta}}{[2]}\right)e^{ni\theta}d\theta. \quad (24)$$

7. *Addition Theorems.*

Many of the expressions used in the following analysis are rather complicated: it will be useful, therefore, to compare the expressions with

those which correspond to them in the case of the ordinary Bessel function; the work will, therefore, be on the same lines as the analysis on pp. 25, 26, 27 of Gray and Mathews' *Treatise on Bessel Functions*, and an analogous notation will be used. References will be made in the form [*G.M.*, p. 25, (62)].

We know that

$$E_p(a)E_{p^{-1}}(b) = 1+\frac{(a+b)}{[1]!}+\frac{(a+b)(a+pb)}{[2]!}+\dots.$$

From this we deduce that

$$E_{p^2}\left(\frac{xt}{[2]}\right)E_{p^{-2}}\left(\frac{yt}{[2]}\right) = 1+\frac{(x+y)t}{\{2\}!}+\frac{(x+y)(x+p^2y)t^2}{\{4\}!}+\dots = \mathfrak{E}(x, y, t) \quad (25)$$

and

$$E_{p^2}\left(-\frac{x}{[2]t}\right)E_{p^{-2}}\left(-\frac{y}{[2]t}\right) = 1-\frac{(x+y)t}{\{2\}!}+\frac{(x+y)(x+p^2y)t^2}{\{4\}!}-\dots$$

$$= \mathfrak{E}(-x, -y, t). \quad (26)$$

Forming the product of these, we have

$$E_{p^2}\left(\frac{xt}{[2]}\right)E_{p^2}\left(-\frac{x}{[2]t}\right)E_{p^{-2}}\left(\frac{yt}{[2]}\right)E_{p^{-2}}\left(-\frac{y}{[2]t}\right) = \mathfrak{E}(x, y, t)\,\mathfrak{E}(-x, -y, t). \quad (27)$$

By means of results (8) and (9) the left side of expression (27) may be written in the form

$$\sum_{m=-\infty}^{+\infty} t^m J_{[m]}(x)\times\sum_{r=-\infty}^{+\infty} t^r p^{r^2}\mathfrak{J}_{[r]}\left(\frac{y}{p}\right) = \mathfrak{E}(x, y, t)\,\mathfrak{E}(-x, -y, t). \quad (28)$$

Equating the coefficients of t^n on both sides of this, we obtain

$$\sum_{m=-\infty}^{m=+\infty} p^{(n-m)^2}J_{[m]}(x)\,\mathfrak{J}_{[n-m]}\left(\frac{y}{p}\right) = \frac{(x+y)(x+yp^2)\dots(x+yp^{2n-2})}{\{2n\}!}$$

$$\left\{1-\frac{(x+y)(x+yp^{2n})}{[2n+2][2]}+\frac{(x+y)(x+p^2y)(x+yp^{2n})(x+yp^{2n+2})}{[2n+2][2n+4][2][4]}+\dots\right\}$$

$$= \mathrm{J}_n(x, y). \quad (29)$$

The series on the right side of the above reduces to $J_n(x+y)$, in case $p=1$. A particular case of this theorem is

$$J_{[0]}(x)\,\mathfrak{J}_{[0]}\left(\frac{y}{p}\right)-2pJ_{[1]}(x)\,\mathfrak{J}_{[1]}\left(\frac{y}{p}\right)+\dots+(-1)^n 2p^{n^2}J_{[n]}(x)\,\mathfrak{J}_{[n]}\left(\frac{y}{p}\right)+\dots$$

$$= 1-\frac{(x+y)^2}{\{2\}!\,\{2\}!}+\frac{(x+y)^2(x+p^2y)^2}{\{4\}!\,\{4\}!}-\dots. \quad (30)$$

8.

Neumann has shown that

$$J_0(\sqrt{b^2+2bc\cos a+c^2}) = J_0(b)\,J_0(c)+2\Sigma\,(-1)^s J_s(b)\,J_s(c)\cos(sa).$$

We proceed to obtain the analogous theorem for the function $J_{[n]}$.

We have $$E_{p^2}\left(\frac{\kappa x t}{[2]}\right) E_{p^2}\left(-\frac{x}{[2]\,\kappa t}\right) = \sum_{-\infty}^{+\infty} \kappa^n t^n J_{[n]}(x).$$

Now

$$E_{p^2}\left(\frac{\kappa x t}{[2]}\right) E_{p^2}\left(-\frac{x}{[2]\,\kappa t}\right) = E_{p^2}\left(\frac{\kappa x t}{[2]}\right) E_{p^2}\left(-\frac{\kappa x}{[2]\,t}\right) E_{p^{-2}}\left(\frac{\kappa x}{[2]\,t}\right) E_{p^2}\left(-\frac{x}{[2]\,\kappa t}\right),$$

because the product of the two middle E functions in the expression on the right side of the equation is unity, by theorem (4). The product of the four E functions may also be written

$$\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x)\, E_{p^{-2}}\left(\frac{\kappa x}{[2]\,t}\right) E_{p^2}\left(-\frac{x}{[2]\,\kappa t}\right);$$

so that $$\sum_{n=-\infty}^{+\infty} \kappa^n t^n J_{[n]}(x) = E_{p^{-2}}\left(\frac{\kappa x}{[2]\,t}\right) E_{p^2}\left(-\frac{x}{[2]\,\kappa t}\right) \sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x).$$

[Cf. *G.M.*, p. 25, (62).]

Transforming the E functions in this equation by means of (4), we have

$$\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x) = E_{p^2}\left(-\frac{\kappa x}{[2]\,t}\right) E_{p^{-2}}\left(\frac{x}{[2]\,\kappa t}\right) \sum_{n=-\infty}^{+\infty} \kappa^n t^n J_{[n]}(x). \tag{31}$$

The product of the E functions in this equation is the series

$$1-\left(\kappa-\frac{1}{\kappa}\right)\frac{x}{[2]\,t}+\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)\frac{x^2}{[2][4]\,t^2}-\dots,$$

the analogue of $$\exp\left\{\frac{x}{2t}\left(\kappa-\frac{1}{\kappa}\right)\right\};$$

therefore $$\sum_{n=-\infty}^{+\infty} t^n J_{[n]}(\kappa x) = \left\{1-\left(\kappa-\frac{1}{\kappa}\right)\frac{x}{[2]\,t}+\dots\right\} \sum_{n=-\infty}^{+\infty} \kappa^n t^n J_{[n]}(x).$$

Put $x = r$, $\kappa = e^{i\theta}$, and let

$$i^n \sin_n\theta = \frac{(e^{i\theta}-e^{-i\theta})(e^{i\theta}-p^2e^{-i\theta})(e^{i\theta}-p^4e^{-i\theta})\dots}{(1+p)(1+p^2)(1+p^3)\dots} \text{ to } n \text{ factors};$$

then

$$\sum_{-\infty}^{+\infty} J_{[n]}(re^{i\theta})\,t^n = \left\{1-\frac{ir\sin\theta}{[1]\,t}+\frac{i^2r^2\sin_2\theta}{[2]!\,t^2}-\frac{i^3r^3\sin_3\theta}{[3]!\,t^3}+\dots\right\} \sum_{-\infty}^{+\infty} e^{ni\theta} J_{[n]}(r)\,t^n. \tag{32}$$

Equating coefficients of t^n, we have

$$J_{[n]}(re^{i\theta}) = e^{ni\theta}\left\{J_{[n]}(r) - \frac{ir\sin\theta}{[1]}e^{i\theta}J_{[n+1]}(r) + \frac{i^2r^2\sin_2\theta}{[2]!}e^{2i\theta}J_{[n+2]}(r) - \dots\right\}, \quad (33)$$

analogous to *G.M.*, p. 26, (64), ed. 1895.

If, in this equation, we put $e^{i\theta} = i$, then we obtain

$$J_{[n]}(ir) = i^n\left\{J_{[n]}(r) + \frac{2r}{[2]}J_{[n+1]}(r) + \frac{2(1+p^2)r^2}{[2][4]}J_{[n+2]}(r) + \dots\right\}. \quad (34)$$

9.

In this article we shall obtain briefly the theorems for the function $\mathfrak{J}_{[n]}$ corresponding to those obtained in the last article for $J_{[n]}$.

We have shown in result (9) that

$$E_{p^{-2}}\left(\frac{x\kappa t}{[2]}\right)E_{p^{-2}}\left(-\frac{x}{[2]\kappa t}\right) = \sum_{n=-\infty}^{n=\infty} p^{n^2}\kappa^n\mathfrak{J}_{[n]}\left(\frac{x}{p}\right)t^n.$$

Now

$$E_{p^{-2}}\left(\frac{x\kappa t}{[2]}\right)E_{p^{-2}}\left(-\frac{x}{[2]\kappa t}\right)$$

$$= E_{p^{-2}}\left(\frac{x\kappa t}{[2]}\right)E_{p^{-2}}\left(-\frac{\kappa x}{[2]t}\right)E_{p^2}\left(\frac{\kappa x}{[2]t}\right)E_{p^{-2}}\left(-\frac{x}{[2]\kappa t}\right),$$

and from this we deduce, as in the last article, that

$$\sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{\kappa x}{p}\right)t^n = E_{p^2}\left(\frac{x}{[2]\kappa t}\right)E_{p^{-2}}\left(-\frac{\kappa x}{[2]t}\right)\sum_{n=-\infty}^{+\infty} p^{n^2}\kappa^n\mathfrak{J}_{[n]}\left(\frac{x}{p}\right).$$

The product of the two E functions is, by theorem (8), after some obvious reductions,

$$1 - \left(\kappa - \frac{1}{\kappa}\right)\frac{x}{[2]t} + \left(\kappa - \frac{1}{\kappa}\right)\left(p^2\kappa - \frac{1}{\kappa}\right)\frac{x^2}{[2][4]t^2} - \dots.$$

This is analogous to $\qquad \exp\left\{\frac{x}{2t}\left(\kappa - \frac{1}{\kappa}\right)\right\},$

but differs from the corresponding series obtained previously in connection with $J_{[n]}$. We have now

$$\sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{\kappa x}{p}\right)t^n = \left\{1 - \left(\kappa - \frac{1}{\kappa}\right)\frac{x}{[2]t}\right.$$

$$\left. + \left(\kappa - \frac{1}{\kappa}\right)\left(p^2\kappa - \frac{1}{\kappa}\right)\frac{x^2}{[2][4]t^2} - \dots\right\}\sum_{-\infty}^{+\infty} p^{n^2}\kappa^n\mathfrak{J}_{[n]}\left(\frac{x}{p}\right). \quad (35)$$

Put $x = r$, $\kappa = e^{\iota\theta}$, and let $i^n \sin_n(-\theta)$ denote, as before,

$$\frac{(e^{-\iota\theta}-e^{\iota\theta})(e^{-\iota\theta}-p^2e^{\iota\theta})\dots(e^{-\iota\theta}-p^{2n-2}e^{\iota\theta})}{(1+p)(1+p^2)\dots(1+p^n)};$$

then

$$\sum_{n=-\infty}^{+\infty} p^{n^2}\mathfrak{J}_{[n]}\left(\frac{re^{\iota\theta}}{p}\right)t^n$$

$$=\left\{1+\frac{ir\sin(-\theta)}{[1]\,t}+\frac{i^2r^2\sin_2(-\theta)}{[2]!\,t^2}+\dots\right\}\sum_{n=-\infty}^{+\infty} p^{n^2}e^{n\iota\theta}\mathfrak{J}_{[n]}\left(\frac{r}{p}\right). \quad (36)$$

Equating coefficients of t^n, we obtain

$$\mathfrak{J}_{[n]}\left(\frac{re^{\iota\theta}}{p}\right)=e^{n\iota\theta}\mathfrak{J}_{[n]}\left(\frac{r}{p}\right)+\sum_0^\infty\frac{i^sr^s\sin_s(-\theta)}{[s]!}e^{(n+s)\iota\theta}p^{s(s+2n)}\mathfrak{J}_{[n+s]}\left(\frac{r}{p}\right), \quad (37)$$

analogous to result (33) of the preceding article, and corresponding to *G.M.*, p. 26, (64), ed. 1895.

If in (37) we put $e^{\iota\theta} = i$, we obtain

$$\mathfrak{J}_{[n]}\left(\frac{ir}{p}\right)=\sum_{s=0}^{s=\infty} i^n\frac{2^sr^s}{\{2s\}!}p^{s(s+2n)}\mathfrak{J}_{[n+s]}\left(\frac{r}{p}\right). \quad (38)$$

10.

In result (32) put $r = b$, $\theta = \beta$, and in (36) put $r = c$, $\theta = \gamma$; then multiply the results together: we obtain

$$\sum_{n=-\infty}^{+\infty} J_{[n]}(be^{\iota\beta})t^n\sum_{n=-\infty}^{+\infty}p^{n^2}\mathfrak{J}_{[n]}\left(\frac{ce^{\iota\gamma}}{p}\right)t^n$$

$$=\left\{1-if'(b,c,\beta,\gamma)\frac{t^{-1}}{[1]}+i^2f''(b,c,\beta,\gamma)\frac{t^{-2}}{[2]!}-\dots\right\}$$

$$\times\sum_{-\infty}^{+\infty}e^{n\iota\beta}J_{[n]}(b)t^n\sum_{-\infty}^{+\infty}e^{n\iota\gamma}p^{n^2}\mathfrak{J}_{[n]}\left(\frac{c}{p}\right)t^n, \quad (39)$$

in which

$$f^{(n)}(b,c,\beta,\gamma)=b^n\sin_n\beta-[n]\,b^{n-1}c\sin_{n-1}\beta\sin(-\gamma)$$

$$+\frac{[n][n-1]}{[2]!}b^{n-2}c^2\sin_{n-2}\beta\sin_2(-\gamma)-\dots+(-)^nc^n\sin_n(-\gamma). \quad (40)$$

This expression (40) reduces, when $p = 1$, to $(b\sin\beta+c\sin\gamma)^n$. If $\beta=\gamma=0$, then $f^{(n)}(b,c,\beta,\gamma)=0$. Equation (39) is analogous to *G.M.*, p. 26, (66).

11.

Consider now the expression on the left side of (39), which we wil write for convenience

$$\Sigma J_{(n)}(b\kappa)t^n\,\Sigma p^{n^2}\mathfrak{J}_{[n]}\left(\frac{c\kappa_1}{p}\right)t^n \quad (\kappa=e^{\iota\beta},\ \kappa_1=e^{\iota\gamma})$$

Expressed in terms of the E functions, this is

$$E_{p^2}\left(\frac{b\kappa t}{[2]}\right)E_{p^2}\left(-\frac{b\kappa}{[2]t}\right)E_{p^{-2}}\left(\frac{c\kappa_1 t}{[2]}\right)E_{p^{-2}}\left(-\frac{c\kappa_1}{[2]t}\right).$$

Taking now the product of the first and third E functions, also the product of the second and fourth, we have by (8), after some obvious reductions,

$$\left\{1+\frac{(b\kappa+c\kappa_1)}{[2]}t+\frac{(b\kappa+c\kappa_1)(b\kappa+p^2c\kappa_1)}{[2][4]}t^2+\dots\right\}$$

$$\times\left\{1-\frac{(b\kappa+c\kappa_1)}{[2]}t^{-1}+\frac{(b\kappa+c\kappa_1)(b\kappa+p^2c\kappa_1)}{[2][4]}t^{-2}-\dots\right\}, \tag{41}$$

analogous to $\qquad \exp\left\{\frac{b\kappa+c\kappa_1}{2}\left(t-\frac{1}{t}\right)\right\}.$

The product (41) gives us a Laurent series, in which the coefficients of the powers of t are functions analogous to $J_n(b\kappa+c\kappa_1)$, which we will denote by the symbol $\mathrm{J}_n(b\kappa, c\kappa_1)$. We see that

$$\mathrm{J}_0(b\kappa, c\kappa_1) = 1-\frac{(b\kappa+c\kappa_1)^2}{[2]^2}+\frac{(b\kappa+c\kappa_1)^2(b\kappa+p^2c\kappa_1)^2}{[2]^2[4]^2}-\dots, \tag{42}$$

and in general

$$\mathrm{J}_n(b\kappa, c\kappa_1) = \frac{(b\kappa+c\kappa_1)(b\kappa+p^2c\kappa_1)\dots(b\kappa+p^{2n-2}c\kappa_1)}{[2][4]\dots[2n]}$$

$$\times\left\{1-\frac{(b\kappa+c\kappa_1)(b\kappa+p^{2n}c\kappa_1)}{[2][2n+2]}+\dots\right\}. \tag{43}$$

When n is not integral the finite product in (43) must be replaced by a suitable infinite product, as

$$\frac{1}{\{2n\}!}\prod_{m=0}^{m=\infty}\frac{(b\kappa+p^{2m}c\kappa_1)}{(b\kappa+p^{2n+2m}c\kappa_1)}b^n\kappa^n \qquad (p<1).$$

We have now

$$\sum_{n=0}^{n=\infty}\mathrm{J}_n(be^{i\beta}, ce^{i\gamma})t^n+\sum_{n=1}^{n=\infty}(-1)^n\mathrm{J}_n(be^{i\beta}, ce^{i\gamma})t^{-n}$$

$$=\left\{1-if'(b, c, \beta, \gamma)\frac{t^{-1}}{[1]}+i^2f''(b, c, \beta, \gamma)\frac{t^{-2}}{[2]!}-\dots\right\}$$

$$\times\sum_{n=-\infty}^{+\infty}e^{ni\beta}J_{[n]}(b)t^n\sum_{-\infty}^{+\infty}p^{n^2}e^{ni\gamma}\mathfrak{J}_{[n]}\left(\frac{c}{p}\right)t^n. \tag{44}$$

Equating the coefficients of the various powers of t, we obtain

$$\mathrm{J}_n(be^{i\beta}, ce^{i\gamma}) = C_0-C_1if'(b, c, \beta, \gamma)+C_2i^2f''(b, c, \beta, \gamma)-\dots, \tag{45}$$

which is the analogue of *G. M.*, p. 26, (67).

12.

In this article we shall write down the expressions for $C_0, C_1, \ldots$. Since (*Trans. R.S. Edin.*, Vol. XLI., Part 1, pp. 110–113)

$$J_{[n]}(a) = (-1)^n J_{[-n]}(a) \qquad \text{and} \qquad \mathfrak{J}_{[n]}\left(\frac{b}{p}\right) = (-1)^n \mathfrak{J}_{[-n]}\left(\frac{b}{p}\right),$$

we can write by means of these equations

$$\begin{aligned} C_0 = \;& e^{n\iota\beta} J_{[n]}(b)\, \mathfrak{J}_{[0]}\left(\frac{c}{p}\right) + e^{(n-1)\iota\beta+\iota\gamma} p J_{[n-1]}(b)\, \mathfrak{J}_{[1]}\left(\frac{c}{p}\right) + \ldots \\ & \qquad + e^{n\iota\gamma} p^{n^2} J_{[0]}(b)\, \mathfrak{J}_{[n]}\left(\frac{c}{p}\right) \\ & - e^{(n+1)\iota\beta-\iota\gamma} p J_{[n+1]} \mathfrak{J}_{[1]} + e^{(n+2)\iota\beta-2\iota\gamma} p^4 J_{[n+2]} \mathfrak{J}_{[2]} - \ldots \text{ ad inf.} \\ & - e^{(n+1)\iota\gamma-\iota\beta} p^{(n+1)^2} J_{[1]} \mathfrak{J}_{[n+1]} + e^{(n+2)\iota\gamma-2\iota\beta} p^{(n+2)^2} J_{[2]} \mathfrak{J}_{[n+2]} - \ldots \text{ ad inf.,} \end{aligned} \tag{46}$$

and in general

$$\begin{aligned} C_s = \;& e^{(n+s)\iota\beta} J_{[n+s]} \mathfrak{J}_{[0]} + e^{(n+s-1)\iota\beta+\iota\gamma} p J_{[n+s-1]} \mathfrak{J}_{[1]} + \ldots \\ & \qquad + e^{(n+s)\iota\gamma} p^{(n+s)^2} J_{[0]} \mathfrak{J}_{[n+s]} \\ & - e^{(n+s+1)\iota\beta-\iota\gamma} p J_{[n+s+1]} \mathfrak{J}_{[1]} + e^{(n+s+2)\iota\beta-2\iota\gamma} p^4 J_{[n+s+2]} \mathfrak{J}_{[2]} - \ldots \text{ ad inf.} \\ & - p^{(n+s+1)^2} e^{(n+s+1)\iota\gamma-\iota\beta} J_{[1]} \mathfrak{J}_{n+s+1]} \\ & + p^{(n+s+2)^2} e^{(n+s+2)\iota\gamma-2\iota\beta} J_{[2]} \mathfrak{J}_{[n+s+2]} - \ldots \text{ ad inf.} \end{aligned} \tag{47}$$

This formula is very complicated, but simplifies in two cases, viz., if $\beta = \gamma = 0$; then $f^{(n)}(\beta, \gamma, b, c) = 0$. We have in this case

(i.)

$$\begin{aligned} \mathrm{J}_n(b, c) = J_{[n]}(b)\, \mathfrak{J}_{[0]}\left(\frac{c}{p}\right) + p J_{[n-1]}(b)\, \mathfrak{J}_{[1]}\left(\frac{c}{p}\right) + \ldots + p^{n^2} J_{[0]}(b)\, \mathfrak{J}_{[n]}\left(\frac{c}{p}\right) \\ - \left[p^{(n+1)^2} J_{[1]} \mathfrak{J}_{[n+1]} + p J_{[n+1]} \mathfrak{J}_{[1]}\right] \\ + \left[p^{(n+2)^2} J_{[2]} \mathfrak{J}_{[n+2]} + p^4 J_{[n+2]} \mathfrak{J}_{[2]}\right] - \ldots \\ + (-1)^r \left[p^{(n+r)^2} J_{[r]} \mathfrak{J}_{[n+r]} + p^{r^2} J_{[n+r]} \mathfrak{J}_{[r]}\right] + \ldots \end{aligned} \tag{48}$$

A more general form of this result will be found in *Trans. R.S. Edin.*, Vol. XLI., Part 1, 1904, p. 117.

(ii.) If $p = 1$ and $be^{\iota\beta} + ce^{\iota\gamma}$ be a real quantity, then

$$f^{(n)}(b, c, \beta, \gamma) = (b \sin\beta + c \sin\gamma)^n = 0,$$

Putting $\beta - \gamma = a$, we have the addition theorem for Bessel functions.

(iii.) If we put $be^{\iota\beta}+ce^{\iota\gamma} = a$ (a real quantity),

$$a^2 = b^2+c^2+2bc\cos(\beta-\gamma);$$

also

$$b\sin\beta+c\sin\gamma = 0;$$

therefore $if'(b, c, \beta, \gamma) = \dfrac{bi}{p+1}(e^{\iota\beta}-e^{-\iota\beta})-\dfrac{ci}{p+1}(e^{-\iota\gamma}-e^{\iota\gamma}) = 0$

and

$$i^2f''(b, c, \beta, \gamma) = i^2\left\{b^2\frac{(e^{\iota\beta}-e^{-\iota\beta})(e^{\iota\beta}-p^2e^{-\iota\beta})}{(1+p)(1+p^2)}-bc(1+p)\frac{(e^{\iota\beta}-e^{-\iota\beta})(e^{-\iota\gamma}-e^{\iota\gamma})}{(1+p)(a+p)}\right.$$
$$\left.+c^2\frac{(e^{-\iota\gamma}-e^{\iota\gamma})(e^{-\iota\gamma}-p^2e^{\iota\gamma})}{(1+p)(1+p^2)}\right\}.$$

This expression reduces, since $be^{\iota\beta}-be^{-\iota\beta}+ce^{\iota\gamma}-ce^{-\iota\gamma} = 0$, to

$$\frac{2i^3(1-p^2)\,bc\sin(\beta-\gamma)}{(1+p)(1+p^2)}. \tag{49}$$

The expressions $f^{(2)}, f^{(4)}, \ldots$ are capable of similar reductions, but I defer this, merely remarking that the problem of reduction is to find the simplest form of expressions homogeneous in x and y derived from the product of the two series

$$\left\{1-\frac{x}{[2]t}\left(\kappa-\frac{1}{\kappa}\right)+\frac{x^2}{[2][4]t^2}\left(\kappa-\frac{1}{\kappa}\right)\left(\kappa-\frac{p^2}{\kappa}\right)-\ldots\right\}$$
$$\times\left\{1-\frac{y}{[2]t}\left(\kappa_1-\frac{1}{\kappa_1}\right)+\frac{y^2}{[2][4]t^2}\left(\kappa_1-\frac{1}{\kappa_1}\right)\left(p^2\kappa_1-\frac{1}{\kappa_1}\right)-\ldots\right\}.$$

subject to the condition

$$x\left(\kappa-\frac{1}{\kappa}\right)+y\left(\kappa_1-\frac{1}{\kappa_1}\right) = 0.$$

18.

Referring to the expressions given in Art. 12 for $C_0, C_1, C_2, \ldots$, we see hat, if $n = 0$ and $be^{\iota\beta}+ce^{\iota\gamma}$ be a real quantity, then, since $f'(b, c, \beta, \gamma) = 0$,

$$J_0(be^{\iota\beta}, ce^{\iota\gamma}) = C_0-C_2f''(b, c, \beta, \gamma)+\ldots, \tag{50}$$

nd in this expression

$$C_0 = J_{[0]}(b)\,\mathfrak{J}_{[0]}\left(\frac{c}{p}\right)-2p\cos(\beta-\gamma)J_{[1]}(b)\,\mathfrak{J}_{[1]}\left(\frac{c}{p}\right)+\ldots$$
$$+(-1)^n\,2p^{n^2}\cos n(\beta-\gamma)\,J_{[n]}(b)\,\mathfrak{J}_{[n]}\left(\frac{c}{p}\right)+\ldots,$$

$$C_2 f'' = -\frac{2i(1-p^2)\,bc\sin(\beta-\gamma)}{(1+p)(1+p^2)}$$

$$\times\{[e^{2\iota\beta}J_{[2]}\mathfrak{J}_{[0]}+e^{\iota(\beta+\gamma)}pJ_{[1]}\mathfrak{J}_{[1]}+e^{2\iota\gamma}p^4J_{[0]}\mathfrak{J}_{[2]}$$

$$-p\,(e^{4\iota\beta}J_{[3]}\mathfrak{J}_{[1]}+p^8e^{4\iota\gamma}J_{[1]}\mathfrak{J}_{[3]})\,e^{-\iota(\beta+\gamma)}$$

$$+p^4\,(e^{6\iota\gamma}J_{[4]}\mathfrak{J}_{[2]}+p^{12}e^{6\iota\gamma}J_{[2]}\mathfrak{J}_{[4]})e^{-2\iota(\beta+\gamma)}-\dots\}+\dots$$

$$+i^r f^{(r)}C_{(r)}+\dots, \tag{51}$$

which is the extension of Neumann's addition theorem, we may write the theorem

$$\mathrm{J}_0(be^{\iota\beta},\, ce^{\iota\gamma}) = J_{[0]}(b)\,\mathfrak{J}_{[0]}\left(\frac{c}{p}\right)-2p\cos a\,J_{[1]}(b)\,\mathfrak{J}_{[1]}\left(\frac{c}{p}\right)$$

$$+2p^4\cos 2a\,J_{[2]}(b)\,\mathfrak{J}_{[2]}\left(\frac{c}{p}\right)-\dots+(p-1)\,\phi(b,c,\beta,\gamma)$$

$$(\beta-\gamma=a). \tag{52}$$

Finally we may remark that the addition theorem investigated above is one of an infinite number of such addition theorems. The terms in the general theorem being of the form

$$p^{n(n-\nu)}\frac{[4n\nu]}{[2n\nu]}\cos n(\beta-\gamma)\,J_{[n]}(b)\,\mathfrak{J}_{[n]}\left(\frac{c}{p^{1-\nu}}\right), \tag{52A}$$

ν being an arbitrary integer, if $\nu = 0$, we have the set of theorems which we have been discussing.

PART II.

14. *Transformations of the Function* $P_{[n]}$.

It is well known that

$$F(a,\beta,\gamma,x) = (1-x)^{\gamma-a-\beta}F(\gamma-a,\gamma-\beta,\gamma,x)$$

$$= (1-x)^{-a}F\left(a,\gamma-\beta,\gamma,\frac{x}{x-1}\right)$$

$$= (1-x)^{-\beta}F\left(\beta,\gamma-a,\gamma,\frac{x}{x-1}\right). \tag{53}$$

These expressions are a set of four equal particular integrals of the differential equation of the hypergeometric series denoted*

$$y_1 = y_2 = y_{17} = y_{18}.$$

* Forsyth, *Treatise on Differential Equations*, 2nd ed., pp. 192, 194.

Let $F([a][\beta][\gamma]x)$ denote the series

$$1+\frac{[a][\beta]}{[1][\gamma]}x+\frac{[a][a+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}x^2+\dots;$$

then we shall show that

$$1+\frac{[a][\beta]}{[1][\gamma]}p^{\gamma-a-\beta}x+\frac{[a][a+1][\beta][\beta+1]}{[1][2][\gamma][\gamma+1]}p^{2(\gamma-a-\beta)}x^2+\dots$$

$$=(1-x)_{\gamma-a-\beta}\left\{1+\frac{[\gamma-a][\gamma-\beta]}{[1][\gamma]}x\right.$$

$$\left.+\frac{[\gamma-a][\gamma-a+1][\gamma-\beta][\gamma-\beta+1]}{[2]!\,[\gamma][\gamma+1]}x^2+\dots\right\} \quad (54)$$

$$=(1-x)_{-a}\left\{1-\frac{[a][\gamma-\beta]}{[1][\gamma]}\frac{px}{(p^{a+1}-x)}\right.$$

$$\left.+\frac{[a][a+1][\gamma-\beta][\gamma-\beta+1]}{[2]!\,[\gamma][\gamma+1]}\frac{p^2x^2}{(p^{a+1}-x)(p^{a+2}-x)}-\dots\right\} \quad (55)$$

$$=(1-x)_{-\beta}\left\{1-\frac{[\beta][\gamma-a]}{[1][\gamma]}\frac{px}{(p^{\beta+1}-x)}\right.$$

$$\left.+\frac{[\beta][[\beta+1][\gamma-a][\gamma-a+1]}{[2]!\,[\gamma][\gamma+1]}\frac{p^2x^2}{(p^{\beta+1}-x)(p^{\beta+2}-x)}-\dots\right\}, \quad (56)$$

If $p=1$, these series reduce to the series in expression (53).

Consider series (55): we may write this series

$$(1-x)_{-a}\left\{1+\frac{[-a][\gamma-\beta]}{[1][\gamma]}\frac{x}{1-p^{-a-1}x}+\dots\right\},$$

and again, by means of result (2), we are able to write this in the form

$$(1-x)_{-a}+\frac{[-a][\gamma-\beta]}{[1][\gamma]}x(1-x)_{-a-1}$$

$$+\frac{[-a][-a-1][\gamma-\beta][\gamma-\beta-1]}{[1][2][\gamma][\gamma+1]}x^2(1-x)_{-a-2}+\dots. \quad (57)$$

Replacing $(1-x)_{-a-r}$ by expansions in convergent series (1), we obtain the double series

$$\left\{1-\frac{[-a]}{[1]}x'+\ldots+(-1)^r p^{\frac{1}{2}r(r-1)}\frac{[-a][-a-1]\ldots[-a-r+1]}{[r]!}x^r+\ldots\right\}$$

$$+\frac{[-a][\gamma-\beta]}{[1][\gamma]}x\left\{1-\frac{[-a-1]}{[1]}x+\ldots\right.$$

$$\left.+(-1)^{r-1}p^{\frac{1}{2}(r-1)(r-2)}\frac{[-a-1][-a-2]\ldots[-a-r+1]}{[r]!}x^{r-1}+\ldots\right\}$$

$$+\frac{[-a][-a-1][\gamma-\beta][\gamma-\beta+1]}{[2]![\gamma][\gamma+1]}x^2\left\{1-\frac{[-a-2]}{[1]}x+\ldots\right.$$

$$\left.+(-1)^{r-2}p^{\frac{1}{2}(r-2)(r-3)}\frac{[-a-2][-a-3]\ldots[-a-r+1]}{[r]!}x^{r-2}+\ldots\right\}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \;. \tag{58}$$

Collect the terms according to powers of x. We obtain the series

$$1+p^{\gamma-a-\beta}\frac{[a][\beta]}{[1][\gamma]}x+\ldots,$$

because the coefficient of x^r, after some obvious reduction, may be written

$$p^{-ra}\frac{[a][a+1]\ldots[a+r-1]}{[r]!}$$

$$\times\left\{1+p\frac{[-r][\gamma-\beta]}{[1][\gamma]}+p^2\frac{[-r][-r+1][\gamma-\beta][\gamma-\beta+1]}{[1][2][\gamma][\gamma+1]}+\ldots\right\}, \tag{59}$$

and the series within the large brackets is, for all values of r, by theorem (7),

$$p^{r(\gamma-\beta)}\frac{\Gamma_p([\gamma])\,\Gamma_p([\beta+r])}{\Gamma_p([\gamma+r])\,\Gamma_p([\beta])};$$

so that when r is a positive integer the expression (59), which is the coefficient of x^r, becomes

$$p^{r(\gamma-a-\beta)}\frac{[a][a+1]\ldots[a+r-1][\beta]\ldots[\beta+r-1]}{[r]![\gamma][\gamma+1]\ldots[\gamma+r-1]}.$$

This establishes the theorem

$$F([a][\beta][\gamma]p^{\gamma-a-\beta}x)=(1-x)_{-a}\left\{1+\frac{[a][\gamma-\beta]}{[1][\gamma]}\,\frac{px}{(x-p^{a+1})}+\ldots\right\}, \tag{60}$$

which is the analogue of $y_1=y_{17}$, among the twenty-four particular integrals of the hypergeometric differential equation.

Since the series F is symmetrical in α and β, we have, by an interchange of α and β,

$$F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}x) = (1-x)_{-\beta}\left\{1+\frac{[\beta][\gamma-\alpha]}{[1][\gamma]}\frac{px}{(x-p^{\beta+1})}+\dots\right\},$$

which corresponds to $y_1 = y_{18}$.

The transformation analogous to

$$F(\alpha, \beta, \gamma, x) = (1-x)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma, x)$$

is due to Heine, being easily deduced from his theorem,

$$\phi[a, b, c, q, x] = \prod_{n=0}^{n=\infty}\frac{\left(1-\frac{abx}{c}q^n\right)}{(1-q^n)}\phi\left[\frac{c}{a}, \frac{c}{b}, c, q, \frac{abx}{c}\right],$$

ϕ denoting Heine's series

$$1+\frac{(1-a)(1-b)}{(1-q)(1-c)}x+\dots,$$

if we put $a = p^{\alpha}$, $b = p^{\beta}$, $c = p^{\gamma}$, $q = p$, $x = p^{\gamma-\alpha-\beta}x$.

The result is

$$F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}x) = (1-x)_{\gamma-\alpha-\beta}F([\gamma-\alpha][\gamma-\beta][\gamma]x). \tag{61}$$

We now proceed to apply these and other special transformations to the functions $P_{[n]}$ and $Q_{[n]}$.

15. *The Function* $P_{[n]}$.

If in theorem (60) of the last article we make

$$\alpha = -\tfrac{1}{2}n, \qquad \beta = \tfrac{1}{2}(n+1), \qquad \gamma = \tfrac{1}{2},$$

and change p into p^2, also making x into x^2/p, we have, after some obvious reductions,

$$\begin{aligned}1-p^{1-n}&\frac{[n][n+1]}{[2]!}x^2+p^{-n}\frac{[n-2][n][n+1][n+3]}{[4]!}x^4-\dots\\ &= \prod_{r=0}^{r=\infty}\frac{(1-p^{2r-1}x^2)}{(1-p^{n+2r-1}x^2)}\left\{1+p^{-n}\frac{[n]^2}{[2]!}\frac{x^2}{(p^{n-3}x^2-1)}\right.\\ &\qquad\left.+p^{-2n}\frac{[n]^2[n-2]^2}{[4]!}\frac{x^4}{(p^{n-3}x^2-1)(p^{n-5}x^2-1)}-\dots\right\}.\end{aligned} \tag{62}$$

We compare the series on the left side of the above expression with the series

$$1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)\,n\,(n+1)(n+3)}{4!}x^4-\dots,$$

which is a particular solution of Legendre's differential equation; I have discussed this series in the more general form in connection with the differential equation in (*Trans. R.S. Edin.*, "Generalized Functions of Legendre and Bessel," Vol. XLI., Art. 5, p. 20). In the case when n is an even positive integer it is easily shown that the series on the left side of (62) is $C.P_{[n]}(x)$,

$$P_{[n]}(x) = \frac{[2n]!}{[n]!\,[n]!\,(2)_n}\left\{x^n - p^3\frac{[n][n-1]}{[2][2n-1]}x^{n-2} + \ldots\right\}.$$

If in result (60) we put $\alpha = -\tfrac{1}{2}n$, $\beta = -\tfrac{1}{2}(n-1)$, $\gamma = \tfrac{1}{2}-n$, and change p into p^2, also x into p^3/x^2, we obtain

$$1 - p^3\frac{[n][n-1]}{[n][2n-1]}x^{-2} + p^8\frac{[n][n-1][n-2][n-3]}{[2][4][2n-1][2n-3]}x^{-4} - \ldots$$

$$= \prod_{r=0}^{r=\infty}\frac{(x^2-p^{2r+3})}{(x^2-p^{2r-n+3})}\left\{1 - \frac{[n]^2}{[2][2n-1]}\frac{p^{n+2}}{(p^{n+1}-x^2)}\right.$$

$$\left. + \frac{[n][n-2]^2}{[2][4][2n-1][2n-3]}\frac{p^{2n+4}}{(p^{n+1}-x^2)(p^{n-1}-x^2)} - \ldots\right\}. \quad (63)$$

Multiplying both sides of this equation by $\dfrac{\Gamma_p([2n+1])}{\Gamma_p([n+1])\,\Gamma_p([n+1])\,(2)_n}$, we obtain an interesting transformation of $P_{[n]}(x)$. For the differential equation of the functions $P_{[n]}$, $Q_{[n]}$, and its properties, I refer to *Trans. R.S. Edin.*, Vol. XLI. Two other similar theorems may be obtained in the same way, if we substitute the particular values of α, β, γ, given above, in equation (56). Theorem (61) reduces to an identity if we substitute special values, so as to obtain $P_{[n]}(x)$, both sides of the equation becoming $CP_{[n]}(x)$.

16.

In this article the transformations corresponding to

$P_n(\cos\theta)$

$$= \cos^n\theta\left\{1 - \frac{n(n-1)}{2^2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{2^2.4^2}\tan^4\theta - \ldots\right\} \quad (64)$$

$$= \frac{1.3\ldots(2n-1)}{2^n.n!}\left\{\cos n\theta + \frac{1.n}{1.(2n-1)}\cos(n-2)\theta\right.$$

$$\left. + \frac{1.3.n(n-1)}{1.2(2n-1)(2n-3)}\cos(n-4)\theta + \ldots\right\} \quad (65)$$

will be considered.

Take a series

$$x^n\left\{1+\frac{[n][n-1]}{[2][2]}\left(p^2-\frac{p^3}{x^2}\right)\right.$$

$$\left.+\frac{[n][n-1][n-2][n-3]}{[2][4][2][4]}\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)+\dots\right\}, \tag{66}$$

of which the general term is

$$x^n\frac{[n][n-1][n-2]\dots[n-2r+1]}{\{[2][4][6]\dots[2r]\}^2}\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)\dots\left(p^{4r-2}-\frac{p^{2r+1}}{x^2}\right).$$

If now we expand* the products which involve x in each term, we obtain that the coefficient of x^n in (66) is

$$1+p^2\frac{[n][n-1]}{[2][2]}+p^8\frac{[n][n-1][n-2][n-3]}{[2][4][2][4]}+\dots$$

$$+p^{2r^2}\frac{[n]\dots[n-2r+1]}{\{[2]\dots[2r]\}^2}+\dots,$$

and generally that the coefficient of x^{n-2r} is

$$(-1)^r p^{r(r+2)}\frac{[n]\dots[n-2r+1]}{\{[2][4]\dots[2r]\}^2}$$

$$\times\left\{1+p^{2(r+2)}\frac{[n-2r][n-2r-1]}{[2r+2][2r+2]}\frac{[2r+2]}{[2]}+\dots\right\}. \tag{67}$$

The general term of the series within the large brackets in (67) is

$$p^{2s(r+s)}\frac{[n-2r]\dots[n-2r-2s+1]}{\{[2r+2]\dots[2r+2s]\}^2}\frac{[2r+2]\dots[2r+2s]}{[2]\dots[2s]}.$$

When n is positive and integral these series are finite, but for all values of n they are particular cases of series $F([\alpha][\beta][\gamma]x)$. The application of

$$F([\alpha][\beta][\gamma]p^{\gamma-\alpha-\beta}) \doteq \prod_0^\infty\frac{[\gamma-\beta+n][\gamma-\alpha+n]}{[\gamma-\alpha-\beta+n][\gamma+n]}$$

in the case $\alpha=r-\frac{1}{2}n$, $\beta=s-\frac{1}{2}(n-1)$, $\gamma=r+1$, the base p being changed into p^2, gives us

$$1+p^{2(r+2)}\frac{[n-2r][n-2r-1]}{[2][2r+2]}+\dots$$

$$=\frac{[2][4]\dots[2r]}{[2n-1][2n-3]\dots[2n-2r+1]}\frac{[1][3]\dots[2n-1]}{[n]!} \tag{68}$$

or an equivalent expression in infinite products, when n is not integral.

---

* "Series connected with the Enumeration of Partitions," p. 69, (11); (12).

The series (67), which is the coefficient of x^{n-2r}, is then, by (68) and (67),

$$(-1)^r p^{r(r+2)} \frac{[n]\dots[n-2r+1]}{\{[2][4]\dots[2r]\}^2} \frac{[2]\dots[2r]}{[2n-1]\dots[2n-2r+1]} \frac{[1][3]\dots[2n-1]}{[n]!},$$

which is

$$(-1)^r p^{r(r+2)} \frac{[2n]!}{[n]!\,[n]!\,(2)_n} \frac{[n][n-1]\dots[n-2r+1]}{[2]\dots[2r][2n-1]\dots[2n-2r+1]}, \tag{69}$$

namely, the coefficient of x^{n-2r} in the standard form of $P_{[n]}(x)$. When n is not positive and integral we must replace $[2n]!$ and $(2)_n$ by infinite products defined in Art. 8. We have finally

$$P_{[n]}(x) = \sum_{r=0}^{r=\infty} x^n \frac{[n][n-1][n-2]\dots[n-2r+1]}{\{[2][4][6]\dots[2r]\}^2}$$

$$\times\left(p^2-\frac{p^3}{x^2}\right)\left(p^6-\frac{p^5}{x^2}\right)\dots\left(p^{4r-2}-\frac{p^{2r+1}}{x^2}\right). \tag{70}$$

The extension of (64) is thus determined; for, putting $x = \cos\theta$, $p = 1$, the theorem reduces to

$$P_n(\cos\theta) = \cos^n\theta\left\{1-\frac{n(n-1)}{2^2}\tan^2\theta+\dots\right\}.$$

17.

Consider the series

$$\frac{[2n]!}{[n]!\,[n]!\,(2)_n}\left[\frac{1}{(2)_n}\left\{x+\frac{1}{x}\right\}_n-\frac{p}{(2)_{n-2}}\frac{[n][n-1]}{[n][2n-1]}\left\{x+\frac{p^2}{x}\right\}_{n-2}+\dots\right], \tag{71}$$

which, if $p = 1$, reduces term by term to $P_n\left\{\frac{1}{2}\left(x+\frac{1}{x}\right)\right\}$.

The general term of the series within the brackets is

$$(-1)^r p^{r(2r+1)} \frac{1}{(2)_{n-2r}} \frac{[n][n-1]\dots[n-2r+1]}{[2][4]\dots[2r][2n-1]\dots[2n-2r+1]} \left\{x+\frac{p^{2r}}{x}\right\}_{n-2r}.$$

If n be a positive integer,

$$\left\{x+\frac{1}{x}\right\}_n = \left(x+\frac{1}{x}\right)\left(x+\frac{p^2}{x}\right)\left(x+\frac{p^4}{x}\right)\dots \text{ to } n \text{ factors},$$

$$\left\{x+\frac{p^2}{x}\right\}_n = \left(x+\frac{p^2}{x}\right)\left(x+\frac{p^4}{x}\right)\left(x+\frac{p^6}{x}\right)\dots \text{ to } n \text{ factors}.$$

In case n be not a positive integer,

$$\left\{x+\frac{1}{x}\right\}_n=\prod_{r=1}^{\infty}\frac{\left(x+\frac{p^{2n-2r}}{x}\right)}{\left(x+\frac{p^{-2r}}{x}\right)}x^n\quad(p>1)\quad\text{or}\quad\prod_{r=0}^{\infty}\frac{\left(x+\frac{p^{2r}}{x}\right)}{\left(x+\frac{p^{2n+2r}}{x}\right)}x^n\quad(p<1).$$

For all values of n we have shown* that

$$\left\{x+\frac{1}{x}\right\}_n$$
$$=x^n+\frac{[2n]}{[2]}x^{n-2}+p^2\frac{[2n][2n-2]}{[2][4]}x^{n-4}+p^6\frac{[2n][2n-2][2n-4]}{[2][4][6]}x^{n-6}+\dots.$$

If, now, we expand $\left\{x+\frac{1}{x}\right\}_n$, &c., in expression (71), and collect the terms according to powers of x, we find that x^n arises only from the first term of (71), and the term which involves it is $x^n/(2)_n$, and in general the terms involving x^{n-2r} form the series

$$p^{r(r-1)}\frac{[2n]\dots[2n-2r+2]}{[2]\dots[2r].(2)_n}\left\{1-p\frac{[2n-2r][2r]}{[2][2n-1]}\right.$$
$$\left.+p^4\frac{[2n-2r][2n-2r-2][2r][2r-2]}{[2][4][2n-1][2n-3]}-\dots\right\}x^{n-2r}.\qquad(72)$$

The series within the large brackets is a particular case of the general hypergeometric series (7).

When n is positive and integral the sum of (72) is

$$p^{r(r-1)}\frac{\{[2n]\dots[2n-2r+2]\}\{[1][3]\dots[2r+1]\}x^{n-2r}}{\{[2][4]\dots[2r]\}\{[2n-1]\dots[2n-2r+1]\}(2)_n}.$$

If n be not a positive integer, the finite products in this expression will be replaced by appropriate expressions in terms of Γ_p functions, which, for n integral and positive, will reduce to the expression given above. We have then transformed (71) into

$$\frac{[2n]!}{[n]!\,[n]!\,(2)_n(2)_n}\left\{x^n+\frac{[2n]}{[2][2n-1]}x^{n-2}+p^2\frac{[1][3][2n][2n-2]}{[2][4][2n-1][2n-3]}x^{n-4}+\dots\right\},\qquad(73)$$

* *Supra*, Ser. 2, Vol. 1, p. 69, (11).

which is the extension of

$$P_n\left\{\tfrac{1}{2}\left(x+\frac{1}{x}\right)\right\} = \frac{2n!}{n!\,n!\,2^{2n}}\left\{x^n+\frac{1.2n}{2(2n-1)}x^{n-2}+\ldots\right\}.$$

$[n]! = \Gamma_p([n+1])$.—If we make $p=1$ and $x+x^{-1} = 2\cos\theta$, we obtain

$$P_n(\cos\theta) = \frac{2n!}{n!\,n!\,2^{2n}}\left\{\cos n\theta+\frac{1.2n}{2(2n-1)}\cos(n-2)\theta+\ldots\right\}.$$

18.

It is easily established that

$$\lambda^n = \frac{(2)_n[n]!\,[n]!}{[2n+1]!}\left\{[2n+1]P_{[n]}(\lambda)+p^3[2n-3]\frac{[2n+1]}{[2]}P_{[n-2]}(\lambda)\right.$$
$$\left.+p^6[2n-7]\frac{[2n+1][2n-1]}{[2][4]}P_{[n-4]}(\lambda)+\ldots\right\}, \quad (74)$$

for, replacing the functions P by their series expansions, we find that the terms involving λ^{n-2r} give the series

$$p^{r(r+2)}\frac{[2n-2r]!}{\{2n-2r\}!\,\{2r\}!\,[n-2r]!}[2n+1]\lambda^{n-2r}\left[1+\sum_{s=1}^{\infty}(-)^s p^{s(s+1)-2rs}\right.$$
$$\left.\times\frac{\{2r\}!}{\{2r-2s\}!\,\{2s\}!}\,\frac{[2n-1]\ldots[2n-2s+3]}{[2n-2r-1]\ldots[2n-2r-2s+1]}[2n-4s+1]\right].$$

This series reduces identically to zero, as may be seen if we sum the series term by term; for we find that the sum of r terms is for all values of r a factor of the $(r+1)$-th term, and the expression for the sum will vanish by reason of the presence of a zero factor: the above reasoning establishes a more general theorem (cf. *Trans. R.S. Edin., loc. cit.*).

$$\underset{\ldots}{\lambda^n x^{[n]}} = \frac{(2)_n[n]!\,[n]!}{[2n+1]!}\left\{[2n+1]P_{[n]}(\lambda, x)\right.$$
$$\left.+p^3[2n-3]\frac{[2n+1]}{[2]}P_{[n-2]}(\lambda, x)+\ldots\right\}. \quad (75)$$

We compare this with the theorem in Art. 36, p. 20, Todhunter, *Functions of Laplace, Lamé, and Bessel*, and proceed to show that

$$\frac{1}{y-x} = \sum_0^{\infty}(2n+1)Q_n(y)P_n(x)$$

may be extended in the form

$$\frac{1}{\mu-\lambda} = \sum_{n=0}^{n=\infty} [2n+1] Q_{[n]}(\mu) P_{[n]}(\lambda). \tag{76}$$

If $\mu > \lambda$, then we have

$$\frac{1}{\mu-\lambda} = \frac{1}{\mu} + \frac{\lambda}{\mu^2} + \frac{\lambda^2}{\mu^3} + \dots .$$

Now express each power of λ in a series of $P_{[n]}$ functions by means of the theorem (75), and then collect all the terms which involve the same coefficient. Thus $P_{[n]}(\lambda)$ will arise from $\frac{\lambda^n}{\mu^{n+1}}, \frac{\lambda^{n+2}}{\mu^{n+3}}, \dots$, and, for the multiplier of it,

from $\frac{\lambda^n}{\mu^{n+1}}$ we get $\frac{1}{\mu^{n+1}} \frac{[n]!\,[n]!\,(2)_n}{[2n+1]!}[2n+1]$,

from $\frac{\lambda^{n+2}}{\mu^{n+3}}$ we get $\frac{1}{\mu^{n+3}} \frac{[n+2]!\,[n+2]!\,(2)_{n+2}}{[2n+5]!}[2n+1]p^3\frac{[2n+5]}{[2]}$,

from $\frac{\lambda^{n+4}}{\mu^{n+5}}$ we get $\frac{1}{\mu^{n+5}} \frac{[n+4]!\,[n+4]!\,(2)_{n+4}}{[2n+9]!}[2n+1]p^6\frac{[2n+9][2n+7]}{[2][4]}$.

From this we see that the multiplier of $P_{[n]}(\lambda)$ is

$$[2n+1]\frac{[n]!\,[n]!\,(2)_n}{[2n]!}\left\{\mu^{-n-1}+p^3\frac{[n+1][n+2]}{[2][2n+3]}\mu^{-n-3}+\dots\right\}$$
$$= [2n+1]\, Q_{[n]}[\mu].$$

The series $Q_{[n]}$ reduces term by term to Legendre's series Q_n if we make the base p unity. The series

$$Q_{[n]}(\mu, x) = \frac{[n]!\,[n]!\,(2)_n}{[2n+1]}\left\{\mu^{-n-1}x^{[-n-1]}+p^3\frac{[n+1][n+2]}{[2][2n+3]}\mu^{-n-3}x^{[-n-3]}+\dots\right\}$$

will be found discussed as a solution along with $P_{[n]}(\lambda, x)$ of a certain differential equation analogous to Legendre's in *Trans. R.S. Edin.*, Vol. XLI., 1904. We write now

$$\frac{1}{\mu-\lambda} = \sum_{n=0}^{n=\infty} [2n+1] P_{[n]}(\lambda) Q_{[n]}(\mu) \quad (\mu > \lambda,\ \mu > 1). \tag{77}$$

Subject to convergence of the series, we may also write down a more general theorem, viz.,

$$\sum_{n=0}^{n=\infty}[2n+1]P_{[n]}(\lambda,x)\,Q_{[n]}(\mu,y)=\frac{1}{\mu y^{[1]}}+\frac{\lambda x^{[1]}}{\mu^2 y^{[2]}}+\dots. \tag{78}$$

19. *A Case of Summation of* $F([a][\beta][\gamma][\delta][\epsilon]x)$.

In Art. 14 we have shown that

$$F([a][\beta][\gamma]p^{\gamma-a-\beta}x)=(1-x)_{\gamma-a-\beta}F([\gamma-a][\gamma-\beta][\gamma]x). \tag{79}$$

Now $(1-x)_{\gamma-a-\beta}=1-\frac{[\gamma-a-\beta]}{[1]}x+\dots$

$$+(-1)^r p^{\frac{1}{2}r(r-1)}\frac{[\gamma-a-\beta]\dots[\gamma-a-\beta-r+1]}{[r]!}x^r-\dots.$$

Replacing the functions in (79) by their expansions in series of powers of x, and equating the coefficients of equal powers of x, we obtain from the terms involving x^n

$$\frac{[\gamma-a]_n[\gamma-\beta]_n}{[n]!\,[\gamma]_n}+\frac{[\gamma-a]_{n-1}[\gamma-\beta]_{n-1}}{[n-1]!\,[\gamma]_{n-1}}\frac{[a+\beta-\gamma]}{[1]}p^{\gamma-a-\beta}+\dots+\frac{[a+\beta-\gamma]_n}{[n]!}$$
$$=\frac{[a]_n[\beta]_n}{[n]!\,[\gamma]_n}p^{n(\gamma-a-\beta)},$$

in which $\quad [\gamma]_n=[\gamma][\gamma+1][\gamma+2]\dots[\gamma+n-1].$

Change $\gamma-a$ to x, $\gamma-\beta$ to y, γ to z; then

$$\frac{[x]_n[y]_n}{[z]_n}+\Sigma\frac{[n]!}{[r]!\,[n-r]!}\frac{[x]_{n-r}[y]_{n-r}}{[z]_{n-r}}[z-x-y]_r p^{r(x+y-z)}$$
$$=\frac{[z-x]_n[z-y]_n}{[z]_n}p^{n(x+y-z)}. \tag{80}$$

Divide throughout by $[x]_n[y]_n/[z]_n$, and put

$$x=a-\delta+1,\quad y=a-\epsilon+1,\quad z=a-\beta+1,\quad n=-a;$$

the series (80) now becomes

$$1+p\frac{[a][\beta][\delta+\epsilon-a-\beta-1]}{[1][\delta][\epsilon]}$$
$$+p^2\frac{[a][a+1][\beta][\beta+1][\delta+\epsilon-a-\beta-1][\delta+\epsilon-a-\beta]}{[2][\delta][\delta+1][\epsilon][\epsilon+1]}+\dots$$
$$=p^{a\beta}\frac{\Pi_p([a-\delta])\,\Pi_p([\beta-\delta])\,\Pi_p([a-\epsilon])\,\Pi_p([\beta-\epsilon])}{\Pi_p([a+\beta-\delta])\,\Pi_p([a+\beta-\epsilon])\,\Pi_p([-\delta])\,\Pi_p([-\epsilon])}. \tag{81}$$

In this equation a is a negative integer owing to the manner in which we obtained the identity (80), but both the product and the series are symmetrical in a and β, and β is not restricted to integral values; hence we can write (81) as valid in general, subject to convergence conditions.

The following are two examples of summation :—

$$p^{-n^2}\left\{\frac{\Gamma_p([2n+1])}{\Gamma_p([n+1])\,\Gamma_p([n+1])}\right\}^2$$

$$=1+\sum_{r=1}^{\infty} p^{r(r-2n)}\left\{\frac{[n][n-1]\dots[n-r+1]}{[r]!}\right\}^2\frac{[2n+1][2n+2]\dots[2n+r]}{[r]!}, \tag{82}$$

$$J_{[1]}(a)\,\mathfrak{J}_{[1]}(a)=\frac{[4]}{[2]}J_{[0]}(a)\,\mathfrak{J}_{[2]}(a)+p^2\frac{[8]}{[4]}J_{[1]}(a)\,\mathfrak{J}_{[3]}(a)+\dots$$

$$+p^{r(r-1)}\frac{[4r]}{[2r]}J_{[r-1]}(a)\,\mathfrak{J}_{[r+1]}(a)+\dots. \tag{83}$$

PERPETUANT SYZYGIES

By A. YOUNG *and* P. W. WOOD.

[Received and Read May 12th, 1904.]

THE methods used in the following discussion are entirely based on the symbolical notation, perpetuants linear in the coefficients of each quantic concerned being alone considered (perpetuant types). The symbolical forms of perpetuant types (which had previously all been identified by other methods) were calculated by Grace.* His result—"Any perpetuant linear in the coefficients of each of δ quantics, denoted by the letters $a_1, a_2, \ldots, a_\delta$, can be expressed linearly in terms of products and of perpetuants $(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \ldots (a_1 a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \geqslant 2^{\delta-2}$, $\lambda_2 \geqslant 2^{\delta-3}$, ..., $\lambda_{\delta-1} \geqslant 1$, the order of the letters being fixed beforehand"—is fundamental to our present purpose and will for convenience be quoted as "The Perpetuant Type Theorem."

The difficulty of dealing with the actual syzygies is avoided as follows: all possible products of irreducible forms, for a given degree and weight, are arranged in a predetermined sequence, so that each individual product has a definite place in that sequence. Any syzygy may then be regarded as expressing that one of its products, which comes first in the predetermined sequence, in terms of products which come after it. The first product will be called reducible, by virtue of its being linearly expressible in terms of later products.

Thus, instead of actually finding syzygies, we seek to discover what products are reducible and what products are irreducible. Now all products of perpetuant types of degree δ can be expressed in terms of the forms

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \ldots (a_1 a_\delta)^{\lambda_{\delta-1}}, \ldots ;$$

where the indices λ are only restricted to be positive integers or zeros, the order of the letters being fixed; such forms are all linearly independent, for, if there is a relation between them, there must be a purely algebraic relation between the symbolical products written down, and this is

* *Proc. London Math. Soc.*, Vol. XXXIV,

easily* shown to be impossible. The generating function for all linearly independent perpetuant types and products of perpetuant types of degree δ is therefore $\frac{1}{(1-x)^{\delta-1}}$; it is known† that the generating function for actually irreducible forms of degree δ is $\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}$. So the generating function for irreducible products must be

$$\frac{1-x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}.$$

It is our object to identify these irreducible products: when this has been done for any degree, all the independent syzygies of this degree will have been identified, there being one such syzygy for each reducible product of irreducible forms. Now there cannot be fewer irreducible forms (types) and products of degree δ than the number enumerated by the generating function $\frac{1}{(1-x)^{\delta-1}}$; hence, when we have reduced all forms except this number, we shall have a proof of the irreducibility of the remaining products, as well as a new proof of the irreducibility of the forms already classified as perpetuant types.

Our results are only complete as far as degree 8: a large class of products has been discussed in general, and, though it cannot be asserted that the generating functions obtained for these products are exact, yet the fact that the particular cases of them are exact as far as degree 8 establishes a strong presumption that such is always the case.

The syzygies are all derived from those of degree 4 due to Stroh,‡ and the Jacobian identity

$$(bc)+(ca)+(ab) \equiv 0.$$

The syzygies of degree 4 have been completely discussed already§ by a different method, but we have discussed them here fully in illustration of the general methods employed in the present paper.

---

* For, if there is a relation, let $(a_1a_2)^{\lambda_1}\Sigma N(a_1a_3)^{\lambda_2}\dots(a_1a_\delta)^{\lambda_{\delta-1}}$ be those terms for which the index of (a_1a_2) is lowest; then we may divide out by $(a_1a_2)^{\lambda_1}$ and put $a_1 = a_2$ after division; we get $\Sigma N(a_1a_3)^{\lambda_2}\dots(a_1a_\delta)^{\lambda_{\delta-1}} = 0$. Proceeding in this way, we ultimately see that every coefficient N is zero.

† The irreducibility of these forms has been finally demonstrated by Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Vol. 1, Ser. 2.

‡ *Math. Ann.*, Bd. XXXIII., S. 61–107, § 18.

§ Wood, *Proc. London Math. Soc.*, Vol. 2, Ser. 2.

The above methods are apparently insufficient to obtain all the syzygies for the reduction of products of three forms each of degree 3, and here arises the difficulty in the treatment of products of degree 9.

It seems probable that the generating function for all irreducible products and types of degree δ, where the products have no factor of degree less than $(n+1)$, is

$$\frac{x^{\binom{\delta}{1}+\binom{\delta}{2}+\dots+\binom{\delta}{\kappa}}}{(1-x)^{\delta-1}}, \qquad \kappa \leqslant \frac{\delta}{2};$$

and this result has been proved true for the cases $\kappa = 1$, 2, and $\kappa \geqslant \frac{\delta}{3}$. The question will be discussed at the end of Section VI.

The paper has been divided into sections as follows:—

CONTENTS.

Previous Literature on Perpetuant Syzygies.

Previously published papers, of which none date since 1887, differ fundamentally from the present in three respects: (1) by the use of the literal notation and the theory of partitions; (2) by making the discovery of the syzygies a basis for the enumeration of the irreducible perpetuants; (3) by actual expression, apart from mere enumeration, of the syzygies.

The present paper starts from a knowledge of the symbolical form of the perpetuants and deduces a means of enumerating the syzygies; their actual expressions can be obtained from Section III. All the papers quoted are to be found in the *American Journal of Mathematics.*

Sylvester, Vol. v., "On Subinvariants."

This paper was the starting point of all investigations on perpetuants and treated of certain syzygies of degrees 5, 6, and 7: an error in degree 7 was subsequently corrected by Hammond.

Cayley, Vol. vii., "A Memoir on Seminvariants."

This deals with a method of suitably expressing certain syzygies of degrees 5 and 6.

Hammond, Vol. v., "On the Solution of the Differential Equation of Sources."

This paper obtains the results of Sylvester afresh, and corrects the error in the treatment of degree 7.

——— Vol. VIII., "On Perpetuants, with Applications to the Theory of Finite Quantics."

This paper sketches a general method of classifying and expressing certain syzygies, but gives no definite result for any degree greater than 8.

MacMahon, Vol. x., "Expression of Syzygies among Perpetuants by means of Partitions."

The paper contains, *inter alia*, the actual expression of a number of sextic syzygies enumerated by the generating function

$$\frac{x^6+x^9+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+x^{17}}{(1-x^3)(1-x^4)(1-x^6)}.$$

Several sextic syzygies, however, remain unexpressed.

I. EXTENSION OF THE PERPETUANT TYPE THEOREM.

1. Consider any perpetuant

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\kappa)^{\lambda_{\kappa-1}} \equiv C_\kappa .$$

We assume throughout that the letters are taken in a definite order fixed beforehand. If C_κ appears as a factor of a product P, we shall find certain conditions, ultimately affecting the indices λ, for the "irreducibility" of P; here we use the word "irreducibility" in a perfectly general sense, as implying that P cannot be expressed in terms of other forms obeying certain definite laws (*v.* Note, p. 225).

Notation.—The symbol $[a_{r_1} a_{r_2} \dots a_{r_l}]$ denotes any covariant, reducible or otherwise, involving the symbolical letters $a_{r_1}, a_{r_2}, \dots, a_{r_l}$; and the symbol $[a_{r_1} a_{r_2} \dots a_{r_l}]'$ denotes any covariant, reducible or otherwise, involving all the symbolical letters $a_1, a_2, \dots, a_\kappa$ concerned except $a_{r_1}, a_{r_2}, \dots, a_{r_l}$.

If the transvectant $([a_{r_1} \dots a_{r_l}], [a_{r_1} \dots a r_l]')^\lambda$ replace C_κ in P, we may assume from our previous knowledge that, unless $\lambda \geqslant \mu_{r_1 r_2 \dots r_l}$, P is, according to some definition, reducible; here $\mu_{r_1 r_2 \dots r_l}$ is a quantity supposed known from other investigations: its value will depend on our definition of "reducibility."

We shall obtain all such transvectants of covariants involving some of the letters of C_κ with covariants involving the remaining letters, if we suppose that $a_{r_1}, a_{r_2}, \dots, a_{r_l}$ are some or all of $a_2, a_3, \dots, a_\kappa$, so that a_1 appears always in $[a_{r_1} a_{r_2} \dots a_{r_l}]'$.

To every choice of the letters $a_{r_1}, a_{r_2}, \dots, a_{r_l}$ corresponds a quantity $\mu_{r_1 r_2 \dots r_l}$, so that the number of such quantities $\mu_{r_1 r_2 \dots r_l}$ is $2^{\kappa-1}-1$; for

there are $\binom{\kappa-1}{l}$ quantics $\mu_{r_1 r_2 \dots r_l}$ with l suffixes, and l may have any value from 1 to $\kappa-1$.

The order of the suffixes may be conveniently fixed by the rule

$$r_1 < r_2 < r_3 \dots < r_l.$$

We shall use σ_r to denote the sum of all the μ's whose first suffix is r, $\sigma_{r,s}$ to denote the sum of all the μ's whose first two suffixes are r, s, and so on.

Thus $$\sigma_\kappa = \mu_\kappa, \qquad \sigma_{\kappa-1} = \mu_{\kappa-1} + \mu_{\kappa-1,\kappa},$$

and so on. We shall establish the following

THEOREM. — The conditions of irreducibility* of the product P are, as far as the factor $C_\kappa \equiv (a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\kappa)^{\lambda_{\kappa-1}}$ is concerned,

$$\lambda_{\kappa-1} \geqslant \sigma_\kappa, \quad \lambda_{\kappa-2} \geqslant \sigma_{\kappa-1}, \quad \dots, \quad \lambda_1 \geqslant \sigma_2.$$

We have

$$C_\kappa \equiv ([a_\kappa]', [a_\kappa])^{\lambda_{\kappa-1}} + \Sigma \nu ([a_\kappa]', [a_\kappa])^{\lambda_{\kappa-1}-\epsilon},$$

and therefore by hypothesis C_κ is reducible unless $\lambda_{\kappa-1} \geqslant \mu_\kappa$.

Put $$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_{\kappa-2})^{\lambda_{\kappa-3}} \equiv a \equiv [a_{\kappa-1} a_\kappa]'.$$

Then C_κ differs from $(aa_{\kappa-1})^{\lambda_{\kappa-2}} (aa_\kappa)^{\lambda_{\kappa-1}}$ by terms which we have just defined as reducible.

If $\lambda_{\kappa-2} + \lambda_{\kappa-1}$ is less than either of $\mu_{\kappa-1}$ or $\mu_{\kappa-1,\kappa}$, then obviously C_κ is reducible: if $\lambda_{\kappa-2} + \lambda_{\kappa-1}$ is greater than each of $\mu_{\kappa-1}$ and $\mu_{\kappa-1,\kappa}$, then, by Stroh's series, the covariant

$$(aa_{\kappa-1})^{\lambda_{\kappa-2}} (aa_\kappa)^{\lambda_{\kappa-1}}$$

is linearly expressible in terms of

$$(a_{\kappa-1} a_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}}, \ \left((a_{\kappa-1} a_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}-1}, a\right), \ \dots, \ \left((a_{\kappa-1} a_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}-\mu_{\kappa-1,\kappa}+1}, a\right)^{\mu_{\kappa-1,\kappa}-1};$$

$$(aa_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}}, \ \left((aa_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}-1}, [a_{\kappa-1}]\right), \ \dots, \left((aa_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}-\mu_{\kappa-1}+1}, [a_{\kappa-1}]\right)^{\mu_{\kappa-1}-1};$$

$$(aa_{\kappa-1})^{\lambda_{\kappa-2}+\lambda_{\kappa-1}}, \ (aa_{\kappa-1})^{\lambda_{\kappa-2}+\lambda_{\kappa-1}-1} (aa_\kappa), \ \dots, \ (aa_{\kappa-1})^{\mu_{\kappa-1}+\mu_{\kappa-1,\kappa}} (aa_\kappa)^{\lambda_{\kappa-2}+\lambda_{\kappa-1}-\sigma_{\kappa-1}}.$$

Now by hypothesis each form in the first two rows is reducible, and therefore $(aa_{\kappa-1})^{\lambda_{\kappa-2}} (aa_\kappa)^{\lambda_{\kappa-1}}$ is reducible unless $\lambda_{\kappa-2} \geqslant \mu_{\kappa-1} + \mu_{\kappa-1,\kappa}$.

---

* We always consider $(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_\kappa)^{\lambda_{\kappa-1}}$ reducible if we can express it in terms of forms $(a_1 a_2)^{\mu_1} (a_1 a_3)^{\mu_2} \dots (a_1 a_\kappa)^{\mu_{\kappa-1}}$ such that the first of the quantities $\mu_1 - \lambda_1$, $\mu_2 - \lambda_2$, ..., $\mu_{\kappa-1} - \lambda_{\kappa-1}$ which is not zero is positive.

Hence a necessary condition for irreducibility is

$$\lambda_{\kappa-2} \geqslant \sigma_{\kappa-1}.$$

Assume the theorem true for the indices λ_{r-1} where $r > s$; then, as before, if we put $a \equiv [a_1 a_2 \ldots a_{s-1}]$, we have to consider the form

$$(aa_s)^{\lambda_{s-1}} (aa_{s+1})^{\lambda_s} \ldots (aa_\kappa)^{\lambda_{\kappa-1}}.$$

By Stroh's series this can be expressed in terms of

1. $(a_s a_{s+1})^{\lambda_s+\lambda_{s-1}} [a_s a_{s+1}]', ([a_s a_{s+1}], [a_s a_{s+1}]'), \ldots, ([a_s a_{s+1}], [a_s a_{s+1}]')^{\sigma_{s,\,s+1}-1}$;

2. $(aa_{s+1})^{\lambda_s+\lambda_{s-1}} \ldots, \; ([a_s]', [a_s]), \; \ldots, \; ([a_s]', [a_s])^{\sigma_s-\sigma_{s,\,s+1}-1}$;

3. $(aa_s)^{\lambda_s+\lambda_{s-1}} \ldots, (aa_s)^{\lambda_s+\lambda_{s-1}-1} (aa_{s+1}) \ldots, \ldots, (aa_s)^{\sigma_s} (aa_{s+1})^{\lambda_s+\lambda_{s-1}-\sigma_s} (aa_{s+2})^{\lambda_{s+1}} \ldots$.

We proceed to show that each form in the first two rows is reducible.

(1) Any member of the first row is

$$([a_s a_{s+1}], [a_s a_{s+1}]')^\nu,$$

where $$[a_s a_{s+1}] \equiv (a_s a_{s+1})^{\lambda_{s-1}+\lambda_s-\nu} \equiv \beta, \text{ say,}$$

and $$[a_s a_{s+1}]' \equiv (aa_{s+2})^{\lambda_{s+1}} \ldots (aa_\kappa)^{\lambda_{\kappa-1}},$$

so that we need only consider the form

$$(a\beta)^\nu (aa_{s+2})^{\lambda_{s+1}} \ldots (aa_\kappa)^{\lambda_{\kappa-1}}.$$

In addition to the actual reductions implied by hypothesis, we must also consider those reductions which consist in increasing the earlier indices at the expense of the later ones. (This is a common feature of all inductive proofs, since these reduced indices may be raised to their proper limits by virtue of the inductive proof; *v.* Note, p. 225.)

So, since we have assumed the theorem true for the index λ_s (of a factor followed by not more than $(\kappa-s-1)$ other factors), we must have $\nu \geqslant \sigma_{s,\,s+1}$, if the form last written down is irreducible. Hence every form in the first row is reducible.

(2) If we put $$\beta \equiv (aa_{s+1})^{\lambda_s+\lambda_{s-1}-\nu},$$

$$[a_s]' \equiv (\beta a_{s+2})^{\lambda_{s+1}} (\beta a_{s+3})^{\lambda_{s+2}} \ldots (\beta a_\kappa)^{\lambda_{\kappa-1}};$$

and we have to consider forms

$$(\beta a_s)^\nu (\beta a_{s+2})^{\lambda_{s+1}} \ldots (\beta a_\kappa)^{\lambda_{\kappa-1}}.$$

The condition for the irreducibility of this form is by hypothesis

$\nu \geqslant$ sum of μ's whose first suffix is s, and which do not contain a suffix $s+1$,

i.e., $$\nu \geqslant \sigma_s - \sigma_{s,\,s+1}.$$

Hence every form in the second row is reducible: and, since the index of (aa_s) in the last row is in every term $\geqslant \sigma_s$, we have as the condition, affecting λ_{s-1}, of irreducibility,

$$\lambda_{s-1} \geqslant \sigma_s.$$

We have demonstrated the truth of the conditions for the indices $\lambda_{\kappa-1}$ and $\lambda_{\kappa-2}$, and therefore, by induction, the theorem is universally true.

2. COR. 1.—The perpetuant type theorem is immediately deducible from the preceding result: for, if C_κ is the only factor of the product P, every quantity μ is unity, since in this case the transvectants are reducible only if they are of zero order.

Hence

$$\begin{aligned}\sigma_s &= \text{number of ways of choosing all, none, or any of the suffixes } s+1, \ldots, \kappa, \\ &= 2^{\kappa-s},\end{aligned}$$

and therefore the conditions for the irreducibility of C_κ are

$$\lambda_{s-1} \geqslant 2^{\kappa-s}, \qquad s = 2, \ldots, \kappa.$$

COR. 2.—An especially important case arises when the product P is (for any reason) reducible if C_κ is a Jacobian: in this case every quantity μ is 2, and

$$\sigma_s = 2^{\kappa-s+1},$$

and therefore $\qquad \lambda_{s-1} \geqslant 2^{\kappa-s+1}, \qquad s = 2, \ldots, \kappa.$

In this case the minimum weight of C_κ for irreducibility is $2^\kappa-2$.

(Reference may be made to a paper by Grace, "Extension of Two Theorems on Covariants": *Proc. London Math. Soc.*, Ser. 2, Vol. 1.)

II. SYZYGIES OF DEGREES 3 AND 4 RESPECTIVELY.

3. The syzygies of degrees 3 and 4 offer no difficulties, and may be actually written down apart from the consideration of their reducing special forms: those of degree 4 have been fully discussed already.* A discussion for degree 4 is here given from a different point of view in order to exhibit in a simple case the methods by which syzygies of higher degree are treated.

Degree 3.—There are no syzygies except for weight unity, for which there is a single syzygy:—

$$(bc)+(ca)+(ab) \equiv 0.$$

* Wood, "On Perpetuant Syzygies of Degree 4," *Proc. London Math. Soc.*, Ser. 2, Vol. 2.

The generating function for irreducible forms is $\dfrac{x^3}{(1-x)^2}$; and, since there are three linearly independent product forms $(bc)^\omega$, $(ca)^\omega$, $(ab)^\omega$ for all values of ω, except $\omega = 1$, 0, for which there are only two and one respectively, the generating function for irreducible product forms is

$$3x^2/(1-x)+2x+1.$$

Hence the generating function for all forms is

$$\frac{x^3}{(1-x)^2}+\frac{3x^2}{1-x}+2x+1=\frac{1}{(1-x)^2}.$$

4. *Degree* 4.—There are five different kinds of type forms and products of forms of degree 4: they may be conveniently written

$$C_4,\ C_2^2,\ C_1C_3,\ C_1^2C_2,\ C_1^4;$$

where C_r denotes any perpetuant of degree r, C_r^2 denotes the product of two perpetuants of degree r, and so on. This notation will be preserved throughout.

A product is in the first place considered reducible if it is expressible in terms of products of other kinds which follow it in the above sequence. For weight ω the syzygies are nine in number and consist of—

(1) The three syzygies, due to Stroh, of the form

$$\{a_1a_2a_3a_4\}_\omega \equiv \{(a_1a_2)+(a_3a_4)\}^\omega - \{(a_1a_4)+(a_3a_2)\}^\omega = 0.$$

(2) The six syzygies, arising from the Jacobian identity, of the form

$$(a_1a_2)^{\omega-1}(a_3a_4) = (a_1a_2)^{\omega-1}(a_3a_4) - (a_1a_2)^{\omega-1}(a_2a_3).$$

These last syzygies reduce all products C_2^2, when one of the factors C_2 is a Jacobian, by expressing each such product in terms of products C_1C_3.

In order to fix what products are reduced by Stroh's syzygies, we arrange the products C_2^2 in the following order:—

$$(a_1a_4)^\lambda(a_2a_3)^\mu,\quad (a_1a_3)^\lambda(a_2a_4)^\mu,\quad (a_1a_2)^\lambda(a_3a_4)^\mu,$$

and define—

(i.) $(a_1a_4)^\lambda(a_2a_3)^\mu$ as reducible if it is expressible in terms of products $(a_1a_3)^\lambda(a_2a_4)^\mu$, $(a_1a_2)^\lambda(a_3a_4)^\mu$ and products C_1C_3, &c.

(ii.) $(a_1a_3)^\lambda(a_2a_4)^\mu$ as reducible if it is expressible in terms of products $(a_1a_2)^\lambda(a_3a_4)^\mu$ and products C_1C_3, &c.

Finally, we shall arrange the products $(a_1 a_r)^\lambda (a_s a_t)^\mu$ in ascending values of λ, and define any such form as reducible if it is expressible in terms of similar forms with greater values of λ.

The syzygy $\{a_1 a_2 a_3 a_4\}_\omega \equiv 0$ will reduce the form $(a_1 a_4)^2 (a_2 a_3)^{\omega-2}$, since $(a_1 a_4)(a_2 a_3)^{\omega-1} = \Sigma C_1 C_3$, by the Jacobian identity; the syzygy $\{a_1 a_2 a_4 a_3\}_\omega \equiv 0$ reduces in the same way the form $(a_1 a_3)^2 (a_2 a_4)^{\omega-2}$, and the syzygy $\{a_1 a_3 a_2 a_4\}_\omega \equiv 0$ reduces

$$\binom{\omega}{2}(a_1 a_4)^2 (a_2 a_3)^{\omega-2} + \binom{\omega}{3}(a_1 a_4)^3 (a_2 a_3)^{\omega-3}.$$

Moreover, from $\{a_1 a_2 a_3 a_4\}_\omega \equiv 0$, we have reduced

$$\binom{\omega}{2}(a_1 a_4)^2 (a_3 a_2)^{\omega-2} + \binom{\omega}{3}(a_1 a_4)^3 (a_3 a_2)^{\omega-3},$$

i.e.,
$$\binom{\omega}{2}(a_1 a_4)^2 (a_2 a_3)^{\omega-2} - \binom{\omega}{3}(a_1 a_4)^3 (a_2 a_3)^{\omega-3}.$$

Hence both $(a_1 a_4)^2 (a_2 a_3)^{\omega-2}$ and $(a_1 a_4)^3 (a_2 a_3)^{\omega-3}$ are, in accordance with our definition, reducible products.

Hence the irreducible products C_2^2 are

$$(a_1 a_2)^\lambda (a_3 a_4)^\mu, \qquad \lambda \geqslant 2, \quad \mu \geqslant 2;$$

$$(a_1 a_3)^\lambda (a_2 a_4)^\mu, \qquad \lambda \geqslant 3, \quad \mu \geqslant 2;$$

$$(a_1 a_4)^\lambda (a_2 a_3)^\mu, \qquad \lambda \geqslant 4, \quad \mu \geqslant 2;$$

and so the generating function for all products C_2^2 is

$$\frac{x^4}{(1-x)^2} + \frac{x^5}{(1-x)^2} + \frac{x^6}{(1-x)^2} = \frac{x^4 - x^7}{(1-x)^3}.$$

Finally, when the weight is unity we have only forms $C_1^2 C_2$; by the Jacobian transformation, we can compel any one definite letter to appear in the factor C_2; so for weight unity there are only three such forms, while for all other weights there are six: the generating function for the products $C_1^2 C_2$ is therefore

$$6x^2/(1-x) + 3x.$$

The results for degree 4 may be summarized thus :—

Forms.	Generating Functions.
C_4	$\dfrac{x^7}{(1-x)^8}$
C_2^2	$\dfrac{x^4-x^7}{(1-x)^8}$
$C_1 C_3$	$\dfrac{4x^3}{(1-x)^2}$
$C_1^2 C_2$	$\dfrac{6x^2}{1-x}+3x$
C_1^4	1

Hence the generating function for the total number of forms of degree 4 is

$$\frac{x^7+x^4-x^7}{(1-x)^8}+\frac{4x^3}{(1-x)^2}+\frac{6x^2}{1-x}+3x+1=\frac{1}{(1-x)^3}.$$

III. Definition of Reducibility.—General Methods of Reduction by Differential Operators.

5. The product forms will be arranged in a definite sequence to be particularized immediately, and a product will be defined as reducible if it is linearly expressible in terms of product forms which follow it in this definite sequence. The sequence of the *letters* involved in a symbolical expression will be that defined by their suffixes, viz., $a_1, a_2, \ldots, a_\delta$.

The criteria for determining the relative positions of any two products A and B in the sequence will be made to depend successively on :—

(i.) The partial degrees of the factors of each product.

(ii.) The arrangement of the letters among the factors of each product.

(iii.) The weights of the factors of each product.

(iv.) The indices of the symbolical determinants of the factors of each product.

(i.) If $\quad A \equiv C_{m_1} C_{m_2} \ldots C_{m_\kappa}, \quad m_1 \leqslant m_2 \leqslant m_3 \ldots \leqslant m_\kappa;$

$$B \equiv C_{n_1} C_{n_2} \ldots C_{n_l}, \quad n_1 \leqslant n_2 \leqslant n_3 \ldots \leqslant n_l;$$

then A precedes B, if the first of the quantities

$$m_1-n_1,\ m_2-n_2,\ \ldots,\ m_r-n_r,\ \ldots,$$

which is not zero, is positive.

(ii.) If
$$\left.\begin{aligned} A &\equiv C_{m_1} C_{m_2} \ldots C_{m_\kappa} \\ B &\equiv C'_{m_1} C'_{m_2} \ldots C'_{m_\kappa} \end{aligned}\right\}, \quad m_1 \leqslant m_2 \leqslant m_3 \ldots \leqslant m_\kappa,$$

so that the factors of A and B are of the same partial degrees, then the arrangement of the letters is taken into consideration.

Case I.—$m_1 < m_2$.

Suppose C_{m_1} contains the letters $a_{r_1}, a_{r_2}, \ldots, a_{r_{m_1}}$,

and C'_{m_1} contains the letters $a_{s_1}, a_{s_2}, \ldots, a_{s_{m_1}}$;

where $r_1, r_2, \ldots, r_{m_1}$, or $s_1, s_2, \ldots, s_{m_1}$ are any m_1 of the suffixes 1, 2, 3, ..., δ, such that

$$r_1 < r_2 < r_3 \ldots < r_{m_1},$$

and

$$s_1 < s_2 < s_3 \ldots < s_{m_1}.$$

Then A precedes B, if the first of the quantities

$$r_1-s_1,\ r_2-s_2,\ \ldots,\ r_{m_1}-s_{m_1}$$

which is not zero is negative.

This assumes that C_{m_1} and C'_{m_1} do not contain the same set of m_1 letters.

Case II.—$m = m_1 = m_2 = \ldots = m_\theta < m_{\theta+1} \leqslant m_{\theta+2} \ldots \leqslant m_\kappa$.

In the first place the sequence is to be determined as in Case I. by all the letters occurring in the factors $C_{m_1} C_{m_2} \ldots C_{m_\theta}$.

When the letters in $C_{m_1} C_{m_2} \ldots C_{m_\theta}$ are the same as those in $C'_{m_1} C'_{m_2} \ldots C'_{m_\theta}$, it is necessary to distinguish between the factors $C_{m_1}, C_{m_2}, \ldots, C_{m_\theta}$, in order to determine whether A precedes B or not. Let a_ϕ be the first of the set of letters which occurs in $C_{m_1} C_{m_2} \ldots C_{m_\theta}$; then we define C_{m_1} to be that factor which contains a_ϕ; similarly C'_{m_1} is that factor of the product $C'_{m_1} C'_{m_2} \ldots C'_{m_\theta}$ which contains a_ϕ. Let the letters of C_{m_1}, C'_{m_1} be respectively $a_\phi, a_{r_2}, a_{r_3}, \ldots, a_{r_m}$ and $a_\phi, a_{s_2}, a_{s_3}, \ldots, a_{s_m}$, where

$$\phi < r_2 < r_3 \ldots < r_m,$$

$$\phi < s_2 < s_3 \ldots < s_m.$$

Then A precedes B if the first of the differences

$$r_2 - s_2,\ r_3 - s_3,\ \ldots,\ r_m - s_m$$

which is not zero is *positive*.

(As an example, see the arrangement of the sequence of products C_2^2 at the foot of p. 228.)

If the letters of C_{m_1} and C'_{m_1} are the same, we apply the same test to the products $C_{m_2} C_{m_3} \ldots C_{m_\theta}$, $C'_{m_2} C'_{m_3} \ldots C'_{m_\theta}$. These two products now contain the same set of letters; we choose C_{m_2}, C'_{m_2} to be the two factors which contain the first letter of this set. Then, unless the letters of C_{m_2}, C'_{m_2} are all the same, the sequence is determined by these letters, in the same way as before. Otherwise we must apply the same test to the products $C_{m_3} \ldots C_{m_\theta}$, $C'_{m_3} \ldots C'_{m_\theta}$.

Finally, when the letters of the various corresponding factors of $C_{m_1} C_{m_2} \ldots C_{m_\theta}$ and $C'_{m_1} C'_{m_2} \ldots C_{m_\theta}$ are in every case the same, the tests which have just been laid down must be applied to the products $C_{m_{\theta+1}} \ldots C_{m_\kappa}$, $C'_{m_{\theta+1}} \ldots C'_{m_\kappa}$ in order to determine whether A precedes B or not.

(iii.) If A and B each have their factors of the same partial degrees in the same sets of letters, let

$$A \equiv C_{m_1} C_{m_2} \ldots C_{m_\kappa}, \qquad B \equiv C'_{m_1} C'_{m_2} \ldots C'_{m_\kappa}, \qquad m_1 \leqslant m_2 \ldots \leqslant m_\kappa,$$

where, if certain of the m's are equal, the sequence of the factors $C_{m_1}, C_{m_2}, \ldots, C_{m_\kappa}$ is determined as in (ii.), and further let

$C_{m_1}, C_{m_2}, \ldots, C_{m_\kappa}$ be of total weights $\omega_1, \omega_2, \ldots, \omega_\kappa$ respectively,

and $C'_{m_1}, C'_{m_2}, \ldots, C'_{m_\kappa}$ be of total weights $\omega'_1, \omega'_2, \ldots, \omega'_\kappa$ respectively.

Then A precedes B if the first of the quantities

$$\omega_1 - \omega'_1,\ \omega_2 - \omega'_2,\ \ldots,\ \omega_\kappa - \omega'_\kappa$$

which is not zero is negative.

(iv.) Finally, we have to consider the case of products whose factors are of the same weights in the same sets of letters. Suppose

$$\left.\begin{array}{l} C_{m_r} \equiv (a_{(r,1)} a_{(r,2)})^{\lambda(r,1)} (a_{(r,1)} a_{(r,3)})^{\lambda(r,2)} \ldots (a_{(r,1)} a_{(r,m_r)})^{\lambda(r,m_r-1)} \\ C'_{m_r} \equiv (a_{(r,1)} a_{(r,2)})^{\lambda'(r,1)} (a_{(r,1)} a_{(r,3)})^{\lambda'(r,2)} \ldots (a_{(r,1)} a_{(r,m_r)})^{\lambda'(r,m_r-1)} \end{array}\right\},$$

$$r = 1, 2, \ldots, \kappa$$

where $$(r, 1) < (r, 2) \ldots < (r, m_r-1) < (r, m_r).$$

Then A precedes B, if the first of the quantities

$$\begin{array}{llllllll}
(1) & \lambda_{(1,1)}-\lambda'_{(1,1)}, & \lambda_{(1,2)}-\lambda'_{(1,2)}, & \ldots, & \lambda_{(1,m_1)}-\lambda'_{(1,m_1)}, \\
(2) & \lambda_{(2,1)}-\lambda'_{(2,1)}, & \lambda_{(2,2)}-\lambda'_{(2,2)}, & \ldots, & \lambda_{(2,m_2)}-\lambda'_{(2,m_2)}, \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(r) & \lambda_{(r,1)}-\lambda'_{(r,1)}, & \lambda_{(r,2)}-\lambda'_{(r,2)}, & \ldots, & \lambda_{(r,m_r)}-\lambda'_{(r,m_r)}, \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(\kappa) & \lambda_{(\kappa,1)}-\lambda'_{(\kappa,1)}, & \lambda_{(\kappa,2)}-\lambda'_{(\kappa,2)}, & \ldots, & \lambda_{(\kappa,m_\kappa)}-\lambda'_{(\kappa,m_\kappa)}
\end{array}$$

which is not zero is negative.

A reference to the reductions in the case of degree 5 (in the succeeding section) should go some way towards making the above definition of reducibility quite clear.

General Methods of Reduction by the use of Differential Operators.

6. All the reductions made use of depend ultimately on Stroh's syzygies of degree 4: consider the syzygy

$$\{(ab)+(cd)\}^\lambda - \{(ad)+(cb)\}^\lambda = 0,$$

or $$\sum_{r=0}^{r=\lambda} \binom{\lambda}{r} (ab)^r (cd)^{\lambda-r} - \sum_{r=0}^{r=\lambda} \binom{\lambda}{r} (ad)^r (cb)^{\lambda-r} = 0.$$

Here the letters a, b, c, d denote any of the letters $a_1, a_2, \ldots, a_8$ used elsewhere in the present paper. This syzygy may be written

$$e^{(ab)D}(cd)^\lambda - e^{(ad)B}(cb)^\lambda = 0,$$

where $$B \equiv x_1 \frac{\partial}{\partial b_2} - x_2 \frac{\partial}{\partial b_1}, \qquad D \equiv x_1 \frac{\partial}{\partial d_2} - x_2 \frac{\partial}{\partial d_1};$$

and therefore $$B \,.\, b_x^m = 0, \quad D(cd) = c_x, \quad \ldots.$$

Let Γ_1 be any covariant containing the letters a, b, but not the letters c, d, and let Γ_2 be any covariant containing the letters c, d, but none of the letters of Γ_1. Then we may write

$$\Gamma_1 = \Sigma P_1 (ab)^{\mu_1}, \qquad \Gamma_2 = \Sigma P_2 (cd)^{\mu_2},$$

where P_1 does not contain the letter b in any determinantal factor, and P_2 does not contain the letter d in any determinantal factor.

Then we have a syzygy

$$P_1 . e^{(ab)D}\{P_2(cd)^\lambda\} = P_1 . e^{(ad)B}\{P_2(cb)^\lambda\}\,; \tag{I.}$$

and this syzygy is a relation between products of covariants of the form $\Gamma_1\Gamma_2$ and the same products having the letters b and d interchanged.

Now we have arranged our products in a fixed sequence, we shall suppose that the products on the right-hand side of the relation (I.) come after those on the left in this fixed sequence. The syzygy may then be conveniently written

$$P_1 e^{(ab)D}\{P_2(cd)^\lambda\} = R,$$

where R denotes any products which come in the fixed sequence after the products on the left-hand side: the sum of the products on the left-hand side is, in fact, according to our definition, reducible.

7. In the first place we shall consider the reduction of products $C_2 C_\kappa$: let $C_2 C_\kappa = (ab)^\mu (d_\kappa d_1)^{\lambda_1} (d_\kappa d_2)^{\lambda_2} \dots (d_\kappa d_{\kappa-1})^{\lambda_{\kappa-1}}$: the sequence of such products is primarily determined by the letters contained in the factor C_2: we suppose that the product in which the letters are ab precedes any of the products in which the letters of C_2 are

$$ad_1,\ ad_2,\ \dots,\ ad_r \quad \text{respectively.}$$

This is in accordance with our defined sequence of products when all the letters $d_1, d_2, \dots, d_r$ come after the letter b. In this case we have a series of relations of the form

$$e^{(ab)D_1} C_\kappa = R, \quad e^{(ab)D_2} C_\kappa = R, \quad \dots, \quad e^{(ab)D_r} C_\kappa = R,$$

where

$$D_r = x_1 \frac{\partial}{\partial d_{r,2}} - x_2 \frac{\partial}{\partial d_{r,1}}.$$

We proceed to show that *these r relations are all linearly independent*: this result is of fundamental importance.

The operators $D_1, D_2, \dots, D_r$ are obviously all commutative. The first relation may be written

$$C_\kappa + (ab).D_1 C_\kappa + \frac{(ab)^2}{2!} D_1^2 C_\kappa + \dots = R,$$

or, since the first two terms are obviously reducible, the second by the Jacobian identity, it may be written

$$\frac{(ab)^2}{2!} D_1^2 C_\kappa + \frac{(ab)^3}{3!} D_1^3 C_\kappa + \dots = R, \tag{II.}$$

and this relation is true, whatever covariant C_κ may be involving the letters $d_1, d_2, \dots, d_\kappa$.

8. Consider the equation $D_1 C_\kappa = C'_\kappa$
as an equation to determine C_κ.

If
$$C'_\kappa = (d_\kappa d_1)^{\mu_1} (d_\kappa d_2)^{\mu_2} \ldots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}},$$
then one solution is obviously
$$C_\kappa = \frac{1}{\mu_1+1} (d_\kappa d_1)^{\mu_1+1} (d_\kappa d_2)^{\mu_2} \ldots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}}.$$

Let $C_\kappa^{(1)}$ be any other solution; then we can always write
$$C_\kappa^{(1)} = \sum_\rho N_\rho (d_\kappa d_1)^{\mu_{\rho,1}} (d_\kappa d_2)^{\mu_{\rho,2}} \ldots (d_\kappa d_{\kappa-1})^{\mu_{\rho,\kappa-1}},$$
and hence
$$\sum_\rho N_\rho \mu_{\rho,1} (d_\kappa d_1)^{\mu_{\rho,1}-1} (d_\kappa d_2)^{\mu_{\rho,2}} \ldots (d_\kappa d_{\kappa-1})^{\mu_{\rho,\kappa-1}} \equiv (d_\kappa d_1)^{\mu_1} \ldots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}}.$$

The covariants in this identity are all independent: it follows, therefore, that C_κ can differ from $\frac{1}{\mu_1+1} (d_\kappa d_1)^{\mu_1+1} (d_\kappa d_2)^{\mu_2} \ldots (d_\kappa d_{\kappa-1})^{\mu_{\kappa-1}}$ only by covariants which are products of the quantic d_1 with covariants of degree $\kappa-1$. And so, neglecting reducible terms, $D_1^{-1} C_\kappa$ has a unique meaning: in the same way $D_1^{-2} C_\kappa$, $D_1^{-3} C_\kappa$, ... have each a unique meaning. So in (II.) we may replace C_κ by $D_1^{-2} C_\kappa$: the resulting relation is
$$\frac{(ab)^2}{2!} C_\kappa + \frac{(ab)^3}{3!} D_1 C_\kappa + \ldots = R,$$
and, in the same way, from the relation $e^{(ab)D_2} C_\kappa = R$ we obtain
$$\frac{(ab)^2}{2!} C_\kappa + \frac{(ab)^3}{3!} D_2 C_\kappa + \ldots = R.$$
Therefore, on subtraction,
$$\frac{(ab)^3}{3!} (D_1 - D_2) C_\kappa + \frac{(ab)^4}{4!} (D_1^2 - D_2^2) C_\kappa + \ldots = R,$$
or
$$\left[\frac{(ab)^3}{3!} + \frac{(ab)^4}{4!} (D_1 + D_2) + \frac{(ab)^5}{5!} (D_1^2 + D_1 D_2 + D_2^2) \ldots\right] (D_1 - D_2) C_\kappa = R.$$
Writing this
$$\chi (D_1 - D_2) C_\kappa = R,$$
and replacing C_κ by $D_1^{-1} C_\kappa$, we have
$$\chi (1 - D_2 D_1^{-1}) C_\kappa = R.$$
In this replace C_κ by $D_2 D_1^{-1} C_\kappa$: we have
$$\chi (D_2 D_1^{-1} - D_2^2 D_1^{-2}) C_\kappa = R.$$

So, adding, we get $\qquad \chi(1-D_2^2 D_1^{-2})\, C_\kappa = R.$

Proceeding in this way, we obtain

$$\chi(1-D_2^\rho D_1^{-\rho})\, C_\kappa = R.$$

Since $D_1^{-\rho}$ has no effect on the index of $(d_\kappa d_2)$ in the covariant C_κ, ρ may be taken so large that $D_2^\rho D_1^{-\rho}\, C_\kappa = 0$: so the last relation is

$$\chi\, C_\kappa = R,$$

or

$$\left[\frac{(ab)^3}{3!}+\frac{(ab)^4}{4!}(D_1+D_2)+\dots\right] C_\kappa = R.$$

9. Now let us assume that, in general, we can deduce from the σ relations

$$e^{(ab)D_1}\, C_\kappa = R, \quad e^{(ab)D_2}\, C_\kappa = R, \quad \dots, \quad e^{(ab)D_\sigma}\, C_\kappa = R$$

the relation :—

$$\left\{\frac{(ab)^{\sigma+1}}{(\sigma+1)!}+\frac{(ab)^{\sigma+2}}{(\sigma+2)!}(D_1+D_2+\dots+D_\sigma)+\dots \right.$$
$$\left. +\frac{(ab)^{\sigma+\tau}}{(\sigma+\tau)!}\Big(\sum_{\varpi_1+\dots+\varpi_\sigma=\tau-1} D_1^{\varpi_1}D_2^{\varpi_2}\dots D_\sigma^{\varpi_\sigma}\Big)+\dots\right\} C_\kappa = R. \qquad \text{(III.)}$$

We shall show that, if we take another relation $e^{(ab)D_{\sigma+1}}\, C_\kappa = R$, then the resulting relation will be of the same form as (III.) with σ changed to $\sigma+1$.

On our assumption we can deduce from the σ relations

$$e^{(ab)D_1}\, C_\kappa = R, \quad \dots, \quad e^{(ab)D_{\sigma-1}}\, C_\kappa = R, \quad e^{(ab)D_{\sigma+1}}\, C_\kappa = R$$

the relation :—

$$\left\{\frac{(ab)^{\sigma+1}}{(\sigma+1)!}+\frac{(ab)^{\sigma+2}}{(\sigma+2)!}(D_1+D_2+\dots+D_{\sigma-1}+D_{\sigma+1})+\dots \right.$$
$$\left. +\frac{(ab)^{\sigma+\tau}}{(\sigma+\tau)!}\Big(\sum_{\varpi_1+\dots+\varpi_{\sigma-1}+\varpi_{\sigma+1}=\tau-1} D_1^{\varpi_1}\dots D_{\sigma-1}^{\varpi_{\sigma-1}}D_{\sigma+1}^{\varpi_{\sigma+1}}\Big)+\dots\right\} C_\kappa = R.$$

Subtracting this last relation from (III.), we obtain

$$\left\{\frac{(ab)^{\sigma+2}}{(\sigma+2)!}+\frac{(ab)^{\sigma+3}}{(\sigma+3)!}\Big(\sum_{\rho=1}^{\rho=\sigma+1} D_\rho\Big)+\dots \right.$$
$$\left. +\frac{(ab)^{\sigma+\tau}}{(\sigma+\tau)!}\Big(\sum_{\varpi_1+\dots+\varpi_\sigma+\varpi_{\sigma+1}=\tau-2} D_1^{\varpi_1}\dots D_{\sigma+1}^{\varpi_{\sigma+1}}\Big)+\dots\right\}(D_{\sigma-1}-D_{\sigma+1})\, C_\kappa = R,$$

and by the same argument as before we can show that $(D_{\sigma-1}-D_{\sigma+1})\, C_\kappa$

may be replaced by C_κ. Now we have seen that (III.) is true when $\sigma = 1, 2$: it is therefore true for all values of σ.

An immediate consequence of this is the

THEOREM.—The r relations

$$e^{(ab)D_1} C_\kappa = R, \quad e^{(ab)D_2} C_\kappa = R, \quad \ldots, \quad e^{(ab)D_r} C_\kappa = R$$

are in general linearly independent, and together reduce the product $(ab)^\lambda C_\kappa$, when $\lambda \leqslant r+1$.

10. Certain special cases remain for consideration.

(i.) The relation $e^{(ab)D_\kappa} C_\kappa = R$ is true if the product $C_2 C_\kappa$, in which the letters of C_2 are (ab), precedes the product $C_2 C_\kappa$, in which the letters of C_2 are (ad_κ). We have previously written C_κ in the form

$$(d_\kappa d_1)^{\lambda_1} (d_\kappa d_2)^{\lambda_2} \ldots (d_\kappa d_{\kappa-1})^{\lambda_{\kappa-1}},$$

but it may be equally well written in the form

$$\Sigma N (d_1 d_\kappa)^{\mu_1} (d_1 d_2)^{\mu_2} \ldots (d_1 d_{\kappa-1})^{\mu_{\kappa-1}}.$$

Now the result of operating on each of the terms of this sum is R, and therefore the result of operating on the whole sum, or C_κ, is also R: in other words, our preceding results are not affected by the form of C_κ.

(ii.) If the product $(ab)^\lambda C_\kappa$ comes before the product $(d_\sigma b)^\lambda C_\kappa$, we have in the same way another syzygy

$$e^{(ba)D_\sigma} C_\kappa = R$$

or

$$e^{(ab)(-D_\sigma)} C_\kappa = R;$$

all such syzygies are independent of those already considered: we may, in fact, write $D_{\kappa+1} \equiv -D_\sigma$, and then the extension of the result embodied in (III.) to the case where the operators are $D_1, D_2, \ldots, D_\sigma, D_{\kappa+1}$ is almost identical with the previous investigation. Hence we deduce that, if r_1 letters of C_κ come after a, and r_2 letters of C_κ come after b, then

$$(ab)^\lambda C_\kappa = R, \quad \text{if } \lambda \leqslant r_1 + r_2 + 1.$$

(iii.) We may suppose that the letters a and b refer to covariants of the original quantics and not to the quantics themselves, and that one of the letters—say, d_1—refers to a covariant involving the letters, say, $e_1, e_2, \ldots, e_l$. Then, if the product $(ab)^\lambda C_\kappa$ precedes the product $(ad)^\lambda C'_\kappa$, we have a syzygy

$$e^{(ab)D_1} C_\kappa = R,$$

or, from the identity $(D_1+D_2+\ldots+D_\kappa)\,C_\kappa = 0$, which is easily verified,

$$e^{(ab)(-D_2-D_3-\ldots-D_\kappa)}\,C_\kappa = R.$$

Let P be the covariant obtained from C_κ by replacing d_1 by the letters $e_1, e_2, \ldots, e_l$; then

$$(D_2+D_3+\ldots+D_\kappa+E_1+E_2+\ldots+E_l)\,P = 0,$$

and hence the syzygy may be written

$$e^{(ab)(E_1+E_2+\ldots+E_l)}\,C_\kappa = R.$$

Now C_κ may be represented as a sum of transvectants of covariants of a definite set of r of its letters with covariants of the remainder, and so we may also obtain syzygies of the form

$$e^{(ab)(D_1+D_2+\ldots+D_r)}\,C_\kappa = R,$$

where $d_1, d_2, \ldots, d_r$ are any of the letters of C_κ (the existence of any such relation will, of course, depend on the sequence of the products). We have to show that all such syzygies are linearly independent. We write $D_{\kappa+1}$ for any such composite operator as $(D_1+D_2+\ldots+D_r)$, and observe that all operators, simple or composite, are commutative. The proof of the relation (III.) then proceeds on the same lines as before for both simple and composite operators: only one point arises :—Has the operation

$$D_{\kappa+1}^{-1}\,C_\kappa \equiv (D_1+D_2+\ldots+D_r)^{-1}\,C_\kappa$$

a perfectly definite meaning?

To prove that it has, we write C_κ in the form $\Sigma\, N(a, \beta)^\lambda$, where β is a covariant of the letters $d_1, d_2, \ldots, d_r$, and a is a covariant of the remaining letters: then, in the same way as before, we see that

$$(D_1+D_2+\ldots+D_r)^{-1}\,(a\beta)^\lambda$$

differs from $(\lambda+1)^{-1}(a\beta)^{\lambda+1}$ only by products of covariants of $d_1, d_2, \ldots, d_r$ with covariants of the remaining letters, and so the meaning is for our purpose definite and unique.

IV. SYZYGIES OF DEGREE 5.

11. We shall treat the syzygies of degree 5 at some length in explanation of the general principles set forth in Section III.

The product forms to be considered are of the following classes :—

$$C_2C_3,\quad C_1C_4,\quad C_1C_2^2,\quad C_1^2C_3,\quad C_1^3C_2,\quad C_1^5.$$

The generating functions for all the irreducible products, except C_2C_3 and $C_1^3C_2$, are known from the results for degrees 3 and 4.

(i.) *Generating Function for Products* $C_2 C_8$.

The $C_2 C_8$ products are to be arranged in a sequence to be determined thus:

(*a*) by virtue of the letters involved in the factor C_2: the following is the sequence:—

(i.)	(ii.)	(iii.)	(iv.)	(v.)
$(a_1 a_2)^r C_8$,	$(a_1 a_3)^r C_8$,	$(a_1 a_4)^r C_8$,	$(a_1 a_5)^r C_8$,	$(a_2 a_3)^r C_8$,
(vi.)	(vii.)	(viii.)	(ix.)	(x.)
$(a_2 a_4)^r C_8$,	$(a_2 a_5)^r C_8$,	$(a_3 a_4)^r C_8$,	$(a_3 a_5)^r C_8$,	$(a_4 a_5)^r C_8$.

(*b*) If two products $C_2 C_8$ have the same letters in the C_2 factor, that product whose C_2 factor is of smaller weight precedes the other.

Any $C_2 C_8$ product is then defined as reducible, if it is expressible linearly in terms of products of other classes and of $C_2 C_8$ products which follow it in the sequence just determined.

First consider the products $(a_1 a_2)^r C_8$: using the notation of the preceding section, we have six linearly independent relations (§ 9)

$$e^{\pm(a_1 a_2) D_3} C_8 = R, \qquad e^{\pm(a_1 a_2) D_4} C_8 = R, \qquad e^{\pm(a_1 a_2) D_5} C_8 = R,$$

where, for instance, in the relation $e^{(a_1 a_2) D_3} C_8 = R$, R represents $C_2 C_8$ product forms such as $(a_1 a_3)^r C_8$ together with product forms $C_1 C_5^2$, $C_1^2 C_8$, and, in the relation $e^{-(a_1 a_2) D_3} C_8 = R$, R represents $C_2 C_8$ product forms such as $(a_2 a_3)^r C_8$ with product forms $C_1 C_5^2$, $C_1^2 C_8$. So, for irreducibility, we must have $\nu > 6+1$, that is, $\nu \geqslant 8$.

Moreover, C_8 must not be a Jacobian of the form

$$\Big((a_r a_s)^\lambda,\ a_t\Big), \text{ where } r,\ s,\ t \text{ are 3, 4, 5 in any order;}$$

for
$$(a_1 a_2)^\nu \Big((a_r a_s)^\lambda,\ a_t\Big) = (a_r a_s)^\lambda \Big((a_1 a_2)^\nu,\ a_t\Big) - a_t\Big((a_1 a_2)^\nu,\ (a_r a_s)^\lambda\Big)$$
$$= R, \quad \text{whatever } r \text{ and } s \text{ may be.}$$

Hence, by the second Corollary of Section I. (§ 2), the weight of C_8 is at least 6, and therefore the minimum weight of an irreducible product $(a_1 a_2)^r C_8$ is 14.

The treatment of the products $(a_1 a_3)^r C_8$ is similar: here (by § 9) there are reductions corresponding to the interchange of a_3 with a_4, a_5, and the interchange of a_1 with a_2, a_4, a_5. The five linearly independent relations are

$$e^{(a_3 a_1) D_2} C_8 = R, \qquad e^{\pm(a_1 a_3) D_4} C = R, \qquad e^{\pm(a_1 a_3) D_5} C_8 = R.$$

Hence $\nu \geqslant 7$, and, as before, C_8 must not be a Jacobian of the form $\Big((a_r a_s)^\lambda,\ a_t\Big)$, r, s, t being 2, 4, 5 in any order: therefore the minimum

weight of C_8 is 6, and the minimum weight of an irreducible product $(a_1a_3)^\nu C_8$ is 13.

Similarly for an irreducible product $(a_1a_4)^\nu C_8$, $\nu \geqslant 6$, and the minimum weight of C_8 is 6; while, for an irreducible product $(a_1a_5)^\nu C_8$, $\nu \geqslant 5$, and the minimum weight of C_8 is 6.

Next consider the products $(a_2a_3)^\nu C_8$: here there are reductions corresponding to the interchanges of a_2 with a_4, a_5 and the interchanges of a_3 with a_4, a_5. The four linearly independent relations are

$$e^{\pm(a_2a_3)D_4}C_8 = R, \qquad e^{\pm(a_2a_3)D_5}C_8 = R; \qquad \text{so that } \nu \geqslant 6.$$

Moreover, C_8 must not be a Jacobian of the form $\left((a_4a_5)^\lambda, a_1\right)$. Hence, by the second Corollary of Section I. (§ 2), the weight of C_8 is at least 4: therefore the minimum weight of an irreducible product $(a_2a_3)^\nu C_8$ is 10. In the same way, for an irreducible product $(a_2a_4)^\nu C_8$, $\nu \geqslant 5$, and the weight of $C_8 \geqslant 4$, while, for an irreducible product $(a_2a_5)^\nu C_8$, $\nu \geqslant 4$, and the weight of $C_8 \geqslant 4$.

Now consider the products $(a_3a_4)^\nu C_8$: we have for the reduction of such forms two linearly independent relations

$$e^{\pm(a_3a_4)D_5}C_8 = R;$$

so that $\nu \geqslant 4$, while, by the perpetuant type theorem, C_8 is of weight 3 at least. So, for $(a_3a_5)^\nu C_8$, $\nu \geqslant 3$, and C_8 is of weight 3 at least.

Finally, for the products $(a_4a_5)^\nu C_8$, we have $\nu \geqslant 2$, for, if $\nu = 1$, there is a Jacobian identity; also the weight of C_8 is 3 at least.

12. So the irreducible products C_2C_8 may be tabulated thus:—

$$
\begin{array}{ll}
(a_1a_2)^\nu (a_3a_4)^\lambda (a_3a_5)^\mu, & \nu \geqslant 8, \ \lambda \geqslant 4, \ \mu \geqslant 2; \\
(a_1a_3)^\nu (a_2a_4)^\lambda (a_2a_5)^\mu, & \nu \geqslant 7, \ \lambda \geqslant 4, \ \mu \geqslant 2; \\
(a_1a_4)^\nu (a_2a_3)^\lambda (a_2a_5)^\mu, & \nu \geqslant 6, \ \lambda \geqslant 4, \ \mu \geqslant 2; \\
(a_1a_5)^\nu (a_2a_3)^\lambda (a_2a_4)^\mu, & \nu \geqslant 5, \ \lambda \geqslant 4, \ \mu \geqslant 2; \\
(a_2a_3)^\nu (a_1a_4)^\lambda (a_1a_5)^\mu, & \nu \geqslant 6, \ \lambda \geqslant 3, \ \mu \geqslant 1; \\
(a_2a_4)^\nu (a_1a_3)^\lambda (a_1a_5)^\mu, & \nu \geqslant 5, \ \lambda \geqslant 3, \ \mu \geqslant 1; \\
(a_2a_5)^\nu (a_1a_3)^\lambda (a_1a_4)^\mu, & \nu \geqslant 4, \ \lambda \geqslant 3, \ \mu \geqslant 1; \\
(a_3a_4)^\nu (a_1a_2)^\lambda (a_1a_5)^\mu, & \nu \geqslant 4, \ \lambda \geqslant 2, \ \mu \geqslant 1; \\
(a_3a_5)^\nu (a_1a_2)^\lambda (a_1a_4)^\mu, & \nu \geqslant 3, \ \lambda \geqslant 2, \ \mu \geqslant 1; \\
(a_4a_5)^\nu (a_1a_2)^\lambda (a_1a_3)^\mu, & \nu \geqslant 2, \ \lambda \geqslant 2, \ \mu \geqslant 1.
\end{array}
$$

Here the minimum value of ν is determined by the reductions of

Authors of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies	50 copies	75 copies	100 copies
		s. d.	s. d.	s. d.
4 pages	Free	3 [illegible]	[illegible] 6	[illegible] 0
8 ,,	,,	4 [illegible]	[illegible]	[illegible] 0
12 ,,	,,	7 [illegible]	10 [illegible]	1[illegible] 0
16 ,,	,,	8 [illegible]	12 [illegible]	[illegible] 0
24 ,,	,,	14 [illegible]	21 0	2[illegible] 0

It was resolved by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1s. 6d. per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price (i.e., at 1s. 4d. per sheet)."

It was further resolved by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete set* of *Proceedings up to the last completed volume*."

Complete Volumes of the *Proceedings*, bound in cloth, 8vo., can be purchased by the public from the Publisher, Mr. Francis Hodgson, 89 Farringdon Street, E.C., at the following rates:—

Vol.	I.	Nos.	1–8	Jan. 1865 to Nov. 1866			10s.
,,	II.	,,	9–20	Nov. 1866 ,, 1869			16s.
,,	III.	,,	21–40	,, 1869 ,, 1871			20s.
,,	IV.	,,	41–[illegible]	,, 1871 ,, 1873			31s. 6d.
,,	V.	,,	[illegible]	,, 1873 ,, 1874			15s.
,,	VI.	,,	76–[illegible]	,, 1874 ,, 1875			21s.
,,	VII.	,,	[illegible]	,, 1875 ,, 1876			21s.
,,	VIII.	,,	104–125	,, 1876 ,, 1877			21s.
,,	IX.	,,	126–140	,, 1877 ,, 1878			21s.
,,	X.	,,	141–155	,, 1878 ,, 1879			18s.
,,	XI.	,,	156–165	,, 1879 ,, 1880			12s. 6d.
,,	XII.	,,	166–179	,, 1880 ,, 1881			16s.
,,	XIII.	,,	180–194	,, 1881 ,, 1882			18s.
,,	XIV.	,,	195–214	,, 1882 ,, 1883			20s.
,,	XV.	,,	[illegible]	,, 1883 ,, 1884			20s.
,,	XVI.	,,	[illegible]	,, 1884 ,, 1885			20s.
,,	XVII.	,,	[illegible]	,, 1885 ,, 1886			25s.
,,	XVIII.	,,	[illegible]	,, 1886 ,, 1887			25s.
,,	XIX.	,,	[illegible]–312	,, 1887 ,, 1888			30s.
,,	XX.	,,	[illegible]	,, 1888 ,, 1889			25s.
,,	XXI.	,,	[illegible]	,, 1889 ,, 1890			25s.
,,	XXII.	,,	[illegible]–429	,, 1890 ,, 1891			27s. 6d.
,,	XXIII.	,,	430–449	,, 1891 ,, 1892			20s.
,,	XXIV.	,,	450–474	,, 1892 ,, 1893			22s.
,,	XXV.	,,	475–499	,, 1893 ,, 1894			22s.
,,	XXVI.	,,	500–[illegible]	,, 1894 ,, 1895			30s.
,,	XXVII.	,,	[illegible]	,, 1895 ,, 1896			35s.
,,	XXVIII.	,,	[illegible]–611	,, 1896 ,, 1897			2[illegible]s. 6d.
,,	XXIX.	,,	[illegible]	,, 1897 ,, 1898	Pt. I		20s.
					Pt. II		20s.
,,	XXX.	,,	[illegible]–678	,, 1898 to Mar. 1899			30s.
,,	XXXI.	,,	679–[illegible]	April 1899 ,, Dec. 1899			2[illegible]s.
,,	XXXII.	,,	[illegible]–737	Jan. 1900 ,, June 1900			20s.
,,	XXXIII.	,,	[illegible]–763	Nov. 1900 ,, Feb. 1901			20s.
,,	XXXIV.	,,	764–[illegible]	Mar. 1901 ,, April 1902			20s.
,,	XXXV.	,,	[illegible]–819	May 1902 ,, Jan. 1903			20s.

Cases for binding the Sets or Volumes can also be procured from the Publisher, price 1s. each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25s. each Volume.

The price of cases for binding the Volumes of the Second Series is 1s. 3d.

Vol. 1 of Series 2 (1904) has been published.

// PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SERIES 2.—VOL. 2.—PART 4.

CONTENTS.

Issued November 11, 1904.

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.;
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. Love. / W. Burnside.

34 St. Margaret's Road, Oxford.—A. E. H. Love.

The Croft, Bromley Road, Catford, S.E.—W. Burnside.

Section III., while the minimum values of the indices λ, μ are determined by the theorem of Section I. (§§ 1, 2).

The generating functions for the products $C_2 C_3$ are, therefore, as follows :—

Product.	G. F.	Product.	G. F.
$(a_1 a_2)^r C_3$	$\dfrac{x^{14}}{(1-x)^3}$	$(a_2 a_4)^r C_3$	$\dfrac{x^9}{(1-x)^3}$
$(a_1 a_3)^r C_3$	$\dfrac{x^{13}}{(1-x)^3}$	$(a_2 a_5)^r C_3$	$\dfrac{x^8}{(1-x)^3}$
$(a_1 a_4)^r C_3$	$\dfrac{x^{12}}{(1-x)^3}$	$(a_3 a_4)^r C_3$	$\dfrac{x^7}{(1-x)^3}$
$(a_1 a_5)^r C_3$	$\dfrac{x^{11}}{(1-x)^3}$	$(a_3 a_5)^r C_3$	$\dfrac{x^6}{(1-x)^3}$
$(a_2 a_3)^r C_3$	$\dfrac{x^{10}}{(1-x)^3}$	$(a_4 a_5)^r C_3$	$\dfrac{x^5}{(1-x)^3}$

Hence the generating function for all products $C_2 C_3$ is

$$\frac{x^5+x^6+\ldots+x^{13}+x^{14}}{(1-x)^3} = \frac{x^5-x^{15}}{(1-x)^4}.$$

(ii.) *Generating Function for Products* $C_1^3 C_2$.

Here the only possible reductions arise from the Jacobian identity, when the weight is unity. In this case we can insist on a_1 appearing in the symbolical determinant factor; so that there are only four linearly independent products of unit weight, while there are ten of any other weight.

Hence the generating function is $10x^2/(1-x)+4x$.

The generating functions for types and products of degree 5 may be tabulated thus :—

Forms.	G. F.	Forms.	G. F.
C_5	$\dfrac{x^{15}}{(1-x)^4}$	$C_1^2 C_3$	$\dfrac{10x^3}{(1-x)^3}$
$C_2 C_3$	$\dfrac{x^5-x^{15}}{(1-x)^4}$	$C_1^3 C_2$	$\dfrac{10x^2}{1-x}+4x$
$C_1 C_4$	$\dfrac{5x^7}{(1-x)^3}$	C_1^5	1
$C_1 C_2^2$	$\dfrac{5(x^4-x^7)}{(1-x)^3}$		

The sum of all these generating functions is $\frac{1}{(1-x)^4}$, which is known to be the generating function for the enumeration of the linearly independent covariants of degree 5: from this it follows that the perpetuant type of theorem is exact, and that we have found all the syzygies of degree 5.

V. (i.) PRODUCTS HAVING NO FACTOR OF DEGREE LESS THAN 2.—(ii.) PRODUCTS HAVING NO FACTOR OF DEGREE LESS THAN 3.

13. In the present section we shall show that—

(i.) The generating function for types and irreducible products of degree δ having no factor C_1 is

$$\frac{x^\delta}{(1-x)^{\delta-1}}.$$

(ii.) The generating function for types and irreducible products of degree δ having no factor C_1 or C_2 is

$$\frac{x^{\binom{\delta}{1}+\binom{\delta}{2}}}{(1-x)^{\delta-1}}.$$

It seems probable that, in general, the generating function for types and irreducible products of degree δ having no factor of degree κ or less is

$$\frac{x^{\binom{\delta}{1}+\binom{\delta}{2}+\dots+\binom{\delta}{\kappa}}}{(1-x)^{\delta-1}}.$$

This general result is verified in Section VI. in the case of

$$3\kappa+2 \geqslant \delta \geqslant 2\kappa.$$

(i.) *Products having no Factor of Degree less than* 2.

14. Those products which do contain a factor C_1 are all of the form

$$C_1^r . P_{\delta-r}, \quad r = 1, 2, \dots, \delta-2, \delta.$$

Let V_δ be the generating function for all types and products not having a factor C_1; then the expression

$$V_\delta+\binom{\delta}{1}V_{\delta-1}+\binom{\delta}{2}V_{\delta-2}+\dots+\binom{\delta}{r}V_{\delta-r}+\dots+\binom{\delta}{\delta-2}V_2+V_0$$

is clearly the generating function for all types and irreducible products of degree δ, and so we must have

$$\frac{1}{(1-x)^{\delta-1}} = V_\delta + \binom{\delta}{1} V_{\delta-1} + \ldots + \binom{\delta}{r} V_{\delta-r} + \ldots + \binom{\delta}{\delta-2} V_2 + V_0.$$

In the first place $V_0 = 1$: consider $\binom{\delta}{\delta-2} V_2$, the generating function for the products $C_1^{\delta-2} C_2$; if this product is of unit weight, so that C_2 is a Jacobian, we can insist on a definite letter being present in the symbolical determinant; and therefore the coefficient of x in $\binom{\delta}{\delta-2} V_2$ is $(\delta-1)$; for any other weight the products $C_2 C_1^{\delta-2}$ are irreducible, and so

$$\binom{\delta}{\delta-2} V_2 = (\delta-1)\,x + \binom{\delta}{\delta-2} \frac{x^2}{1-x}.$$

By reference to our previous results, we find that

$$V_3 = \frac{x^3}{(1-x)^2}, \quad V_4 = \frac{x^4}{(1-x)^3}, \quad V_5 = \frac{x^5}{(1-x)^4}.$$

Let us assume that

$$V_{\delta'} = \frac{x^{\delta'}}{(1-x)^{\delta'-1}} \quad \text{for the values } 3, 4, 5, \ldots, \delta-1 \text{ of } \delta';$$

then, from the above relation, we have

$$V_\delta = \frac{\{x+(1-x)\}^\delta}{(1-x)^{\delta-1}} - \left\{ \binom{\delta}{1} \frac{x^{\delta-1}}{(1-x)^{\delta-2}} + \ldots \right.$$

$$\left. + \binom{\delta}{r} \frac{x^{\delta-r}}{(1-x)^{\delta-r-1}} + \ldots + (\delta-1)\,x + 1 \right\}$$

$$= \frac{x^\delta}{(1-x)^{\delta-1}}, \text{ on expanding the first term.}$$

Hence by induction the result is true in general.

Therefore:—

The generating function for all types and irreducible products of degree δ which contain no factor of unit degree is $\dfrac{x^\delta}{(1-x)^{\delta-1}}$.

(ii.) *Products having no Factor of Degree less than 3.*

15. We proceed to find the generating functions for the following products:—

(1) C_2^m, (2) $C_2^m P_{\delta-2m}$, (3) $C_2^m C_3$, (4) $C_2^m C_4$;

where $P_{\delta-2m}$ is any product of degree $\delta-2m$ which contains no factor of degree < 3 and is neither C_3 nor C_4.

It will be seen that the peculiarity in the treatment of the products $C_2^m C_3$ and $C_2^m C_4$ arises from the Jacobian identity; it may be easily verified that this special identity yields no exceptional reductions of other products $C_2^m P_r$, even if P_r is one of the products C_3^2, C_4^2, $C_3 C_4$, $C_3 C_\kappa$, $C_4 C_\kappa$ (see § 17, Note).

(1) *Generating Function for C_2^m.*

We know that the generating function for C_2^2 is

$$\frac{x^4-x^7}{(1-x)^3} = \frac{x^4(1-x)(1-x^3)}{(1-x)^4}.$$

Consider first the product C_2^3: we have the following products:—

$$(a_1 a_6)^\lambda C_2^2, \quad (a_1 a_5)^\lambda C_2^2, \quad (a_1 a_4)^\lambda C_2^2, \quad (a_1 a_3)^\lambda C_2^2, \quad (a_1 a_2)^\lambda C_2^2.$$

Now any of these products is reducible if it is expressible in terms of products following it: thus for $(a_1 a_2)^\lambda C_2^2$ there is no such reduction, so that $\lambda \geqslant 2$, and the generating function for such products is $\dfrac{x^2 V_2}{1-x}$, where V_2 is the generating function of C_2^2; for $(a_1 a_3)^\lambda C_2^2$ there is a reduction corresponding to the interchange of a_2 and a_3, so that $\lambda \geqslant 3$, and so on; finally, for $(a_1 a_6)^\lambda C_2^2$ there are reductions corresponding to the interchanges of a_6 with a_2, a_3, a_4, a_5 respectively, and therefore $\lambda \geqslant 6$; hence the generating function for all these products is

$$\frac{x^2+x^3+x^4+x^5+x^6}{1-x} V_2 = \frac{x^2-x^7}{(1-x)^2} V_2 = \frac{x^6(1-x)(1-x^3)(1-x^5)}{(1-x)^6}.$$

Assume that the generating function for C_2^{m-1} is

$$V_{m-1} = x^{2m-2}\frac{(1-x)(1-x^3)\dots(1-x^{2m-3})}{(1-x)^{2m-2}}.$$

Then the products C_2^m are

$$(a_1 a_{2m})^\lambda C_2^{m-1}, \quad (a_1 a_{2m-1})^\lambda C_2^{m-1}, \quad \dots, \quad (a_1 a_r)^\lambda C_2^{m-1}, \quad \dots, \quad (a_1 a_2)^\lambda C_2^{m-1}.$$

Here, as before, any product is reducible if it is expressible in terms of products following it: thus for $(a_1 a_r)^\lambda C_2^{m-1}$ there are $(r-2)$ reductions corresponding to the interchanges of a_r with a_2, a_3, ..., a_{r-1} respectively, and so $\lambda \geqslant r$; for, by Section III., these reductions are all independent; the

generating function for this set is $\frac{x^r}{1-x} V_{m-1}$, and therefore the generating function for all products C_2^m is

$$V_m = \frac{x^2+x^3+\ldots+x^{2m}}{1-x} V_{m-1} = \frac{(x^2-x^{2m+1})}{(1-x)^2} V_{m-1}$$

$$= \frac{x^{2m}(1-x)(1-x^3)\ldots(1-x^{2m-1})}{(1-x)^{2m}}.$$

Therefore, by induction, the generating function for products C_2^m is

$$\frac{x^{2m}(1-x)(1-x^3)\ldots(1-x^{2m-1})}{(1-x)^{2m}}.$$

(2) *Generating Function for $C_2^m P_{8-2m}$ (where P_{8-2m} is any form or product, other than C_3 or C_4, which contains neither C_1 nor C_2).*

16. *Lemma.*—If the roots of $y^n - p_1 y^{n-1} + p_2 y^{n-2} - \ldots + (-)^n p_n = 0$ are $1, x, x^2, \ldots, x^{n-1}$, then

(1) $$p_\kappa = x^{\binom{\kappa}{2}} \frac{(1-x^n)(1-x^{n-1})\ldots(1-x^{n-\kappa+1})}{(1-x)(1-x^2)\ldots(1-x^\kappa)}.$$

(2) $$p_n + (1-x)p_{n-2} + (1-x)(1-x^3)p_{n-4} + \ldots$$
$$+ (1-x)(1-x^3)\ldots(1-x^{n-1}) = 1, \quad n \text{ even};$$

$$p_n + (1-x)p_{n-2} + (1-x)(1-x^3)p_{n-4} + \ldots$$
$$+ (1-x)(1-x^3)\ldots(1-x^{n-2})p_1 = 1, \quad n \text{ odd}.$$

(1) Assuming the theorem true for all values up to and including n, we shall show that it is also true for the value $(n+1)$; it is obviously true when $n = 1, 2$.

If $y^{n+1} - q_1 y^n + q_2 y^{n-1} + \ldots + (-)^{n-1} q_{n+1}$

$$\equiv \left(y^n - p_1 y^{n-1} + p_2 y^{n-2} + \ldots + (-)^n p_n\right)(y - x^n),$$

$$q_\kappa = p_\kappa + x^n p_{\kappa-1}, \quad \kappa = 1, 2, \ldots, n,$$

$$q_{n+1} = p_n x^n;$$

and therefore, by hypothesis, if $\kappa = 1, 2, \ldots, n$,

$$q_\kappa = x^{\binom{\kappa}{2}} \frac{(1-x^n)\ldots(1-x^{n-\kappa+1})}{(1-x)\ldots(1-x^\kappa)} + x^{n+\binom{\kappa-1}{2}} \frac{(1-x^n)\ldots(1-x^{n-\kappa+2})}{(1-x)\ldots(1-x^{\kappa-1})}$$

$$= x^{\binom{\kappa}{2}} \frac{(1-x^n)\ldots(1-x^{n-\kappa+2})}{(1-x)\ldots(1-x^\kappa)} \{1-x^{n-\kappa+1}+x^{n-\kappa+1}(1-x^\kappa)\}$$

$$= x^{\binom{\kappa}{2}} \frac{(1-x^{n+1})(1-x^n)\ldots(1-x^{n-\kappa+2})}{(1-x)(1-x^2)\ldots(1-x^\kappa)},$$

and $$q_{n+1} = x^n . x^{\binom{n}{2}} = x^{\binom{n+1}{2}}.$$

Therefore the result is true universally.

(2) Let the roots of

$$y^{n+1}-q_1 y^n+\ldots+(-)^{n+1} q_{n+1} = 0 \quad \text{be} \quad 1, x, x^2, \ldots, x^n,$$

and those of

$$y^{n-1}-r_1 y^{n-2}+\ldots+(-)^{n-1} r_{n-1} = 0 \quad \text{be} \quad 1, x, x^2, \ldots, x^{n-2}.$$

Assuming the result for the p's and the r's, it will be sufficient to prove it for the q's; for brevity we consider only the case of n being even. Then

$$q_{n+1}+(1-x)q_{n-1}+(1-x)(1-x^3)q_{n-3}+\ldots+(1-x)(1-x^3)\ldots(1-x^{n-1})q_1$$
$$= x^n p_n+(1-x)\{p_{n-1}+x^n p_{n-2}\}+\ldots+(1-x)(1-x^3)\ldots(1-x^{n-1})(p_1+x^n),$$

and $$p_\kappa = x^{\binom{\kappa}{2}} \frac{(1-x^n)\ldots(1-x^{n-\kappa+1})}{(1-x)\ldots(1-x^\kappa)} = \frac{1-x^n}{1-x^{n-\kappa}} r_\kappa$$

by the first part of the Lemma. Therefore

$$q_{n+1}+(1-x)q_{n-1}+\ldots+(1-x)(1-x^3)\ldots(1-x^{n-1})q_1$$
$$= x^n\{p_n+(1-x)p_{n-2}+\ldots+(1-x)(1-x^3)\ldots(1-x^{n-1})\}$$
$$+(1-x)p_{n-1}+(1-x)(1-x^3)p_{n-3}+\ldots+(1-x)(1-x^3)\ldots(1-x^{n-1})p_1$$
$$= x^n+(1-x^n)\{r_{n-1}+(1-x)r_{n-3}+\ldots+(1-x)(1-x^3)\ldots(1-x^{n-3})r_1\}$$
$$= x^n+(1-x^n) \text{ (by hypothesis)}$$
$$= 1.$$

17. Let V be the generating function of $P_{\delta-2m}$, $\delta = 2m+r$ being the total degree, and let the letters in C_2^m be

$$a_{\kappa_1}, a_{\kappa_2}, \ldots, a_{\kappa_{2m}}, \quad \text{where} \quad \kappa_1 < \kappa_2 < \kappa_3 \ldots < \kappa_{2m}.$$

Then, by definition, the interchange of the letter $a_{\kappa_{2m}}$ with any one of the letters $a_{\kappa_{2m}+1}, a_{\kappa_{2m}+2}, \ldots, a_{\kappa_\delta}$ implies a reduction, and, by Section III., corre-

sponding to each of these interchanges the weight of C_2^m is increased by unity. So in general, by interchanging $a_{\kappa_{2m-\epsilon}}$ with succeeding letters occurring in $P_{\delta-2m}$, the minimum weight for irreducibility is increased by $(\delta-\kappa_{2m-\epsilon}-\epsilon)$, since this is the possible number of interchanges. Here ϵ may be any one of 0, 1, 2, ..., $2m-1$, and therefore, by interchanging all the letters of C_2^m with the letters of $P_{\delta-2m}$ succeeding, the weight is increased by

$$(\delta-\kappa_1)+(\delta-\kappa_2)+\ldots+(\delta-\kappa_{2m})-\binom{2m}{2}.$$

If the $2m$ letters of C_2^m are fixed, the generating function is

$$\frac{x^{2m}(1-x)(1-x^3)\ldots(1-x^{2m-1})}{(1-x)^{2m-1}}V;$$

and therefore, if these $2m$ letters are chosen in all possible ways and interchanged with the letters of $P_{\delta-2m}$, the generating function required is

$$V.x^{-\binom{2m}{2}}\frac{x^{2m}(1-x)(1-x^3)\ldots(1-x^{2m-1})}{(1-x)^{2m-1}}\Sigma x^{(\delta-\kappa_1)+(\delta-\kappa_2)+\ldots+(\delta-\kappa_{2m})},$$

where the Σ applies to all possible choices of the suffixes $\kappa_1, \kappa_2, \ldots, \kappa_{2m}$ from the suffixes 1, 2, 3, ..., δ.

The numbers $\delta-\kappa_1$, $\delta-\kappa_2$, ..., $\delta-\kappa_{2m}$ are all different and each is one of the numbers 0, 1, 2, ..., $\delta-1$; so, by the Lemma,

$$\Sigma x^{\delta-\kappa_1+\delta-\kappa_2+\ldots+\delta-\kappa_{2m}} = p_{2m} = x^{\binom{2m}{2}}\frac{(1-x^\delta)\ldots(1-x^{\delta-2m+1})}{(1-x)\ldots(1-x^{2m})}.$$

Therefore the generating function for $C_2^m P_{\delta-2m}$ is

$$V\frac{x^{2m}(1-x)(1-x^3)\ldots(1-x^{2m-1})}{(1-x)^{2m}}\frac{(1-x^\delta)(1-x^{\delta-1})\ldots(1-x^{\delta-2m+1})}{(1-x)(1-x^2)\ldots(1-x^{2m})},$$

where V is the generating function for $P_{\delta-2m}$ and $\delta = 2m+r$ is the total degree.

Note.—If $P_{\delta-2m}$ contains a factor C_3 or C_4 which is a Jacobian, any extra reductions due to a Jacobian transformation will be included in V, the generating function for $P_{\delta-2m}$.

(3) *Generating Function for $C_2^m C_3$.*

18. A different kind of reduction is introduced only in the case where C_3 is a Jacobian: three cases must be separately considered—

(*a*) C_3 is a Jacobian and does not contain a_1: here there is a new reduction.

(β) C_3 is a Jacobian and contains a_1, but not a_2: here there is a new reduction if, and only if, C_3 is of the form (C_2, a_1).

(γ) C_3 contains both a_1 and a_2: here there is never any new reduction.

(a) If the product is irreducible, C_3 can be written in the form $(a_{r_1} a_{r_2})^{\lambda} (a_{r_1} a_{r_3})^{\mu}$, $\lambda \geqslant 4$, $\mu \geqslant 2$, by Section I., Cor. 2.

The number of reductions due to replacing the letters of C_3 by earlier letters contained in C_2^m is

$$(r_1-1)+(r_2-2)+(r_3-3) = (r_1-2)+(r_2-2)+(r_3-2),$$

and therefore the generating function for such products is

$$\frac{x^6}{(1-x)^2}\,\frac{x^{2m}(1-x)\dots(1-x^{2m-1})}{(1-x)^{2m}} \sum_{r_1, r_2, r_3=2}^{2m+3} x^{(r_1-2)+(r_2-2)+(r_3-2)}$$

$$= x^{2m+6}\,\frac{(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m+2}}\,\frac{x^3(1-x^{2m+2})(1-x^{2m+1})(1-x^{2m})}{(1-x)(1-x^2)(1-x^3)},$$

by the Lemma (§ 16).

(β) If the product is irreducible, C_3 can be written in the form

$$(a_1 a_{r_1})^{\lambda} (a_1 a_{r_2})^{\mu}, \quad \lambda \geqslant 3, \quad \mu \geqslant 1,$$

by Section I., § 1, and the number of reductions due to replacing a_{r_1}, a_{r_2} by earlier letters contained in C_2^m is

$$(r_1-2)+(r_2-3) = (r_1-3)+(r_2-3)+1;$$

therefore the generating function for such products is

$$\frac{x^5}{(1-x)^2}\,\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}} \sum_{r_1, r_2=3}^{2m+3} x^{(r_1-3)+(r_2-3)}$$

$$= x^{2m+5}\,\frac{(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m+2}}\,\frac{x(1-x^{2m+1})(1-x^{2m})}{(1-x)(1-x^2)}.$$

(γ) Here it makes no difference to the irreducibility of the product whether C_3 is a Jacobian or not; so we may write

$$C_3 = (a_1 a_2)^{\lambda} (a_1 a_r)^{\mu}, \quad \lambda \geqslant 2, \quad \mu \geqslant 1;$$

and the number of reductions due to replacing a_r by earlier letters contained in C_2^m is $(r-3)$; and therefore the generating function for such products is

$$\frac{x^3}{(1-x)^2}\,\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}} \sum_{r=3}^{2m+3} x^{r-3}$$

$$= x^{2m+3}\,\frac{(1-x)(1-x^3)\dots(1-x^{2m+1})}{(1-x)^{2m+3}}.$$

So, finally, the generating function for all products $C_2^m C_3$ is

$$\frac{x^{2m+3}(1-x)(1-x^3)\dots(1-x^{2m+1})}{(1-x)^{2m+3}}\left\{1+\frac{x^3(1-x^{2m})}{1-x^2}+x^6\frac{(1-x^{2m})(1-x^{2m+2})}{(1-x^2)(1-x^3)}\right\}$$

$$=\frac{x^\delta}{(1-x)^{\delta-1}}\left\{(1-x)(1-x^3)\dots(1-x^{\delta-4})p_3+(1-x)(1-x^3)\dots(1-x^{\delta-2})p_1\right\},$$

where $\delta=2m+3$ is the total degree, and (as in the Lemma of § 16)

$$p_\kappa=x^{\binom{\kappa}{2}}\frac{(1-x^\delta)(1-x^{\delta-1})\dots(1-x^{\delta-\kappa+1})}{(1-x)(1-x^2)\dots(1-x^\kappa)}.$$

(4) *Generating Function for $C_2^m C_4$.*

19. A different kind of reduction from those considered in § 17 is introduced when C_4 is a Jacobian of the form (C_2, C_2), and does not contain a_1, and in no other case.

If C_4 does not contain a_1, then, by Section I., if the product is irreducible, it can be written in the form

$$C_4\equiv(a_{r_1}a_{r_2})^{\lambda_1}(a_{r_1}a_{r_3})^{\lambda_2}(a_{r_1}a_{r_4})^{\lambda_3},\quad\text{where}\quad\lambda_1\geqslant 6,\ \lambda_2\geqslant 3,\ \lambda_3\geqslant 1,$$

and, in the same way as before, we find that the generating function for such products is

$$\frac{x^{10}}{(1-x)^3}\,\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}}\sum x^{(r_1-1)+(r_2-2)+(r_3-3)+(r_4-4)}$$

$$=x^{2m+10}\frac{(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m+3}}\,\frac{x^4(1-x^{2m+3})\dots(1-x^{2m})}{(1-x)(1-x^2)(1-x^3)(1-x^4)}.$$

If C_4 does contain the letter a_1, we write

$$C_4\equiv(a_1a_{r_1})^{\lambda_1}(a_1a_{r_2})^{\lambda_2}(a_1a_{r_3})^{\lambda_3},\quad\text{where}\quad\lambda_1\geqslant 4,\ \lambda_2\geqslant 2,\ \lambda_3\geqslant 1;$$

and the generating function for such products is

$$\frac{x^7}{(1-x)^3}\,\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}}\sum x^{(r_1-2)+(r_2-3)+(r_3-4)}$$

$$=x^{2m+7}\frac{(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m+3}}\,\frac{(1-x^{2m+3})(1-x^{2m+2})(1-x^{2m+1})}{(1-x)(1-x^2)(1-x^3)}.$$

Hence, adding and simplifying these two results, we find for the complete

generating function of products $C_2^m C_4$,

$$\frac{x^\delta}{(1-x)^{\delta-1}}\left\{(1-x)(1-x^3)\dots(1-x^{\delta-5})p_4+(1-x)(1-x^3)\dots(1-x^{\delta-3})p_2+(1-x)(1-x^3)\dots(1-x^{\delta-1})\left(1-\frac{1}{1-x}\right)\right\},$$

where $\delta = 2m+4$ is the total degree, and p_2, p_4 are defined above (§ 16).

20. We are now in a position to find V_δ, the generating function for the types and those irreducible products which contain no factor of degree less than 3; for we know that $\frac{x^\delta}{(1-x)^{\delta-1}}$ is the generating function for the irreducible forms with no factor C_1, and every such product is of the form $C_2^m P_{\delta-2m}$. We must consider two cases according as δ is even or odd.

δ even.—

$$\frac{x^\delta}{(1-x)^{\delta-1}} = V_\delta + \frac{x^2(1-x)}{(1-x)^2}\,\frac{(1-x^\delta)(1-x^{\delta-1})}{(1-x)(1-x^2)}\,V_{\delta-2}+\dots$$

$$+\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}}\,\frac{(1-x^\delta)(1-x^{\delta-1})\dots(1-x^{\delta-2m+1})}{(1-x)(1-x^2)\dots(1-x^{2m})}\,V_{\delta-2m}+\dots$$

$$+\frac{x^\delta}{(1-x)^{\delta-1}}\left\{(1-x)(1-x^3)\dots(1-x^{\delta-5})p_4+(1-x)(1-x^3)\dots(1-x^{\delta-3})p_2+(1-x)(1-x^3)\dots(1-x^{\delta-1})\left(1-\frac{1}{1-x}\right)\right\}$$

$$+x^\delta\,\frac{(1-x)(1-x^3)\dots(1-x^{\delta-1})}{(1-x)^\delta};$$

the last two terms being the generating functions for $C_2^n C_4$ and C_2^{n+2} respectively, where $\delta = 2n+4$.

δ odd.—

$$\frac{x^\delta}{(1-x)^{\delta-1}} = V_\delta + \frac{x^2(1-x)}{(1-x)^2}\,\frac{(1-x^\delta)(1-x^{\delta-1})}{(1-x)(1-x^2)}\,V_{\delta-2}+\dots$$

$$+\frac{x^{2m}(1-x)(1-x^3)\dots(1-x^{2m-1})}{(1-x)^{2m}}\,\frac{(1-x^\delta)(1-x^{\delta-1})\dots(1-x^{\delta-2m+1})}{(1-x)(1-x^2)\dots(1-x^{2m})}\,V_{\delta-2m}+\dots$$

$$+\frac{x^\delta}{(1-x)^{\delta-1}}\left\{(1-x)(1-x^3)\dots(1-x^{\delta-4})p_3+(1-x)(1-x^3)\dots(1-x^{\delta-2})p_1\right\};$$

the last term being the generating function for $C_2^n C_3$, where $\delta = 2n+3$.

Now assume that, for $r < \delta$ and > 4,

$$V_r = \frac{x^{\binom{r}{1}+\binom{r}{2}}}{(1-x)^{r-1}},$$

and it is easily verified, from the two results of the preceding Lemma (§ 16), that, whether δ is even or odd,

$$V_\delta = \frac{x^{\binom{\delta}{1}+\binom{\delta}{2}}}{(1-x)^{\delta-1}}.$$

Hence the result is established in general.

VI. Products of the kinds (1) C_κ^2; (2) $C_\kappa C_m$, $\kappa < m < 2\kappa$; (3) $C_\kappa C_{2\kappa}$.

21. The general principle involved by the present application of Stroh's syzygies of degree 4 may be explained thus: Suppose we are seeking for the syzygies in which a product $C_\kappa C_m$ occurs. Let us divide up the letters of C_κ into two sets,

$$a_1, a_2, \ldots, a_{r_1}, \qquad b_1, b_2, \ldots, b_{r_2};$$

and those of C_m into two sets,

$$c_1, c_2, \ldots, c_{s_1}, \qquad d_1, d_2, \ldots, d_{s_2}.$$

Let α, β, γ, δ be covariants, whose symbolical characters are comprised in the sets $a_1, a_2, \ldots, a_{r_1}$: $b_1, b_2, \ldots, b_{r_2}$: $c_1, c_2, \ldots, c_{s_1}$: $d_1, d_2, \ldots, d_{s_2}$ respectively. Then C_κ can be expressed in terms of transvectants of the form $(\alpha, \beta)^\lambda$, and C_m can be expressed in terms of transvectants of the form $(\gamma, \delta)^\mu$.

Consider the Stroh syzygy

$$\{(\alpha\beta)+(\gamma\delta)\}^\omega - \{(\alpha\delta)+(\gamma\beta)\}^\omega = 0,$$

which may be written $\quad e^{(\alpha\beta)D_\delta}(\gamma\delta)^\omega = e^{(\alpha\delta)D_\beta}(\gamma\beta)^\omega,$

where
$$D_\beta = x_1\frac{\partial}{\partial\beta_2} - x_2\frac{\partial}{\partial\beta_1},$$

$$D_\delta = x_1\frac{\partial}{\partial\delta_2} - x_2\frac{\partial}{\partial\delta_1}.$$

Now let us suppose $(\alpha\delta)^\rho(\gamma\beta)^\sigma$ is a product which, by the definition of Section III., succeeds the product $(\alpha\beta)^{\rho'}(\gamma\delta)^{\sigma'}$ in the sequence of products: then we have the relation

$$e^{(\alpha\beta)D_\delta}(\gamma\delta)^\omega = \text{reducible terms}$$

or
$$e^{(\alpha\beta)D_\delta}(c_1, c_2, \ldots, c_{s_1}, \delta) = \text{reducible terms},$$

where $(c_1, c_2, \ldots, c_{s_1}, \delta)$ is the symbolical product obtained by replacing γ by the letters of which it is composed. Moreover, since $c_1, c_2, \ldots, c_{s_1}, \delta$ are the only letters involved in this symbolical product, we know that

$$(D_{c_1}+D_{c_2}+\ldots+D_{c_{s_1}}+D_\delta)(c_1, c_2, \ldots, c_{s_1}, \delta) = 0.$$

Therefore we may write the relation

$$\left.\begin{aligned} &e^{-(\alpha\beta)\{D_{c_1}+D_{c_2}+\ldots+D_{c_{s_1}}\}}\, C_m = \text{reducible terms} \\ \text{or} \qquad &e^{(\alpha\beta)\{D_{d_1}+D_{d_2}+\ldots+D_{d_{s_2}}\}}\, C_m = \text{reducible terms} \end{aligned}\right\}. \tag{1}$$

This relation gives us a reduction for the product $C_\kappa C_m$ by virtue of the fact that the product

$$(a_1, a_2, \ldots, a_{r_1}, \; b_1, b_2, \ldots, b_{r_2})(c_1, c_2, \ldots, c_{s_1}, \; d_1, d_2, \ldots, d_{s_2})$$

precedes, in our fixed arrangement, the product

$$(a_1, a_2, \ldots, a_{r_1}, \; d_1, d_2, \ldots, d_{s_2})(b_1, b_2, \ldots, b_{r_1}, \; c_1, c_2, \ldots, c_{s_1}).$$

In this way we obtain a series of relations such as (1): such relations have been considered in Section III. [§ 10, (iii.)], and have there been shewn to be all independent. It follows, therefore, that we obtain one reduction for $C_\kappa C_m$ corresponding to every product $C_{\kappa'} C_{m'}$ following $C_\kappa C_m$ in the sequence of products already determined, provided that each of $C_{\kappa'}, C_{m'}$ contains letters from each of the factors C_κ, C_m; and for every such reduction the minimum weight of C_κ for irreducibility is increased by unity.

(i.) *Generating Function for C_κ^2.*

22. Let

$$\begin{aligned} C_\kappa^2 &\equiv (a_1 a_{r_1})^{\lambda_1}(a_1 a_{r_2})^{\lambda_2} \ldots (a_1 a_{r_{\kappa-1}})^{\lambda_{\kappa-1}}(b_1 b_2)^{\mu_1}(b_1 b_3)^{\mu_2} \ldots (b_1 b_\kappa)^{\mu_{\kappa-1}}, \\ &\equiv C_\kappa^{(1)} C_\kappa^{(2)}, \end{aligned}$$

where $r_1 < r_2 < \ldots < r_{\kappa-1}$, and the letters involved are $a_1, a_2, \ldots, a_{2\kappa}$ in some order. If either $C_\kappa^{(1)}$ or $C_\kappa^{(2)}$ is a Jacobian, the product is reducible, since it is expressible in terms of products $C_{\kappa-\epsilon} C_{\kappa+\epsilon}$: hence, by the second Corollary of Section I., each of $C_\kappa^{(1)}$ or $C_\kappa^{(2)}$ is of weight $(2^\kappa-2)$ at least.

In the first place we suppose the letters of $C_\kappa^{(1)}$ and of $C_\kappa^{(2)}$ to be fixed and we disregard their arrangement: we then find how many reductions are possible by virtue of expressing $C_\kappa^{(1)} C_\kappa^{(2)}$ in the form $\Sigma C_{\kappa-\epsilon} C_{\kappa+\epsilon}$. To this end we consider all possible products $C_r C_{2\kappa-r}$, such that r takes any value from 2 to $\kappa-1$, where each of $C_r, C_{2\kappa-r}$ contains letters from each of the factors $C_\kappa^{(1)}, C_\kappa^{(2)}$: then we know that, corresponding to every such product, there is a reduction for $C_\kappa^{(1)} C_m^{(2)}$, and that these reductions are all

independent; so that, corresponding to each such product, the minimum weight of $C_\kappa^{(1)}$ for the irreducibility of $C_\kappa^{(1)} C_\kappa^{(2)}$ is increased by unity.

The letters of C_r may be chosen in $\binom{2\kappa}{r}$ ways, but we must exclude every case where these r letters are all included among the letters of $C_\kappa^{(1)}$ or among those of $C_\kappa^{(2)}$, for in such cases there is no corresponding reduction of the product $C_\kappa^{(1)} C_\kappa^{(2)}$; hence the number of choices for $C_r C_{2\kappa-r}$ is

$$\binom{2\kappa}{r} - \binom{\kappa}{r} - \binom{\kappa}{r}.$$

So the total number of reductions of this kind for $C_\kappa^{(1)} C_\kappa^{(2)}$ is

$$\sum_{r=2}^{r=\kappa-1} \left\{\binom{2\kappa}{r} - 2\binom{\kappa}{r}\right\} = \tfrac{1}{2}\left[2^{2\kappa} - 4\kappa - 2 - \binom{2\kappa}{\kappa}\right] - 2[2^\kappa - \kappa - 2]$$

$$= 2^{2\kappa-1} - 2^{\kappa+1} - \tfrac{1}{2}\binom{2\kappa}{\kappa} + 3.$$

Further, by the Jacobian identities, which are independent of these reductions, the minimum weight of $C_\kappa^{(1)}$ is $2^\kappa - 2$. So, whatever are the letters of $C_\kappa^{(1)}$, the minimum weight of $C_\kappa^{(1)}$ to ensure the irreducibility of $C_\kappa^{(1)} C_\kappa^{(2)}$ is

$$2^{2\kappa-1} - 2^\kappa - \tfrac{1}{2}\binom{2\kappa}{\kappa} + 1.$$

It remains to discover how many products $P_1 P_2$ follow $C_\kappa^{(1)} C_\kappa^{(2)}$ in the sequence of products already determined, where P_1 and P_2 are each of degree κ, and P_1 is taken as that factor which contains a_1.

If $r_1 > 2$, there is a reduction corresponding to every product $P_1 P_2$, where P_1 contains a_2, and the number of such products is $\binom{2\kappa-2}{\kappa-2}$.

If $r_1 > 3$, there is a reduction corresponding to every product $P_1 P_2$, where P_1 contains a_3 and not a_2, and the number of such products is $\binom{2\kappa-3}{\kappa-2}$.

Continuing in this way, we obtain

$$\binom{2\kappa-2}{\kappa-2} + \binom{2\kappa-3}{\kappa-2} + \ldots + \binom{2\kappa-r_1+1}{\kappa-2} = \binom{2\kappa-1}{\kappa-1} - \binom{2\kappa-r_1+1}{\kappa-1}$$

reductions.

If we have any further reductions, P_1 must contain both a_1 and a_{r_1}, and we get a reduction if P_1 contains also a_{r_1+1} where $r_1+1 < r_2$; in

this way we have

$$\binom{2\kappa-r_1-1}{\kappa-3}+\binom{2\kappa-r_1-2}{\kappa-3}+\dots+\binom{2\kappa-r_2+1}{\kappa-3}=\binom{2\kappa-r_1}{\kappa-2}-\binom{2\kappa-r_2+1}{\kappa-2}$$

reductions.

The same argument is pursued for each of the letters of $C_\kappa^{(1)}$ in turn: hence the number of reductions, corresponding to the products $C_\kappa^2 = P_1P_2$, which follow $C_\kappa^{(1)}C_\kappa^{(2)}$ in the determined sequence is

$$\binom{2\kappa-1}{\kappa-1}-\binom{2\kappa-r_1+1}{\kappa-1}+\binom{2\kappa-r_1}{\kappa-2}-\binom{2\kappa-r_2+1}{\kappa-2}+\dots-\binom{2\kappa-r_{\kappa-1}+1}{1}$$

$$=\binom{2\kappa-1}{\kappa-1}-\binom{2\kappa-r_1}{\kappa-1}-\binom{2\kappa-r_2}{\kappa-2}-\binom{2\kappa-r_3}{\kappa-3}-\dots-\binom{2\kappa-r_{\kappa-1}}{1}-1.$$

Hence, if the letters of $C_\kappa^{(1)}$ are $a_1, a_{r_1}, \dots, a_{r_{\kappa-1}}$, its minimum weight is

$$2^{2\kappa-1}-2^\kappa-\left[\binom{2\kappa-r_1}{\kappa-1}+\binom{2\kappa-r_2}{\kappa-2}+\dots+\binom{2\kappa-r_{\kappa-1}}{1}\right],$$

and, since the minimum weight of $C_\kappa^{(2)}$ is $(2^\kappa-2)$, the generating function for the irreducible products C_κ^2 is

$$\frac{x^{2^{2\kappa-1}-2}}{(1-x)^{2\kappa-2}}\sum_{r_1, r_2, \dots, r_{\kappa-1}} x^{-\binom{2\kappa-r_1}{\kappa-1}-\binom{2\kappa-r_2}{\kappa-2}-\dots-\binom{2\kappa-r_{\kappa-1}}{1}},$$

where $r_1, r_2, \dots, r_{\kappa-1}$ take all values satisfying

$$1<r_1<r_2<\dots<r_{\kappa-1}<2\kappa+1.$$

We proceed to show that

$$S \equiv \sum_{r_1, r_2, \dots, r_{\kappa-1}} x^{-\binom{2\kappa-r_1}{\kappa-1}-\binom{2\kappa-r_2}{\kappa-2}-\dots-\binom{2\kappa-r_{\kappa-1}}{1}}=\frac{1-x^{-\binom{2\kappa-1}{\kappa-1}}}{1-x^{-1}}.$$

First find the sum for the possible values $r_{\kappa-2}+1, r_{\kappa-2}+2, \dots, 2\kappa$ of $r_{\kappa-1}$; we have

$$\sum_{r_{\kappa-1}} x^{-\binom{2\kappa-r_{\kappa-1}}{1}}=\frac{1-x^{-\binom{2\kappa-r_{\kappa-2}}{1}}}{1-x^{-1}};$$

and therefore

$$S \equiv \frac{1}{1-x^{-1}}\left\{\sum_{r_1, r_2, \dots, r_{\kappa-2}}\left(x^{-\binom{2\kappa-r_1}{\kappa-1}-\dots-\binom{2\kappa-r_{\kappa-2}}{2}} - x^{-\binom{2\kappa-r_1}{\kappa-1}-\dots-\binom{2\kappa-r_{\kappa-3}}{3}-\binom{2\kappa-r_{\kappa-2}+1}{2}}\right)\right\};$$

also
$$\sum_{r_{\kappa-2}=r_{\kappa-3}+1}^{2\kappa-1}\left\{x^{-\binom{2\kappa-r_{\kappa-2}}{2}}-x^{-\binom{2\kappa-r_{\kappa-2}+1}{2}}\right\}=1-x^{-\binom{2\kappa-r_{\kappa-3}}{2}};$$

and therefore

$$S \equiv \frac{1}{1-x^{-1}} \left\{ \sum_{r_1, r_2, \ldots, r_{\kappa-3}} \left(x^{-\binom{2\kappa-r_1}{\kappa-1} - \ldots - \binom{2\kappa-r_{\kappa-3}}{3}} - x^{-\binom{2\kappa-r_1}{\kappa-1} - \ldots - \binom{2\kappa-r_{\kappa-3}+1}{3}} \right) \right\}$$

$$\equiv \frac{1}{1-x^{-1}} \sum_{r_1, r_2, \ldots, r_{\kappa-\epsilon}} \left\{ x^{-\binom{2\kappa-r_1}{\kappa-1} - \ldots - \binom{2\kappa-r_{\kappa-\epsilon}}{\epsilon}} - x^{-\binom{2\kappa-r_1}{\kappa-1} - \ldots - \binom{2\kappa-r_{\kappa-\epsilon}+1}{\epsilon}} \right\},$$

by repeated application of the same method.

Hence, finally,

$$S \equiv \frac{1}{1-x^{-1}} \sum_{r_1=2}^{r_1=\kappa+2} \left\{ x^{-\binom{2\kappa-r_1}{\kappa-1}} - x^{-\binom{2\kappa-r_1+1}{\kappa-1}} \right\} = \frac{1}{1-x^{-1}} \left\{ 1 - x^{-\binom{2\kappa-1}{\kappa-1}} \right\}.$$

So the generating function for C_κ^2 is

$$\frac{x^{2^{2\kappa-1}-2}}{(1-x)^{2\kappa-2}} \frac{1-x^{-\binom{2\kappa-1}{\kappa-1}}}{1-x^{-1}} = \frac{x^{2^{2\kappa-1}-1-\frac{1}{2}\binom{2\kappa}{\kappa}} \left\{1-x^{\frac{1}{2}\binom{2\kappa}{\kappa}}\right\}}{(1-x)^{2\kappa-1}},$$

since $$\binom{2\kappa-1}{\kappa-1} = \binom{2\kappa-1}{\kappa} = \tfrac{1}{2}\binom{2\kappa}{\kappa}.$$

(ii.) *Generating Function for $C_\kappa C_m$, where $\kappa < m < 2\kappa$.*

28. In the first place we suppose the letters of C_κ and C_m to be fixed, and we disregard their arrangement; we seek for products $C_r C_s$ $(r+s = m+\kappa)$, such that r takes all values from 2 to $\kappa-1$, where each of $C_r C_s$ contains letters from each of $C_\kappa C_m$; then, as before, for every such product the minimum weight of C_κ, for the irreducibility of the product $C_\kappa C_m$, is increased by unity.

The letters of C_r may be chosen in $\binom{\kappa+m}{r}$ ways; but we must exclude the cases where these r letters are all included among the letters of C_κ or among those of C_m; hence the total number of such products and therefore of reductions is

$$\sum_{r=2}^{r=\kappa-1} \left\{ \binom{\kappa+m}{r} - \binom{\kappa}{r} - \binom{m}{r} \right\} = \sum_{r=1}^{r=\kappa-1} \left\{ \binom{m+\kappa}{r} - \binom{m}{r} \right\} - (2^\kappa - 2).$$

Independently of these reductions the minimum weight of C_κ must be $2^\kappa - 2$, for, if C_κ is a Jacobian, the product $C_\kappa C_m$ is certainly reducible; hence the minimum weight of C_κ for irreducibility is

$$\sum_{r=1}^{r=\kappa-1} \left\{ \binom{m+\kappa}{r} - \binom{m}{r} \right\}.$$

Again, if C_m is a Jacobian of the form (C_l, C_{m-l}), then

$$C_\kappa C_m = (C_l, C_\kappa)\, C_{m-l} - (C_{m-l}, C_\kappa)\, C_l,$$

and this is a reduction if both l and $m-l$ are less than κ ($m < 2\kappa$). So l may have the values $\kappa-1$, $\kappa-2$, ..., $m-\kappa+1$ (it is to be remarked that the values θ, $m-\theta$ of l lead to the same result); now the letters of C_m can be chosen for C_l in $\binom{m}{l}$ ways, so that the number of ways in which C_m can be chosen to give rise to a Jacobian reduction, whatever be the letters of C_κ, is

$$\tfrac{1}{2}\left\{\binom{m}{\kappa-1}+\binom{m}{\kappa-2}+\dots+\binom{m}{m-\kappa+1}\right\}$$

$$= \tfrac{1}{2}\{2^m-2\} - \left\{\binom{m}{\kappa}+\binom{m}{\kappa+1}+\dots+\binom{m}{m-1}\right\}.$$

Moreover, by the perpetuant type theorem, the minimum weight of C_m is $2^{m-1}-1$, and all these reductions are independent. Hence, without regarding the arrangement of the letters, we see that the minimum weight of the product $C_\kappa C_m$ for irreducibility is

$$\sum_{r=1}^{r=\kappa-1}\left\{\binom{m+\kappa}{r}-\binom{m}{r}\right\}+2^m-2-\sum_{r=\kappa}^{r=m-1}\binom{m}{r}$$

$$= \sum_{r=1}^{r=\kappa-1}\binom{m+\kappa}{r}+2^m-2-(2^m-2)$$

$$= \sum_{r=1}^{r=\kappa-1}\binom{m+\kappa}{r}.$$

In considering the arrangement of the letters, there is one reduction corresponding to each product $C'_\kappa C'_m$ which follows in the sequence of products the product $C_\kappa C_m$ which we are considering: this is obvious if C'_κ contain letters from C_κ and C_m. If, however, C'_κ contain only letters from C_m, there is still one reduction; for, if the product $C_\kappa C_m$ is irreducible, C_m must not be expressible in terms of a Jacobian of which C'_κ is one of the forms.

Now the total number of forms $C_\kappa C_m$ is $\binom{m+\kappa}{\kappa}$, and the extra reductions due to the arrangements of the letters number in the various cases

$$0,\ 1,\ 2,\ \dots,\ \binom{m+\kappa}{\kappa}-1, \text{ respectively};$$

hence the generating function for $C_\kappa C_m$ is

$$\frac{x^{\binom{m+\kappa}{\kappa-1}+\binom{m+\kappa}{\kappa-2}+\dots+\binom{m+\kappa}{1}}\left\{1+x+x^2+\dots+x^{\binom{m+\kappa}{\kappa}-1}\right\}}{(1-x)^{m+\kappa-2}}$$
$$=\frac{x^{\binom{m+\kappa}{\kappa-1}+\binom{m+\kappa}{\kappa-2}+\dots+\binom{m+\kappa}{1}}\left\{1-x^{\binom{m+\kappa}{\kappa}}\right\}}{(1-x)^{m+\kappa-1}}.$$

This result proves that the generating function for products of degree δ having no factor of degree less than $(\kappa+1)$, provided $\kappa \geqslant \frac{1}{3}\delta$ (and of course $\leqslant \frac{1}{2}\delta$), is

$$\frac{x^{\binom{\delta}{1}+\binom{\delta}{2}+\dots+\binom{\delta}{\kappa}}}{(1-x)^{\delta-1}}$$

(see also Section V., § 13).

The case of $\kappa=\frac{1}{2}\delta$ requires a word of explanation: the irreducible forms considered in this case are the perpetuant types of degree δ; their generating function is known to be

$$\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}},$$

and this has been proved to be exact;* when δ is odd and $=2n+1$, the generating function as given by the preceding result is

$$\frac{x^{\binom{2n+1}{1}+\dots+\binom{2n+1}{n-1}+\binom{2n+1}{n}}}{(1-x)^{2n}}=\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}.$$

A slight exception occurs in the case where δ is even and $=2n$; here the generating function is

$$\frac{x^{2^{\delta-1}-1}}{(1-x)^{\delta-1}}=\frac{x^{\binom{2n}{1}+\binom{2n}{2}+\dots+\binom{2n}{n-1}+\frac{1}{2}\binom{2n}{n}}}{(1-x)^{2n-1}}.$$

(iii.) *Generating Function for $C_\kappa C_{2\kappa}$.*

24. Similar methods are applied in the investigation of products $C_\kappa C_{2\kappa}$, but the resulting generating function does not in general admit of ex-

* Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 2.

pression in a simple form; in the special case of products $C_3 C_6$ we shall evaluate the generating function for the purpose of the subsequent discussion of syzygies of degree 9.

The number of reductions possible, where the letters of $C_\kappa C_{2\kappa}$ are fixed, is

$$\sum_{r=2}^{r=\kappa-1}\left\{\binom{3\kappa}{r}-\binom{2\kappa}{r}-\binom{\kappa}{r}\right\}+2^\kappa-2+2^{2\kappa-1}-1;$$

for C_κ must not be a Jacobian, and the minimum weight of $C_{2\kappa}$ is $2^{2\kappa-1}-1$. Hence the minimum weight is, for a fixed choice of letters,

$$\sum_{r=1}^{r=\kappa-1}\binom{3\kappa}{r}+\tfrac{1}{2}\binom{2\kappa}{\kappa}-\tfrac{1}{2}(2^{2\kappa}-2)-(2^\kappa-2)+2^\kappa-2+2^{2\kappa-1}-1$$

$$=\sum_{r=1}^{r=\kappa-1}\binom{3\kappa}{r}+\tfrac{1}{2}\binom{2\kappa}{\kappa}.$$

Now suppose that the letters of C_κ are $a_{r_1}, a_{r_2}, \ldots, a_{r_\kappa}$, where

$$1<r_1<r_2\ldots<r_\kappa;$$

then the number of products $C'_\kappa C'_{2\kappa}$, such that C'_κ contains letters of C_κ and of $C_{2\kappa}$, which follow the product $C_\kappa C_{2\kappa}$ in the sequence already determined is easily seen to be

$$N\equiv\binom{3\kappa-r_1}{\kappa}-\binom{2\kappa-r_1+1}{\kappa}+\binom{3\kappa-r_2}{\kappa-1}+\ldots+\binom{3\kappa-r_\kappa}{1};$$

and, as before, we can show that

$$\sum_{r_1, r_2, \ldots, r_\kappa} x^N=\sum_{r_1} x^{\binom{3\kappa-r_1}{\kappa}-\binom{2\kappa-r_1+1}{\kappa}}\,\frac{1-x^{\binom{3\kappa-r_1}{\kappa-1}}}{1-x};$$

here r_1 may have any value from 2 to $2\kappa+1$.

If C_κ contains a_1 (corresponding to $r_1=1$), we get a term

$$x^{\binom{3\kappa-1}{\kappa}-\frac{1}{2}\binom{2\kappa}{\kappa}}\,\frac{1-x^{\binom{3\kappa-1}{\kappa-1}}}{1-x};$$

where the extra index $\frac{1}{2}\binom{2\kappa}{\kappa}$ arises from the fact that, if $C_{2\kappa}$ does not contain a_1 and can be expressed as a Jacobian (C_κ, C'_κ), there is a reduction.

Hence the generating function for irreducible products $C_\kappa C_{2\kappa}$ is

$$\frac{x^{\binom{3\kappa}{1}+\binom{3\kappa}{2}+\dots+\binom{3\kappa}{\kappa-1}+\frac{1}{2}\binom{2\kappa}{\kappa}}}{(1-x)^{3\kappa-1}}\left\{x^{\binom{3\kappa-1}{\kappa}-\frac{1}{2}\binom{2\kappa}{\kappa}}\left(1-x^{\binom{3\kappa-1}{\kappa-1}}\right)\right.$$

$$\left.+\sum_{r=2}^{r=2\kappa+1} x^{\binom{3\kappa-r}{\kappa}-\binom{2\kappa-r+1}{\kappa}}\left(1-x^{\binom{3\kappa-r}{\kappa-1}}\right)\right\}.$$

Putting $\kappa=3$, we find for the irreducible products $C_3 C_6$ of degree 9 the generating function

$$\frac{x^{55}(1-x^{10}+x^{9}-x^{19}+x^{16}-x^{31}+x^{25}-x^{74})}{(1-x)^{8}}.$$

Summary of Results giving Generating Functions for the Enumeration of certain Classes of Irreducible Products of total Degree δ.

Products.	Generating Functions.
All products containing no factor C_1 (§ 14)	$\dfrac{x^{\delta}}{(1-x)^{\delta-1}}$
C_2^m (where $\delta=2m$) (§ 15)	$\dfrac{x^{\delta}(1-x^3)(1-x^5)\dots(1-x^{\delta-1})}{(1-x)^{\delta-1}}$
$C_2^m C_3$ (where $\delta=2m+3$) (§ 18)	$\dfrac{x^{\delta}(1-x^3)(1-x^5)\dots(1-x^{\delta-2})}{(1-x)^{\delta-1}}\left\{1+\dfrac{x^3(1-x^{\delta-3})}{1-x^2}+x^6\dfrac{(1-x^{\delta-3})(1-x^{\delta-1})}{(1-x^2)(1-x^3)}\right\}$
$C_2^m C_4$ (where $\delta=2m+4$) (§ 19)	$\dfrac{x^{\delta+3}(1-x^3)(1-x^5)\dots(1-x^{\delta-5})(1-x^{\delta-3})(1-x^{\delta-2})(1-x^{\delta-1})}{(1-x)^{\delta-1}(1-x^2)(1-x^3)} \times\left\{1+\dfrac{x^7(1-x^{\delta-4})}{(1-x^4)}\right\}$
$C_2^m P_{\delta-2m}$ ($P_{\delta-2m}$ contains no factor of degree less than 3 and is neither C_3 nor C_4) (§ 17)	$\dfrac{x^{2m}(1-x^3)(1-x^5)\dots(1-x^{2m-1})}{(1-x)^{2m-1}}\dfrac{(1-x^{\delta})(1-x^{\delta-1})\dots(1-x^{\delta-2m+1})}{(1-x)(1-x^2)\dots(1-x^{2m})}V$ (where V is the generating function of $P_{\delta-2m}$)
All products containing no factor of degree less than 3 (§ 20)	$\dfrac{x^{\binom{\delta}{1}+\binom{\delta}{2}}}{(1-x)^{\delta-1}}$
C_κ^2 (where $\delta=2\kappa$) (§ 22)	$\dfrac{x^{2^{\delta-1}-1-\frac{1}{2}\binom{\delta}{\frac{1}{2}\delta}}\left(1-x^{\frac{1}{2}\binom{\delta}{\frac{1}{2}\delta}}\right)}{(1-x)^{\delta-1}}$
$C_\kappa C_m$ (where $\delta=\kappa+m$, $\kappa<m<2\kappa$) (§ 23)	$\dfrac{x^{\binom{\delta}{1}+\binom{\delta}{2}+\dots+\binom{\delta}{\kappa-1}}\left(1-x^{\binom{\delta}{\kappa}}\right)}{(1-x)^{\delta-1}}$

VII. SYZYGIES OF DEGREES 6, 7, AND 8.—NOTE ON SYZYGIES OF DEGREE 9.

25. We are now in a position to write down from the preceding general results the generating functions for the enumeration of the irreducible products of degrees 6, 7, and 8 respectively.

Degree 6.

	Products.	Generating Functions.	How obtained.
Type forms :—	C_6	$\dfrac{x^{31}}{(1-x)^5}$	Perpetuant type theorem
Products :—	C_3^2	$\dfrac{x^{21}-x^{31}}{(1-x)^5}$	C_κ^2, $\kappa=3$ (Section VI., § 22)
	C_2C_4	$\dfrac{x^9+x^{11}-x^{14}-x^{21}}{(1-x)^5}$	$C_2^mC_4$, $m=1$ (Section V., § 19)
	C_2^3	$\dfrac{x^6-x^9-x^{11}+x^{14}}{(1-x)^5}$	C_2^m, $m=3$ (Section V., § 15)
	C_1C_5	$\dfrac{6x^{15}}{(1-x)^4}$	Perpetuant type theorem
	$C_1C_2C_3$	$\dfrac{6(x^5-x^{15})}{(1-x)^4}$	C_2C_3, $m=1$ (Section V., § 18)
	$C_1^2C_4$	$\dfrac{15x^7}{(1-x)^3}$	Perpetuant type theorem
	$C_1^2C_2^2$	$\dfrac{15(x^4-x^7)}{(1-x)^3}$	C_2^m, $m=2$ (Section V., § 15)
	$C_1^3C_3$	$\dfrac{20x^3}{(1-x)^2}$	Perpetuant type theorem
	$C_1^4C_2$	$\dfrac{15x}{1-x}-10x$	$C_1^{n-2}C_2$, $n=6$ (Section V., § 14)
	C_1^6	1	

Also—

(i.) Sum of generating functions for forms C_6, C_3^2 is $x^{21}/(1-x)^5$ (see Section V., § 13).

(ii.) Sum of generating functions for forms C_6, C_3^2, C_2C_4, C_2^3 is $x^6/(1-x)^5$ (see Section V., § 13).

(iii.) Sum of all generating functions is $1/(1-x)^5$, so that all the enumeration of the irreducible forms is exact and all the syzygies for degree 6 have been identified.

One point calls for comment: in reducing the products $C_2 C_4$, if $C_2 = (a_5 a_6)^\nu$, the only restriction on ν is (Section V., § 19) $\nu \geqslant 2$. Now consider the form

$$C_2 C_4 = (a_5 a_6)^2 \{(a_1 a_2)^3, (a_3 a_4)^3\},$$

which arises in products of weight 9. By the ordinary Jacobian identity we have

$$(a_5 a_6)^2 \{(a_1 a_2)^3, (a_3 a_4)^3\} \equiv (a_1 a_2)^3 \{(a_5 a_6)^2, (a_3 a_4)^3\} - (a_3 a_4)^3 \{(a_5 a_6)^2, (a_1 a_2)^3\}.$$

Now each of the Jacobians $\{(a_5 a_6)^2, (a_3 a_4)^3\}$, $\{(a_5 a_6)^2, (a_1 a_2)^3\}$ is of degree 4 and weight 6, and therefore is expressible as a sum of product forms. Hence the product $(a_5 a_6)^2 \{(a_1 a_2)^3, (a_3' a_4)^3\}$ is reducible: this is not in opposition to the general result, as we can shew that the Jacobian $\{(a_1 a_2)^3, (a_3 a_4)^3\}$, although of degree 4 and weight 7, is itself reducible. For, by the perpetuant type theorem,

$$\begin{aligned}\{(a_1 a_2)^3, (a_3 a_4)^3\} &\equiv \kappa (a_1 a_2)^4 (a_2 a_3)^2 (a_3 a_4) + \text{product forms} \\ &\quad \kappa \{(a_1 a_2)^4, (a_3 a_4)\}^2 + \lambda \{(a_1 a_2)^5, (a_3 a_4)\} + \mu \{(a_1 a_2)^4, (a_3 a_4)^2\} \\ &\quad + \text{product forms},\end{aligned}$$

and every term of $\{(a_1 a_2)^5, (a_3 a_4)\}$ is a product form. Hence

$$\{(a_1 a_2)^3, (a_3 a_4)^3\} \equiv \kappa \{(a_1 a_2)^4, (a_3 a_4)\}^2 + \mu \{(a_1 a_2)^4, (a_3 a_4)^2\} + \text{product forms}.$$

In this result interchange a_1 and a_2 and subtract; then

$$2\{(a_1 a_2)^3, (a_3 a_4)^3\} = \text{product forms};$$

and therefore the Jacobian is reducible.

In the treatment of product forms of degree 3κ ($\kappa \geqslant 3$) no such difficulty can arise, for, although we have the analogous result

$$C'_\kappa (C_\kappa, \overline{C}_\kappa) = \Sigma \text{ (product of three factors)},$$

where C'_κ is of weight $2^{2\kappa-2}-2$, and each of C_κ, $\overline{C}_\kappa$ is of weight $2^{2\kappa-2}-1$, it will be seen that the minimum weight, as determined in Section V., § 24, of the factor $C'_\kappa C_{2\kappa}$ is in all cases greater than $2^{2\kappa-2}-2$; so that there is no occasion to investigate the reducibility of the Jacobian form $(C_\kappa, \overline{C}_\kappa)$ of weight $2^{2\kappa-1}-1$ and degree 2κ (see § 24).

Degree 7.

26.

	Products.	Generating Functions.	How obtained.
Type forms:—	C_7	$\dfrac{x^{63}}{(1-x)^6}$	Perpetuant type theorem
Products:—	C_3C_4	$\dfrac{x^{28}-x^{63}}{(1-x)^6}$	$C_\kappa C_m$, $\kappa=3$, $m=4$ (Section VI., § 23)
	C_2C_5	$\dfrac{x^{17}(1-x^7)(1+x^2+x^4)}{(1-x)^6}$	$C_2^mC_r$, $m=1$, $r=5$ (Section V., § 17)
	$C_2^2C_3$	$\dfrac{x^7(1-x^5)(1-x^7)(1+x^5+x^7)}{(1-x)^6}$	$C_2^mC_3$, $m=2$ (Section V., § 18)
	C_1C_6	$\dfrac{7x^{31}}{(1-x)^5}$	Results for degree 6
	$C_1C_3^2$	$\dfrac{7(x^{21}-x^{31})}{(1-x)^5}$	Results for degree 6
	$C_1C_2C_4$	$\dfrac{7(x^9+x^{11}-x^{14}-x^{21})}{(1-x)^5}$	Results for degree 6
	$C_1C_2^3$	$\dfrac{7(x^6-x^9-x^{11}+x^{14})}{(1-x)^5}$	Results for degree 6
	$C_1^2C_5$	$\dfrac{21x^{15}}{(1-x)^4}$	Results for degree 6
	$C_1^2C_2C_3$	$\dfrac{21(x^5-x^{15})}{(1-x)^4}$	Results for degree 6
	$C_1^3C_4$	$\dfrac{35x^7}{(1-x)^3}$	Results for degree 6
	$C_1^3C_2^2$	$\dfrac{35(x^4-x^7)}{(1-x)^3}$	Results for degree 6
	$C_1^4C_3$	$\dfrac{35x^3}{(1-x)^2}$	Results for degree 6
	$C_1^5C_2$	$\dfrac{21x}{1-x}-15x$	$C_1^{n-2}C_2$, $n=7$ (Section V., § 14)
	C_1^7	1	

Also—

(i.) Sum of generating functions for forms C_7, C_3C_4 is $x^{28}/(1-x)^6$ (see Section V., § 13).

(ii.) Sum of generating functions for forms C_7, $C_3 C_4$, $C_2 C_5$, $C_2^2 C_3$ is $x^7/(1-x)^6$ (see Section V., § 13).

(iii.) Sum of all generating functions is $1/(1-x)^6$; so that all the syzygies are enumerated above.

Degree 8.

It is clear that we need only determine the generating functions for the products C_8, C_4^2, $C_3 C_5$, $C_2 C_6$, $C_2 C_3^2$, $C_2^2 C_4$, C_2^4; since all other products have a factor C_1, and the corresponding generating functions may be determined from the results for degree 7 (see Section V., § 14).

	Products.	Generating Functions.	How obtained.
Type forms:—	C_8	$\dfrac{x^{127}}{(1-x)^7}$	Perpetuant type theorem
Products :—	C_4^2	$\dfrac{x^{92}-x^{127}}{(1-x)^7}$	C_κ^2, $\kappa = 4$ (Section VI., § 22)
	$C_3 C_5$	$\dfrac{x^{86}-x^{92}}{(1-x)^7}$	$C_\kappa C_m$, $\kappa = 3$, $m = 5$ (Section VI., § 23)
	$C_2 C_6$	$\dfrac{x^2(1-x^7)(1-x^8)}{(1-x^2)(1-x)^2}\;\dfrac{x^{81}}{(1-x)^5}$	$C_2^m C_r$, $m = 1$, $r = 6$ (Section V., § 17)
	$C_2 C_3^2$	$\dfrac{x^3(1-x^7)(1-x^8)}{(1-x^3)(1-x)^2}\;\dfrac{x^{21}-x^{81}}{(1-x)^5}$	$C_2^m C_r$, $m = 1$, $r = 6$ (Section V., § 17)
	$C_2^2 C_4$	$\dfrac{x^{11}(1-x^5)(1-x^6)(1-x^{14})}{(1-x^2)(1-x)^7}$	$C_2^m C_4$, $m = 2$ (Section V., § 19)
	C_2^4	$\dfrac{x^8(1-x^3)(1-x^5)(1-x^7)}{(1-x)^7}$	C_2^m, $m = 4$ (Section V., § 15)

Also—

(i.) Sum of generating functions for forms C_8, C_4^2 (containing no factor of degree 3 or less) is

$$\frac{x^{92}}{(1-x)^7} = \frac{x^{\binom{8}{1}+\binom{8}{2}+\binom{8}{3}}}{(1-x)^7}.$$

(ii.) Sum of generating functions for forms C_8, C_4^2, $C_3 C_5$ is $x^{86}/(1-x)^7$ (see Section V., § 13).

(iii.) Sum of all generating functions above is $x^8/(1-x)^7$ (see Section V., § 13).

Note on Syzygies of Degree 9.

27. The only types and products of degree 9 which have no factor of degree less than 3 are C_9, C_4C_5, C_3C_6, C_3^3; and therefore the generating function for all such irreducible forms is $x^{45}/(1-x)^8$. Also the generating function for C_9 is $x^{255}/(1-x)^8$: the generating function for C_4C_5 is $\dfrac{x^{129}(1-x^{126})}{(1-x)^8}$, while the generating function for C_3C_6 is (§ 24)

$$\frac{x^{55}(1-x^{10}+x^9-x^{19}+x^{16}-x^{31}+x^{25}-x^{74})}{(1-x)^8}.$$

Hence the generating function for the products C_3^3 should be

$$\frac{x^{45}-x^{255}-x^{129}(1-x^{126})-x^{55}(1-x^{10}+x^9-x^{19}+x^{16}-x^{31}+x^{25}-x^{74})}{(1-x)^8}$$
$$=\frac{x^{45}(1-x^{10}+x^{20}-x^{19}+x^{29}-x^{26}+x^{41}-x^{35})}{(1-x)^8}.$$

This generating function may be written in the form

$$\frac{\begin{array}{l}x^{45}(1+2x+3x^2+4x^3+5x^4+6x^5+7x^6+8x^7+9x^8+10x^9+10x^{10}+10x^{11}\\ \quad+10x^{12}+10x^{13}+10x^{14}+10x^{15}+10x^{16}+10x^{17}+10x^{18}+9x^{19}+9x^{20}\\ \quad+9x^{21}+9x^{22}+9x^{23}+9x^{24}+9x^{25}+8x^{26}+7x^{27}+6x^{28}+6x^{29}+6x^{30}+6x^{31}\\ \quad+6x^{32}+6x^{33}+6x^{34}+5x^{35}+4x^{36}+3x^{37}+2x^{38}+x^{39})\end{array}}{(1-x)^6}.$$

The sum of the coefficients in the numerator is 280, the same as the number of possible different arrangements of the letters in the factors of C_3^3.

The first few terms may be accounted for by the methods already used, thus

$$(a_1a_2a_3)(a_4a_5a_6)(a_7a_8a_9) \quad \text{is of minimum weight } 45,$$

$$\left.\begin{array}{l}(a_1a_2a_4)(a_3a_5a_6)(a_7a_8a_9)\\ (a_1a_2a_3)(a_4a_5a_7)(a_6a_8a_9)\end{array}\right\} \quad ,, \quad ,, \quad 46,$$

$$\left.\begin{array}{l}(a_1a_2a_5)(a_3a_4a_6)(a_7a_8a_9)\\ (a_1a_2a_4)(a_3a_5a_7)(a_6a_8a_9)\\ (a_1a_2a_3)(a_4a_5a_8)(a_6a_7a_9)\end{array}\right\} \quad ,, \quad ,, \quad 47,$$

$$\left.\begin{array}{l}(a_1a_2a_6)(a_3a_4a_5)(a_7a_8a_9)\\ (a_1a_2a_5)(a_3a_4a_7)(a_6a_8a_9)\\ (a_1a_2a_4)(a_3a_5a_8)(a_6a_7a_9)\\ (a_1a_2a_3)(a_4a_5a_9)(a_6a_7a_8\end{array}\right\} \quad ,, \quad ,, \quad 48.$$

But our known reductions give six products of minimum weight 49 (instead of five products): they are

$$(a_1 a_2 a_7)(a_3 a_4 a_5)(a_6 a_8 a_9),$$
$$(a_1 a_2 a_6)(a_3 a_4 a_7)(a_5 a_8 a_9),$$
$$(a_1 a_2 a_5)(a_3 a_4 a_8)(a_6 a_7 a_9),$$
$$(a_1 a_2 a_4)(a_3 a_5 a_9)(a_6 a_7 a_8),$$
$$(a_1 a_2 a_3)(a_4 a_6 a_7)(a_5 a_8 a_9),$$
$$(a_1 a_3 a_4)(a_2 a_5 a_6)(a_7 a_8 a_9),$$

and one of these forms should be of minimum weight 50 at least.

This would seem to imply that Stroh's syzygies of degree 4 as used hitherto are insufficient for the investigation of the products C_3^3: it seems probable that similar difficulties will arise successively in the treatment of products C_4^4, C_5^5, ..., C_m^m, ..., that is, whenever we arrive at a degree m^2.

[*Note.*—The symbol $(a_1 a_2 a_3)$ is used simply to denote any covariant linear in the coefficients of each of the quantics represented by the letters a_1, a_2, a_3.]

NOTE ON THE APPLICATION OF POISSON'S FORMULA TO DISCONTINUOUS DISTURBANCES

By Lord RAYLEIGH.

[Received May 16th, 1904.—Read June 9th, 1904.]

IN a recent paper* Prof. Love draws attention to "the discovery of an oversight in Stokes's justly famous memoir on the 'Dynamical Theory of Diffraction.'" The dilatation Δ satisfies the partial differential equation $d^2\Delta/dt^2 = a^2\nabla^2\Delta$, and is calculated from it by means of Poisson's integral formula. "According to this formula any function f which satisfies an equation of the form $d^2f/dt^2 = a^2\nabla^2 f$ can be expressed in terms of initial values by the equation

$$f = \frac{t}{4\pi}\iint \dot{f}_0(at)\,d\sigma + \frac{d}{dt}\left\{\frac{t}{4\pi}\iint f_0(at)\,d\sigma\right\}, \tag{A}$$

in which the integration refers to angular space about the point at which f is estimated, and $f_0(at)$ and $\dot{f}_0(at)$ denote the initial values of f and df/dt on a sphere of radius at with its centre at the point."

". . . . it will be seen that all Stokes's results depend upon the employment of Poisson's integral formula to express the dilatation and the components of the rotation. In a recent paper I have pointed out that this formula does not in general yield correct expressions for these quantities. In the same paper I identified the formula (A) with one which has been used by Poincaré and others, viz.,

$$f = \frac{1}{4\pi}\iint\left(t\dot{f}_0 + f_0 + r\frac{df_0}{dr}\right)_{r=at} d\sigma, \tag{B}$$

where r denotes distance from the point at which f is estimated."

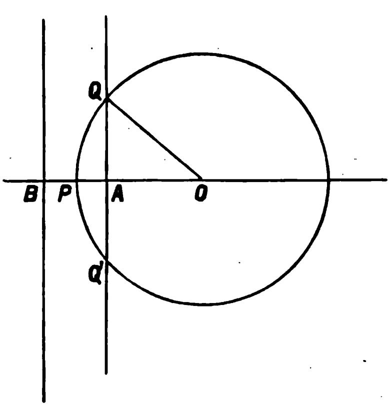
"The reason for the failure of such formulæ as (B) to represent the dilatation and the components of rotation is clear from an inspection of (B). When the point at which the disturbance is estimated is near the front of an advancing wave the sphere described about the point penetrates but a little way into the region within which the initial disturbance is

* *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 291, 1903.

confined, and the part of the sphere which is included in the integration is very small. Thus the formula cannot express any quantity which has a value different from zero at the front of an advancing wave. Now there is no kinematical or dynamical reason why the dilatation and rotation in an elastic solid should be supposed to vanish at the front of an advancing wave, and it appears therefore that Stokes's analysis is adequate to express the effects of particular types of initial disturbance, but not those of an arbitrary initial disturbance confined to a finite portion of the medium."

Having myself on a former occasion* applied Poisson's formula to the forbidden case of a uniform initial condensation limited to the slice bounded by two parallel planes without meeting any difficulty, I was naturally rather taken aback by the above criticism, although it is true that I then contemplated f as representing the *velocity-potential* rather than the dilatation. But the argument for the dilatation assumes much the same form, and it may be desirable to set it out in full.

Let us suppose then that a gaseous medium is initially undisturbed except between the parallel planes A, B, and that within AB there is initially no velocity, but only a uniform dilatation (or condensation). We know, of course, what will happen from the theory of plane waves. The initial state of things is equivalent to the superposition of two progressive waves between which the dilatation is shared. These advance in opposite directions, and in each the particle velocity is uniform and in the direction of propagation. We have now to inquire what account of the matter Poisson's formula will give.



In this f_0 represents the initial dilatation, confined to AB. The initial velocity of dilatation $\dot{f}_0$ is zero both within and without the slice, but this is not of itself sufficient to establish the evanescence of the first integral in (A) after the sphere has reached the slice. We have also to consider what may happen at the boundary planes A and B themselves. Taking the plane A, we see that immediately in front of it the dilatation rises suddenly from 0 to $\frac{1}{2}f_0$, but the effect of this is compensated in the integral by the drop

* *Theory of Sound*, § 274.

from f_0 to $\frac{1}{2}f_0$ which occurs symmetrically behind. In like manner the boundary B can contribute nothing, and we may equate the first integral to zero.

In the second integral, f_0 has a constant value over the portion QPQ' of the sphere and vanishes over the remainder. Also, if θ denote the angle AOQ,

$$\iint d\sigma = 2\pi(1-\cos\theta) = 2\pi\frac{AP}{OP};$$

and $OP = at$. Thus

$$f = \tfrac{1}{2}f_0\frac{d}{dt}\left(t\frac{AP}{at}\right) = \tfrac{1}{2}f_0,$$

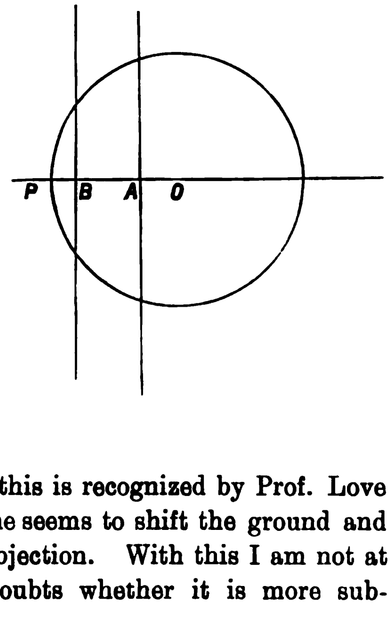
since AP increases with velocity a; and accordingly the dilatation is correctly given by Poisson's formula.

When the wave has passed, the sphere cuts completely through the slice, and

$$\iint d\sigma = 2\pi\frac{AB}{OP} = 2\pi\frac{AB}{at};$$

so that $\qquad t\iint d\sigma =$ constant,

and consequently $f = 0$. In all respects the passage of the wave is correctly represented.



It is clear that the objection is really directed not against (A), but against (B), which, so far as I know, was not used at all by Poisson or by Stokes. And indeed this is recognized by Prof. Love himself in a later passage (p. 321), where he seems to shift the ground and advances against (A) an entirely distinct objection. With this I am not at present concerned, though I have my doubts whether it is more substantial than the first.

Even as regards (B), the charge of failure preferred against it seems to ignore the fact that the integrand becomes infinite in the case supposed of a discontinuous initial condition. Let us apply (B) to the circumstances already defined of a dilatation limited to the slice between two planes A and B. The first term in $\dot{f}_0$ vanishes as before. The second term in f_0, also as before, assumes the value $\frac{1}{2}f_0(1-\cos\theta)$. The only difficulty is in the third term, where df_0/dr, vanishing within and without the slice, becomes

infinite on the circle of transition whose diameter is QQ'. If x be a co-ordinate measured parallel to AB, f_0 is a function of x only, and

$$\frac{df_0}{dr} = \frac{df_0}{dx} \cos\theta.$$

Thus the third integral

$$= \tfrac{1}{2} \int r \frac{df_0}{dx} \cos\theta \sin\theta \, d\theta = \tfrac{1}{2} \int \frac{df_0}{dx} \cos\theta \, dx = \tfrac{1}{2} \cos\theta \, . \, f_0;$$

and altogether (B) gives $\frac{1}{2}f_0$, the correct result. According to Prof. Love's indictment the formula yields zero when the sphere only cuts a little into the slice.

TYPES OF COVARIANTS OF ANY DEGREE IN THE COEFFICIENTS OF EACH OF ANY NUMBER OF BINARY QUANTICS OF FINITE ORDER

By P. W. WOOD.

[Received May 26th, 1904.—Read June 9th, 1904.]

INTRODUCTION (§§ 1–8).

1.

It is proposed to give a method of determining the type forms for the "complete system mod $(ab)^{\frac{1}{2}n}$" of any number of binary quantics. The following theorems are known:—

I.* Any perpetuant linear in the coefficients of each of the binary quantics $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, ..., $a_{\delta_x}^{n_\delta}$, where n_1, n_2, ..., n_δ are all infinite, is linearly expressible in terms of

(i.) symbolical products of the form $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \geqslant 2^{\delta-2}$, $\lambda_2 \geqslant 2^{\delta-3}$, ..., $\lambda_{\delta-1} \geqslant 1$, and the sequence of the letters a_1, a_2, ..., a_δ is fixed beforehand;

(ii.) products of perpetuants of lower total degrees.

II.† Any covariant linear in the coefficients of each of the binary quantics $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, ..., $a_{\delta_x}^{n_\delta}$, where n_1, n_2, ..., n_δ are all finite, is linearly expressible in terms of

(i.) symbolical products of the form $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$, where $\lambda_1 \geqslant 2^{\delta-2}$, $\lambda_2 \geqslant 2^{\delta-3}$, ..., $\lambda_{\delta-1} \geqslant 1$, and the sequence of the letters a_1, a_2, ..., a_δ is fixed beforehand;

(ii.) covariants having a factor of the form $(a_h a_\kappa)^\lambda (a_\kappa a_l)^{n_\kappa-\lambda}$;

(iii.) products of covariants of lower total degrees.

These results depend ultimately *only* on the repeated application of fundamental identities of the form $(bc)+(ca)+(ab) \equiv 0$, a, b, c being any three of the letters a_1, a_2, ..., a_δ, the letters being all taken as referring to different quantics.

Camille Jordan ‡ laid down the dictum that the process of determining type forms must be performed in two distinct steps: first, we must

* Grace, *Proc. London Math. Soc.*, Vol. xxxv.

† A. Young, *Ibid.*, Ser. 2, Vol. 1.

‡ *Liouville*, 1876.

determine the type forms solely by the use of identities such as the above; and, secondly, we must find how the forms thus obtained are modified by the interchanges of letters referring to the same quantic. We know* that in Theorems I. and II. we have obtained all the possible results derivable from the application of the fundamental identity: the next step is to determine how the type forms (i.) in these theorems are modified when certain of the letters $a_1, a_2, \ldots, a_\delta$ refer to the same quantic, and may therefore be freely interchanged.

2.

Grace† has obtained the result for perpetuants, by interchanging equivalent letters in the result of Theorem (I.), in the following form :—

III. (i.) If the symbols $a_1, a_2, \ldots, a_\delta$ all refer to the same quantic. then the type form of an irreducible perpetuant is

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}},$$

where

$$\lambda_{\delta-r} = 2^{r-1} + (\xi_{\delta-1} + \xi_{\delta-2} + \ldots + \xi_{\delta-r}), \qquad r = 1, 2, \ldots, \delta-2,$$
$$\lambda_1 = 2^{\delta-2} + 2(\xi_{\delta-1} + \xi_{\delta-2} + \ldots + \xi_2 + \xi_1),$$

the ξ's being positive integers or zeros.

(ii.) If the symbols $a_1, a_2, \ldots, a_\delta$ refer to one quantic, and the symbols $b_1, b_2, \ldots, b_\epsilon$ refer to a second quantic, then the type form of an irreducible perpetuant of partial degrees δ and ϵ is

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}} (a_\delta b_1)^{\lambda_\delta} (b_1 b_2)^{\mu_1} \ldots (b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}},$$

where

$$\mu_{\epsilon-r} = 2^{r-1} + \eta_\epsilon + \eta_{\epsilon-1} + \ldots + \eta_{\epsilon-r+1}, \qquad r = 1, 2, \ldots, \epsilon-1,$$
$$\lambda_\delta = 2^{\epsilon-1} + \eta_\epsilon + \eta_{\epsilon-1} + \ldots + \eta_2 + \eta_1,$$
$$\lambda_{\delta-r} = 2^{\epsilon+r-1} + \xi_{\delta-1} + \xi_{\delta-2} + \ldots + \xi_{\delta-r}, \qquad r = 1, 2, \ldots, \delta-2,$$
$$\lambda_1 = 2^{\delta+\epsilon-2} + 2(\xi_{\delta-1} + \xi_{\delta-2} + \ldots + \xi_2 + \xi_1),$$

the ξ's and η's being positive integers or zeros.

The corresponding result for the perpetuant type form of any degree in each of any number of binary quantics can be written down at once from the preceding result for two quantics: it should be remarked that

* Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

† *Proc. London Math. Soc.*, Vol. XXXV. Reference should be made to the early part of this paper for a statement of the general principles underlying the investigation.

the generating functions obtained in this way agree with those found independently by Stroh * and MacMahon.†

3.

It is the purpose of the present paper to show that, if in dealing with quantics of finite order we neglect covariants with a factor $(a_h a_\kappa)^\lambda (a_\kappa a_l)^{n_\kappa - \lambda}$ (that is, forms of the second class in Theorem II. above), then we obtain exactly the same results as in Theorem III. for types of covariants of quantics of finite order: we shall, in fact, show that

IV. Any covariant of degree δ in the coefficients of

$$a_x^m (\equiv a_{1_x}^m \equiv a_{2_x}^m \equiv \dots \equiv a_{\delta_x}^m)$$

and of degree ϵ in the coefficients of $b_x^n (\equiv b_{1_x}^n \equiv b_{2_x}^n \equiv \dots \equiv b_{\epsilon_x}^n)$ is linearly expressible in terms of

(i.) symbolical products of the form

$$(a_1 a_2)^{\lambda_1} \dots (a_\delta b_1)^{\lambda_\delta} (b_1 b_2)^{\mu_1} \dots (b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}},$$

where the indices λ, μ satisfy the conditions given in III., (ii.);

(ii.) covariants having a factor $(a\alpha)^\lambda (a\beta)^{m-\lambda}$ or a factor $(b\alpha)^\lambda (b\beta)^{n-\lambda}$;

(iii.) products of covariants of lower total degrees.

It will be seen that the method of proof admits of extension to the case of covariants of any degree in the coefficients of each of any number of quantics: we shall throughout speak of covariants included in class (ii.) of Theorem II. as "covariants of the second class." The succeeding proof consists only of a suitable modification and combination of the proofs of Theorems II. and III. as given by their respective authors.

INTRODUCTORY LEMMAS (§§ 4–6).

4.

It will be sufficient in the first place to give an outline of such portions of Young's proof of Theorem III. as are essential to the present investigation: the two succeeding Lemmas consist only of a modified form of the corresponding Lemmas, as originally given by Grace in the course of the proof of Theorem III.

Summary of Results contained in the Proof of Theorem III.

(i.) Consider the covariant $C_\delta \equiv (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}}$; if we write

---

* *Math. Ann.*, Bd. XXXVI. † *Proc. London. Math. Soc.*, Vol. XXVI.

$$a_{4_x}^{n_4-\lambda_4} q_y^{\rho} \equiv (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5} \dots (a_{\delta-1} a_\delta)^{\lambda_{\delta-1}} a_{4_x}^{n_4-\lambda_4} a_{5_y}^{n_5-\lambda_4-\lambda_5} \dots a_{\delta_y}^{n_\delta-\lambda_{\delta-1}},$$

then C_δ is a term of the transvectant

$$\left\{(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2},\quad a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\right\}_{y=x}^{\lambda_3},$$

and so differs from it or from any other one of its terms by covariants involving more factors in a_1, a_2, a_3 alone: if either λ_1 or λ_2 is less than $2^{\delta-3}$, then it may be shown that this transvectant and each of its terms is linearly expressible in terms of covariants of the second class, of products of covariants, and of covariants involving more factors in a_1, a_2, a_3 alone.

(ii.) If we put $(a_1 a_2)^{\mu} \equiv a_x^{n_1+n_2-2\mu}$, and consider forms of the second class arising from the $(\delta-1)$ quantics $a_x^{n_1+n_2-2\mu}$, $a_{3_x}^{n_3}$, ..., $a_{\delta_x}^{n_\delta}$, we include among such forms those covariants having a factor

$$(a_h a)^{\nu} (a a_\kappa)^{n_1+n_2-2\mu-\nu}:$$

it may be shown that any such covariant is a form of the second class arising from the original δ quantics $a_{1_x}^{n_1}$, $a_{2_x}^{n_2}$, ..., $a_{\delta_x}^{n_\delta}$.

(iii.) Finally, if in the transvectant $\left\{(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2},\ a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\right\}_{y=x}^{\lambda_3}$ any one of n_1, n_2, n_3 is less than $(\lambda_1+\lambda_2)$, then that transvectant gives rise only to covariants of the second class; so that, if we neglect covariants of the second class, the Jordan Lemma or Stroh's generalized form of the Lemma may be applied to the first member

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2}$$

of the transvectant.

5.

As we are concerned only with the type forms (class i.), we shall henceforth neglect covariants of the second class and products of covariants.

LEMMA I.—The covariant

$$(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} (a_3 a_4)^{\nu} \dots \equiv (a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} P,$$

wherein a_1, a_2, a_3, refer to the same quantic, is expressible in the form

$$\Sigma (a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P,$$

where λ' is even and $\geqslant 2\mu'$.

The covariant $(a_1 a_2)^\lambda (a_2 a_3)^\mu P$ is a term of the transvectant

$$T \equiv \left\{(a_1 a_2)^\lambda (a_2 a_3)^\mu, \quad a_{4_x}^{n_4-\lambda_4} q_y^\rho\right\}_{y=x}^\nu,$$

and therefore differs from T only by covariants involving more factors in a_1, a_2, a_3 alone. Also, by § 4, (iii.), $(a_1 a_2)^\lambda (a_2 a_3)^\mu$ is, by means of the Jordan Lemma, linearly expressible in terms of the covariants of the three sets

$$\left.\begin{array}{l} (a_1 a_2)^\alpha (a_2 a_3)^\beta \\ (a_2 a_3)^\alpha (a_3 a_1)^\beta \\ (a_3 a_1)^\alpha (a_1 a_2)^\beta \end{array}\right\}, \quad \alpha \ngtr 2\beta,$$

where $\alpha+\beta = \lambda+\mu$; and therefore T is expressible as a sum of transvectants of the three sets

$$\left.\begin{array}{l} \left\{(a_1 a_2)^\alpha (a_2 a_3)^\beta, \quad a_{4_x}^{n_4-\lambda_4} q_y^\rho\right\}_{y=x}^\nu \\ \left\{(a_2 a_3)^\alpha (a_3 a_1)^\beta, \quad a_{4_x}^{n_4-\lambda_4} q_y^\rho\right\}_{y=x}^\nu \\ \left\{(a_3 a_1)^\alpha (a_1 a_2)^\beta, \quad a_{4_x}^{n_4-\lambda_4} q_y^\rho\right\}_{y=x}^\nu \end{array}\right\}, \quad \alpha \ngtr 2\beta.$$

Since a_1, a_2, a_3 refer to the same quantic, they are interchangeable and the three sets of transvectants are equivalent to the first set alone. Now the transvectant $\left\{(a_1 a_2)^\alpha (a_2 a_3)^\beta, \; a_{4_x}^{n_4-\lambda_4} q_y^\rho\right\}_{y=x}^\nu$ contains either the term $(a_1 a_2)^\alpha (a_2 a_3)^\beta (a_3 a_4)^\nu \ldots$, if $\nu \leqslant n_3-\beta$, or the term $(a_2 a_3)^\beta (a_3 a_4)^{n_3-\beta} Q$, if $\nu \geqslant n_3-\beta$, Q being any symbolical product of all the letters except a_3; the latter of these terms is a covariant of the second class. The whole transvectant differs from one or other of these terms by covariants involving more factors in a_1, a_2, a_3 alone. Hence the covariant

$$(a_1 a_2)^{\lambda} (a_2 a_3)^\mu P$$

is linearly expressible in terms of

(i.) covariants $(a_1 a_2)^\alpha (a_2 a_3)^\beta P$, $\alpha \ngtr 2\beta$;

(ii.) covariants with more factors in a_1, a_2, a_3 alone.

The covariants (ii.) may be treated in the same way, and, since the number of factors in a_1, a_2, a_3 alone cannot be increased indefinitely without giving rise to product forms, it follows that $(a_1 a_2)^\lambda (a_2 a_3)^\mu P$ is, neglecting covariants of the second class and product forms, expressible in terms of covariants

$$(a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P, \quad \text{where } \lambda' \ngtr 2\mu'.$$

Finally, if λ' is odd, it may be increased by unity by interchanging the equivalent letters a_1 and a_2: hence we obtain the result stated.

6.

LEMMA II.—The covariant of degree δ

$$(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} (a_3 a_4)^{\nu} \ldots \equiv (a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} P,$$

wherein a_2, a_3 refer to the same quantic, while a_1 refers to a different quantic, is expressible in the form

$$\Sigma (a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P,$$

where $\lambda' \geqslant \mu' + 2^{\delta-3}$.

The covariant $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} P$ is a term of the transvectant

$$T \equiv \left\{(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu}, \quad a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\right\}_{y=x}^{\nu},$$

and therefore differs from it by covariants involving more factors in a_1, a_2, a_3 alone; also, by § 4, (iii.), $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu}$ may, by means of Stroh's generalized form of the Jordan Lemma, be expressed linearly in terms of covariants

$$(a_1 a_2)^{\omega}, \quad (a_1 a_2)^{\omega-1} (a_2 a_3), \quad \ldots, \quad (a_1 a_2)^{\omega-r} (a_2 a_3)^{r},$$

$$(a_1 a_3)^{\omega}, \quad (a_1 a_3)^{\omega-1} (a_3 a_2), \quad \ldots, \quad (a_1 a_3)^{\omega-r} (a_3 a_2)^{r},$$

$$(a_2 a_3)^{\omega}, \quad (a_2 a_3)^{\omega-1} (a_3 a_1), \quad \ldots, \quad (a_2 a_3)^{\omega-p} (a_3 a_1)^{p},$$

where $\omega = \lambda + \mu$, and, if p is chosen arbitrarily, r is a positive integer given by

$$2(r+1)+p+1 = \omega+1 \text{ or } \omega+2.$$

[If r is given by $2(r+1)+p+1 = \omega+2$, there is a linear relation connecting the $(\omega+2)$ covariants above.]

Hence T is expressible linearly in terms of transvectants of the sets

$$\left.\begin{array}{l} \left\{(a_1 a_2)^{\omega-r'} (a_2 a_3)^{r'}, \quad a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\right\}_{y=x}^{\nu} \\ \left\{(a_1 a_3)^{\omega-r'} (a_3 a_2)^{r'}, \quad a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\right\}_{x=y}^{\nu} \end{array}\right\}, \quad r' = 0, 1, 2, \ldots, r,$$

$$\left\{(a_2 a_3)^{\omega-p'} (a_3 a_1)^{p'}, \quad a_{4_x}^{n_4-\lambda_4} q_y^{\rho}\right\}_{y=x}^{\nu}, \qquad p' = 0, 1, 2, \ldots, p;$$

and, since a_2 and a_3 refer to the same quantic, the first two sets are equivalent to the first set alone.

Take $p = 2^{\delta-3}-1$; then, by § 4, (ii.), each transvectant of the third set

T 2

is expressible as a sum of covariants of the second class, of products of covariants, and of covariants with more factors in a_1, a_2, a_3 alone; also

$$\omega - 2r = p+1 \quad \text{or} \quad p+2,$$

and therefore $$\omega - 2r \geqslant 2^{\delta-3}.$$

Now the transvectant $\{(a_1 a_2)^{\omega - r'} (a_2 a_3)^{r'},\ a_{4_x}^{n_4 - \lambda_4} q_y^{\rho}\}_{y=x}^{\nu}$ contains either the term $(a_1 a_2)^{\omega - r'} (a_2 a_3)^{r'} (a_3 a_4)^{\nu} \ldots$, if $\nu \leqslant n_3 - r'$, or the term $(a_2 a_3)^{r'} (a_3 a_4)^{n_3 - r'} Q$, if $\nu \geqslant n_3 - r'$, Q being any symbolical product of all the letters except a_3; and the latter term is a covariant of the second class. Therefore this transvectant differs from one or other of these terms by covariants involving more factors in a_1, a_2, a_3 alone.

Hence the covariant $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu} P$ is expressible linearly in terms of covariants $(a_1 a_2)^{\lambda'} (a_2 a_3)^{\mu'} P$, where $\lambda' \geqslant \mu' + 2^{\delta-3}$, and of covariants with more factors in a_1, a_2, a_3 alone: the latter may be treated in the same way, and so finally we obtain the result stated.

COVARIANTS OF ANY DEGREE IN THE COEFFICIENTS OF A SINGLE QUANTIC (§§ 7–8).

7.

We shall first establish inductively the results for covariants of a single quantic: the treatment of the general case follows immediately.

Covariants of Degree 3.

The forms to be considered are $(a_1 a_2)^{\lambda} (a_2 a_3)^{\mu}$, and two cases of importance occur :—

(i.) $a_1 \neq a_2 = a_3$.—Here we have $\mu \geqslant 1$, and, by Lemma II. (§ 6), $\lambda \geqslant \mu + 1$; so

$$\mu = 1 + \eta, \qquad \lambda = 2 + \xi + \eta,$$

ξ, η being positive integers or zero.

(ii.) $a_1 = a_2 = a_3$.—Here we have $\mu \geqslant 1$, and, by Lemma I. (§ 5), λ is even and $\geqslant 2\mu$; so

$$\mu = 1 + \eta, \qquad \lambda = 2 + 2(\xi + \eta),$$

ξ, η being positive integers or zero.

8. *Covariants of Degree δ.*

Assume that the type forms $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\dots(a_{\delta-2}a_{\delta-1})^{\lambda_{\delta-2}}$ of degree $(\delta-1)$ are already determined as follows:—

(i.) $a_1 \neq a_2 = \dots = a_{\delta-1}$.—

$$\lambda_{\delta-r} = 2^{r-2}+\xi_{\delta-2}+\xi_{\delta-3}+\dots+\xi_{\delta-r}, \qquad r = 2, 3, \dots, \delta-1.$$

(ii.) $a_1 = a_2 = a_3 = \dots = a_{\delta-1}$.—

$$\lambda_{\delta-r} = 2^{r-2}+\xi_{\delta-2}+\xi_{\delta-3}+\dots+\xi_{\delta-r}, \qquad r = 2, 3, \dots, \delta-2;$$

$$\lambda_1 = 2^{\delta-3}+2(\xi_{\delta-2}+\xi_{\delta-3}+\dots+\xi_2+\xi_1).$$

These are, in the first instance, the two most important cases of equivalence among the letters: it will be sufficient, since we have verified these results for degree 3, to show that the corresponding type forms of degree δ are determined in the same way. The forms to be considered are $C_\delta \equiv (a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$; putting

$$a_x^{n_1+n_2-2\lambda_1} \equiv (a_1a_2)^{\lambda_1},$$

we see that C_δ differs from

$$(aa_3)^{\lambda_2}(a_3a_4)^{\lambda_3}\dots(a_{\delta-1}a_\delta)^{\lambda_{\delta-1}}$$

only by covariants with more factors (a_1a_2). Here $a_3, a_4, \dots, a_\delta$ refer to one quantic and a refers to another, and therefore by hypothesis, from (ii.),

$$\lambda_{\delta-r} = 2^{r-1}+\xi_{\delta-1}+\xi_{\delta-2}+\dots+\xi_{\delta-r}, \qquad r = 1, 2, \dots, \delta-2.$$

[In applying this result for degree $(\delta-1)$, we are neglecting covariants of the second class among which will be included forms with a factor

$$(a_h a)^\nu (aa_x)^{n_1+n_2-2\lambda_1-\nu};$$

but, by § 4, (ii.), such forms are covariants of the second class as originally defined.]

The covariants with more factors in (a_1a_2) may be treated in the same way. The first exponent λ_1 is determined thus:—

(i.) $a_1 \neq a_2 = a_3 = \dots = a_\delta$.—By Lemma II. (§ 6),

$$\lambda_1 \geqslant \lambda_2+2^{\delta-3};$$

and therefore $\lambda_1 = 2^{\delta-2}+\xi_{\delta-1}+\xi_{\delta-2}+\dots+\xi_2+\xi_1$.

(ii.) $a_1 = a_2 = a_3 = \ldots = a_\delta$.—By Lemma I. (§ 5),

$$\lambda_1 \text{ is even and } \geqslant 2\lambda_2;$$

and therefore $\lambda_1 = 2^{\delta-2}+2(\xi_{\delta-1}+\xi_{\delta-2}+\ldots+\xi_2+\xi_1)$.

Hence the results are true universally.

If the order of the binary quantic is n, then every covariant of the second class is of gr. $\geqslant \frac{1}{2}n$; so that the type forms enumerated above constitute the "complete system mod $(ab)^{\frac{1}{2}n}$" for the binary quantic in question.

COVARIANTS OF ANY DEGREE IN THE COEFFICIENTS OF EACH OF TWO QUANTICS (§ 9).

9.

The treatment of covariants of two quantics proceeds inductively on similar lines: assuming that the restrictions on the indices λ and μ as given in Theorem IV. (§ 3) are true for the partial degrees $(\delta-1)$ and ϵ, we show that they are true for the partial degrees δ and ϵ. We have shown (§ 8) that, for the type form of an irreducible perpetuant

$$(a_1 b_1)^{\lambda_1}(b_1 b_2)^{\mu_1}(b_2 b_3)^{\mu_2}\ldots(b_{\epsilon-1} b_\epsilon)^{\mu_{\epsilon-1}}$$

of partial degrees unity and ϵ,

$$\mu_{\epsilon-r} = 2^{r-1}+\eta_\epsilon+\eta_{\epsilon-1}+\ldots+\eta_{\epsilon-r+1}, \qquad r = 1, 2, \ldots, \epsilon-1;$$

$$\lambda_1 = 2^{\epsilon-1}+\eta_\epsilon+\eta_{\epsilon-1}+\ldots+\eta_2+\eta_1.$$

Next consider the form $(a_1a_2)^{\lambda_1}(a_2b_1)^{\lambda_2}(b_1b_2)^{\mu_1}\ldots(b_{\epsilon-1}b_\epsilon)^{\mu_{\epsilon-1}}$; putting $a_x^{n_1+n_2-2\lambda_1} \equiv (a_1a_2)^{\lambda_1}$, we can show, as in § 8, that, by the result just given,

$$\mu_{\epsilon-r} = 2^{r-1}+\eta_\epsilon+\eta_{\epsilon-1}+\ldots+\eta_{\epsilon-r+1}, \qquad r = 1, 2, \ldots, \epsilon-1,$$

$$\lambda_2 = 2^{\epsilon-1}+\eta_\epsilon+\eta_{\epsilon-1}+\ldots+\eta_2+\eta_1;$$

and then, by Theorem II. (§ 1), we have $\lambda_1 \geqslant 2^\epsilon$.

Finally, we deduce the type form for partial degrees δ and ϵ, as given in Theorem IV., from that for partial degrees $(\delta-1)$ and ϵ, by putting $(a_1a_2)^{\lambda_1} \equiv a_x^{n_1+n_2-2\lambda_1}$ as in § 8, and then determining the value of λ_1 by Lemma I. (§ 8).

The type forms enumerated in Theorem IV. constitute the complete system mod $(a'b')^{\frac{1}{2}m}$ or mod $(a'b')^{\frac{1}{2}n}$ of the two quantics a_x^m and b_x^n.

10.

It is obvious that this inductive method admits of extension to the treatment of covariants of any degree in the coefficients of each of any number of binary quantics; the value of the first exponent is always established either by Theorem II. (§ 1), or by one of the two Lemmas (§§ 5–6), and the remaining exponents are determined by induction by putting $a_x^{n_1+n_2-2\lambda_1} \equiv (a_1 a_2)^{\lambda_1}$ and making use of the result for degree one less in the first quantic. It is easily seen that the exponents thus determined are exactly the same as in the case of perpetuants, since Theorem II. (§ 1), and the two Lemmas (§§ 5–6), by means of which the results are established, differ from Theorem I. (§ 1) and the Lemmas given by Grace for the proof of Theorem III. (§ 2) only by the inclusion of the covariants of the second class.

ON FUNCTIONS GENERATED BY LINEAR DIFFERENCE EQUATIONS OF THE FIRST ORDER

By E. W. BARNES.

[Received March 23rd, 1904.—Read April 14th, 1904.—Revised July 7th, 1904.]

1. The most simple solution of the linear difference equation of the first order whose coefficients are meromorphic functions is, in general, a one-valued function with sets of simple sequences of poles tending to infinity. When the coefficients are one-valued functions with essential singularities in the finite part of the plane the solution has, in general, sequences of such singularities.* It is proposed in this paper to show, in connection with the difference equation

$$P(x+1)-P(x) = \chi(x),$$

where $\chi(x)$ is a one-valued analytic function, that, *in general*, its solution cannot be a solution of any differential equation of finite order and dimensions unless either (1) the coefficients of the latter are obtained by differentiation from the solution itself, or (2) from these coefficients and the function $\chi(x)$ and its differentials we can, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, derive the solution itself.

In these cases we shall say that some of the coefficients of the differential equation belong to a type which embraces the function which is the solution of the difference equation.

The cases of exception to the previous general theorem will be considered. It will also be shown that for the more general equation

$$P(x+1)-\psi(x)\,P(x) = \chi(x),$$

where $\psi(x)$ and $\chi(x)$ are one-valued analytic functions, a similar result holds good.

The theorem includes as a special case one proved by Hölder † for

* In connection with these statements reference may be made to Guichard, *Ann. de l'Ecole Normale Supérieure*, 5 Sér., T. IV.; Mellin, *Acta Mathematica*, T. XV., pp. 317-384; Hurwitz, *Acta Mathematica*, T. XX., pp. 285-312, and T. XXI., p. 243. I hope to develop the theory in a future paper.

† Hölder, *Mathematische Annalen*, Bd. XXVIII., pp. 1-13; see also Moore, *Mathematische Annalen*, Bd. XLVIII., pp. 49 *et seq.* Hölder's theorem affirms that the gamma function cannot be a solution of a linear differential equation with algebraic coefficients. He states (*loc. cit.*) that the proposition was communicated to him verbally by Weierstrass.

the gamma function and extended by the author* to G and double gamma functions. It is important as showing that the linear difference equation of the first order gives rise to new classes of transcendants which cannot be generated, as are so many functions, by differential equations.

2. If we have a differential equation of finite order and dimensions, we may write it

$$f\{x, y, y^{(1)}, \ldots, y^{(n)}\} = 0, \tag{A}$$

where f is integral in $y, y^{(1)}, \ldots, y^{(n)}$. If the equation be of order n and dimensions m, the terms of *class* s are defined to be terms of the type

$$y^{m_1}(y^{(1)})^{m_2} \ldots (y^{(n)})^{m_{n+1}},$$

where
$$m_1+2m_2+\ldots+(n+1)m_{n+1} = s.$$

Terms of zero class will be independent of y and functions solely of the independent variable x. We assume that the differential equation has for its coefficients one-valued analytic functions of x. The variable x is assumed to be real or complex without restriction.

3. THEOREM.—If the solution of the previous differential equation (A) be also a solution of the linear difference equation

$$P(x+1)-P(x) = \chi(x), \tag{B}$$

where $\chi(x)$ is a uniform function of x, the equation (A) can be so reduced that terms of the highest class are of the form

$$f_s(x)\sum_k {}_s\phi_k(x)\,{}_sQ_k,$$

where the ϕ's are simply-periodic functions of x of period unity and ${}_sQ_k$ denotes symbolically some product

$$y^{m_1}(y^{(1)})^{m_2} \ldots (y^{(n)})^{m_{n+1}}$$

of class s.

Let the terms of highest class s in the original equation (A) be (r_s+1) in number. They can be written symbolically

$$R_0(x)\,{}_sQ_0+R_1(x)\,{}_sQ_1+\ldots+R_{r_s}(x)\,{}_sQ_{r_s}$$

where the R's are one-valued functions of x.

Divide the equation throughout by $R_0(x)$, and subtract this equation from the one formed by changing x into $(x+1)$. Then, since a solution $f(x)$ satisfies the difference equation (B), the original differential

* *Quarterly Journal of Mathematics*, Vol. XXXI., pp. 310–314; *Phil. Trans. Roy. Soc.*, (A), Vol. CXCVI., pp. 384–387.

equation must be reducible to

$$\sum_{k=1}^{r_s}\left[\frac{R_k(x+1)}{R_0(x+1)}\,{}_sQ_k\{f(x+1)\}-\frac{R_k(x)}{R_0(x)}\,{}_sQ_k\{f(x)\}\right]$$

$$+{}_sQ_0\{f(x+1)\}-{}_sQ_0\{f(x)\}+\text{ terms of lower class }=0,$$

provided this latter equation be not a mere identity.

Now

$${}_sQ_k\{f(x+1)\} = {}_sQ_k\{f(x)+\chi(x)\} = {}_sQ_k\{f(x)\}+\text{ terms of lower class.}$$

Hence $f(x)$ satisfies a differential equation of which terms of highest class are

$$\sum_{k=1}^{r_s}\left[\frac{R_k(x+1)}{R_0(x+1)}-\frac{R_k(x)}{R_0(x)}\right]{}_sQ_k\{f(x)\}.$$

We can now repeat the previous process and reduce the equation to one with fewer terms of class s, unless all the coefficients

$$\frac{R_k(x+1)}{R_0(x+1)}-\frac{R_k(x)}{R_0(x)} \qquad (k=1, 2, \ldots, r_s)$$

vanish identically.

In the latter case the ratios $\dfrac{R_k(x)}{R_0(x)}$ are simply-periodic functions of period unity.

In the former case we either arrive at another alternative of this nature, or obtain a differential equation with a single term of class s.

Finally, therefore, we reduce the differential equation to one in which the terms of the highest class s can be written in the form

$$f_s(x)\sum_k[{}_s\phi_k(x)\,{}_sQ_k(y)],$$

the ϕ's being one-valued simply-periodic functions of x of period unity, and $f_s(x)$ being a one-valued function of x.

4. Suppose now that the reduced differential equation is

$$f(x)\sum_{k=1}^{r_s}\{{}_s\phi_k(x)\,{}_sQ_k(y)\}+\sum_{k=1}^{r_{s-1}}[{}_{s-1}\psi_k(x)\,{}_{s-1}Q_k(y)]+\text{ terms of lower class }=0,$$

where the ψ's are one-valued functions of x.

Since $y=f(x)$ satisfies the difference equation (B), we have

$$\frac{d^n}{dx^n}[f(x+1)] = y^{(n)}+\chi^{(n)}(x).$$

Hence, when x is changed into $x+1$,

$${}_sQ_k(y) = y^{{}_sm_1}(y')^{{}_sm_2}\ldots(y^{(n)})^{{}_sm_{n+1}}$$

becomes

$$\prod_{r=0}^{n}\left[\{y^{(r)}+\chi^{(r)}(x)\}^{{}_sm_{r+1}}\right]$$

$$= {}_sQ_k(y)\left[1+\sum_{r=0}^{n}{}_sm_{r+1}\frac{\chi^{(r)}x}{y^{(r)}}+\sum_{r=0}^{n}\frac{{}_sm_{r+1}({}_sm_{r+1}-1)}{2!}\left\{\frac{\chi^{(r)}(x)}{y^{(r)}}\right\}^2\right.$$

$$\left.+\sum_{\substack{r_1\\(r_1\neq r_2)}}\sum_{r_2}{}_sm_{r_1+1}\,{}_sm_{r_2+1}\frac{\chi^{(r_1)}(x)\,\chi^{(r_2)}(x)}{y^{(r_1)}\,y^{(r_2)}}+\ldots\right].$$

Divide the reduced differential equation by $f_s(x)$, change x into $(x+1)$, and subtract the reduced equation divided by $f_s(x)$. We obtain

$$\sum_{k=1}^{r_{s-1}}\left[\frac{{}_{s-1}\psi_k(x+1)}{f_s(x+1)}\,{}_{s-1}Q_k(y)-\frac{{}_{s-1}\psi_k(x)}{f_s(x)}\,{}_{s-1}Q_k(y)\right]$$

$$+\sum_{k=1}^{r_s}{}_sm_1\,{}_sQ_k(y)\,\frac{\chi(x)}{y}\,{}_s\phi_k(x)+\text{ terms of class }(s-2)$$

$$+\text{terms of lower classes} = 0.$$

5. The terms of class $(s-1)$ in this equation will persist unless we have a series of equations of the type

$$\frac{{}_{s-1}\psi_k(x+1)}{f_s(x+1)}-\frac{{}_{s-1}\psi_k(x)}{f_s(x)} = -{}_sm_1\chi(x)\,{}_s\phi_k(x). \tag{B$_1$}$$

Corresponding to some values of k the terms on the right-hand side will vanish. This cannot, however, happen for all the r_s values of k unless there is no factor $y^{{}_sm_1}$ in any of the terms of class s of the original reduced equation.

Hence, *either* some of the coefficients ${}_{s-1}\psi_k(x)$ are such that $\dfrac{{}_{s-1}\psi_k(x)}{f_s(x)}$ is a function which is a solution of a difference equation of the type

$$P(x+1)-P(x) = -\,{}_sm_1\chi(x)\,{}_s\phi_k(x);$$

or ${}_sm_1 = 0$ for each of the terms ${}_sQ_k(y)$, and the differential equation may be written

$$f_s(x)\left[\sum_{k=1}^{r_s}{}_s\phi_k(x)\,{}_sQ_k(y)+\sum_{k=1}^{r_{s-1}}{}_{s-1}\phi_k(x)\,{}_{s-1}Q_k(y)\right]+\text{ terms of lower class} = 0,$$

where the ϕ's are simply-periodic functions of x of period unity. In this differential equation the terms ${}_sQ_k(y)$ do not involve y apart from its differentials with regard to x.

Take the latter of the two alternatives thus presented, and let terms of class $(s-2)$ in the differential equation just written be

$$\sum_{k=1}^{r_{s-2}} [{}_{s-2}\psi_k(x)\,{}_{s-2}Q_k(y)].$$

If from this differential equation we now form the reduced equation, it will be of lower class than $(s-1)$ and the terms of class $(s-2)$ will be

$$\sum_{k=1}^{r_{s-2}} \left\{\frac{{}_{s-2}\psi_k(x+1)}{f_s(x+1)} - \frac{{}_{s-2}\psi_k(x)}{f_s(x)}\right\} {}_{s-2}Q_k(y) + \sum_{k=1}^{r_s} {}_s\phi_k(x)\,{}_sQ_k(y)\,{}_sm_2\frac{\chi^{(1)}(x)}{y^{(1)}}$$
$$+ \sum_{k=1}^{r_{s-1}} {}_{s-1}\phi_k(x)\,{}_{s-1}Q_k(y)\,{}_{s-1}m_1\frac{\chi(y)}{y}.$$

These terms will persist unless the functions $\dfrac{{}_{s-2}\psi_k(x)}{f_s(x)}$ (or some of them) satisfy a difference equation of the type

$$P(x+1)-P(x) = -{}_sm_2\,{}_s\phi_k(x)\,\chi^{(1)}(x) - {}_{s-1}m_1\,{}_{s-1}\phi_k(x)\,\chi(x).$$

Both the constants ${}_sm_2$ and ${}_{s-1}m_1$ may vanish when, and only when, none of the terms of class s in the original reduced differential equation involve y or y' apart from their differentials with regard to x, and none of the terms of class $(s-1)$ involve y.

6. Repeating the process, we see that ultimately:—(1) Either some of the coefficients of the differential equation, when written in the form

$$\sum_{k=1}^{r_s} {}_s\phi_k(x)\,{}_sQ_k(y) + \text{terms of lower class} = 0,$$

must satisfy a difference equation of the type

$$P(x+1)-P(x) = \sum_{r=0} a_r\chi^{(r)}(x)\,\phi_r(x) \tag{C}$$

where the constants a do not all vanish; or (2) the differential equation must be of the form

$${}_s\phi(x)\,(y^{(n)})^{{}_sm_n+1} + \sum_k {}_{s-1}\phi_k(x)\,{}_{s-1}Q_k\{y^{(n-1)},\ y^{(n)}\}$$
$$+ \sum_k {}_{s-2}\phi_k(x)\,{}_{s-2}Q_k\{y^{(n-2)},\ y^{(n-1)},\ y^{(n)}\} + \dots$$
$$+ \sum_k \frac{{}_{s-n}\psi_k(x)}{f_s(x)}\,{}_{s-n}Q_k\{y,\ y^{(1)},\ \dots,\ y^{(n)}\}$$
$$+ \text{terms of lower class than } (s-n) = 0,$$

where $s = (n+1)\,{}_s m_{n+1}$ and the ϕ's are one-valued simply-periodic functions of period unity; or (3) the differential equation can be reduced to one of lower class, in which terms of highest class persist.

If now we consider case (2) and reduce the equation last written, we get a differential equation of class $(s-n)$, of which the terms of highest class are

$$\sum_k \left\{ \frac{{}_{s-n}\psi_k(x+1)}{f_s(x+1)} - \frac{{}_{s-n}\psi_k(x)}{f_s(x)} \right\} {}_{s-n}Q_k(y) + {}_s\phi(x)\,{}_s m_{n+1}\,\chi^{(n)}(x)\left[y^{(n)}\right]^{m_{n+1}-1}$$

$$+\Sigma_{s-1}\phi(x)\,{}_{s-1}m_n\,\chi^{(n-1)}(x)\left[y^{(n)}\right]^{s-1^{m_{n+1}}}\left[y^{(n-1)}\right]^{s-1^{m_n}-1}+\dots.$$

These terms of highest class exist unless some of the functions $\dfrac{{}_{s-n}\psi_k(x)}{f_s(x)}$ satisfy difference equations of the form

$$P(x+1)-P(x) = -{}_s m_{n+1}\,{}_s\phi(x)\,\chi^{(n)}(x)+\sum_{r=0}^{n-1} a_r\chi^{(r)}(x)\,\phi_r(x),$$

which is of the same type as equation (C).

In this last difference equation the terms on the right-hand side only vanish when $\chi(x)$ satisfies a differential equation of the form

$${}_s m_{n+1}\frac{d^n y}{dx^n}\,{}_s\phi(x) = \sum_{r=0}^{n-1} a_r \frac{d^r y}{dx^r}\,\phi_r(x).$$

Thus either (1) the original differential equation can be reduced to one of lower class in which terms of the highest class persist, or (2) it has coefficients which, when the terms of highest class are written in the form

$$\sum_{k=1}^{r_s} {}_s\phi_k(x)\,{}_sQ_k(y),$$

are solutions of a difference equation of the type

$$P(x+1)-P(x) = \sum_{r=0}^{n} a_r\phi_r(x)\,\chi^{(r)}(x), \tag{C}$$

in which all the coefficients a_r on the right-hand side are certainly not zero.

7. In the first case we can again reduce the equation to one of lower class, and so on indefinitely, unless the second alternative occurs again. Ultimately, we either are forced to the second alternative, or we get an equation whose class is unity, that is, an equation

$$h(x)\,y+k(x) = 0,$$

which is not a differential equation, and whose coefficients must be such that $-\frac{k(x)}{h(x)}$ is a solution of the difference equation (B).

Suppose now that the antepenultimate reduced equation to the one just written is

$$y'+p_1(x)y+p_2(x)y^2+p_3(x)=0, \tag{1}$$

which is of class 2, and the most general form of equation of this class. We reduce it to

$$y^2+\frac{p_1(x+1)-p_1(x)-2p_2(x+1)\chi}{p_2(x+1)-p_2(x)}y$$

$$+\frac{p_2(x+1)\chi^2+p_3(x+1)-p_3(x)+\chi'}{p_2(x+1)-p_2(x)}=0,$$

or (say)

$$y^2+q_1(x)y+q_2(x)=0.$$

And, reducing this, we get

$$y\{q_1(x+1)-q_1(x)-2\chi\}+q_1(x+1)\chi+\chi^2+q_2(x+1)-q_2(x)=0.$$

We see therefore that by taking the coefficients of the equation (1), forming similar functions when $(x+1)$ is substituted for x, and taking rational combinations of these quantities and χ and χ', we can form a solution of the difference equation (B). Hence the coefficients of (1) must, all or some of them, be functions from which, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations (addition, subtraction, multiplication, or division), solutions of the difference equation (B) can, with the aid of $\chi(x)$ and its derivates, be built up. Some of them must therefore be one-valued functions with infinite sequences of zeros or poles arranged in a manner which is in strict correlation to the distribution of zeros and poles which characterises the nature of the solution of the difference equation. The coefficients *may*, of course, be more complex functions than the functions which are solutions of the difference equation; for the process of forming finite differences may materially simplify the distribution of zeros and poles. This would be the case when the coefficients are derived from linear difference equations. And so, for instance, the gamma function might be a solution of a differential equation whose coefficients were built up with the aid of double gamma functions. But the gamma function can be derived, without the intervention of differential equations, by the ordinary process of forming finite differences from the double gamma function. Thus this possibility does not affect the main object of this paper, which is to show *not* that by means of differential equations we can

go from functions which are more to functions which are less complex, but that we cannot proceed *vice versa*, the more complex functions being obtained from the less complex by the process of difference integration. If we proceed successively backwards, we see that all the previous remarks about the nature of the coefficients of the differential equation (1) must be true of the coefficients of the original differential equation.

8. Consider now the second alternative of § 6.

We have the difference equation

$$P(x+1)-P(x) = \sum_{r=0} a_r \phi_r(x)\, \chi^{(r)}(x), \tag{C}$$

in which any but not all of the a's may vanish.

[When the upper limit for r is n, a_n is not zero.]

Let the solution of $\quad P(x+1)-P(x) = \chi(x)$

be

$$G(x)+\phi_0(x),$$

$\phi_0(x)$ being a simply-periodic function of x of period unity. The solution of

$$P(x+1)-P(x) = \chi^{(r)}(x)$$

will be

$$G^{(r)}(x)+\phi_1(x).$$

Hence the solution of the difference equation (C) will be

$$\sum_{r=0} a_r \phi_r(x)\, G^{(r)}(x)+\phi(x).$$

Thus in this case the coefficients of the original differential equation must be functions which may be obtained from the solution of the original difference equation (B) by the process of differentiation.

9. We have, however, still to consider the particular case when the difference equation (C) reduces to

$$P(x+1)-P(x) = 0.$$

This will happen if $\chi(x)$ satisfies the linear differential equation

$$\sum_{r=0}^{n} a_r \phi_r(x) \frac{d^r y}{dx^r} = 0, \tag{C$_1$}$$

in which the functions ϕ are uniform functions simply-periodic of period unity.

Consider the case in which the ϕ's are meromorphic functions with a single essential singularity at infinity. This is equivalent to assuming that the coefficients of the original differential equation (A) are also mero-

morphic. In this case one of the fundamental solutions of (C_1) is* given by an aggregate of the type $\sum_{k=0}^{m-1} x^k \Phi_k(x)$, where $\Phi_k(x)$ denotes a simply-periodic function of the second kind which satisfies the difference equation

$$\Phi_k(x+1) = e^{\theta}\Phi_k(x),$$

where e^{θ} is the l-th root r_l-ply repeated of the fundamental equation of (C_1) and $(1 \leqslant m \leqslant r_l)$.

The number r_l which intervenes must of course be $\leqslant n$. When the roots of the fundamental equation of (C_1) are all different the numbers r_l are all unity.

We see then that it is possible that the solution of the difference equation

$$P(x+1)-P(x) = \sum_{k=0}^{m-1} x^k \Phi_k(x), \tag{D}$$

or a sum of solutions of such equations, may satisfy a differential equation of finite order and dimensions whose coefficients, supposed meromorphic, are not of a type which, in the language of § 1, embraces the solution of the difference equation.

10. Let us consider the solution of this difference equation.

Since $\Phi_k(x)$ satisfies $\quad \Phi_k(x+1) = e^{\theta}\Phi_k(x),$

we have

$$\Phi_k(x) = e^{\theta x}\wp_k(x),$$

where $\wp_k(x)$ is a simply-periodic function of period unity.

Hence a solution of $\quad P(x+1)-P(x) = \Phi_k(x)$

is

$$\frac{\Phi_k(x)}{e^{\theta}-1} + \wp(x).$$

The solution of (D) is composed of the sum of solutions of equations of the type

$$P(x+1)-P(x) = e^{\theta x}x^k\wp_k(x). \tag{E}$$

Put

$$P(x) = e^{\theta x}\wp_k(x)\, Q(x),$$

and this equation becomes

$$e^{\theta}Q(x+1) - Q(x) = x^k,$$

of which a solution is

$$Q(x) = -\frac{\iota\Gamma(1+k)}{2\pi}\int \frac{e^{-xz}}{1-e^{\theta-z}}(-z)^{-k-1}\,dz,$$

* Forsyth, *Differential Equations*, Part III., Vol. IV. (1902), p. 415.

the integral being taken round the usual contour for the Riemann ζ function.* This solution may be readily verified by substitution.

Hence the solution of the equation (E) is

$$\wp_k(x)\, e^{\theta x} \left\{ \frac{-\iota}{2\pi} \int \frac{e^{-xz}\,(k)!}{1-e^{\theta-z}} (-z)^{-k-1}\, dz \right\} + \wp(x).$$

The integral is equal to the residue at the origin of $\dfrac{k!\, e^{-xz}}{\{1-e^{\theta-z}\}\{-z\}^{k+1}}$. It is thus an extended Bernoullian number, which we may denote by $S_k(x, \theta)$. It is evidently, in general, a polynomial in x of degree k. [It is of degree $k+1$ when $e^\theta = 1$ and we have real Bernoullian numbers.]

The solution of (E) is therefore

$$\wp_k(x)\, e^{\theta x} S_k(x, \theta) + \wp(x).$$

Hence the solution of (D) is

$$\sum_{k=0}^{m-1} [e^{\theta x} \wp_k(x)\, S_k(x, \theta)] + \wp(x),$$

which is the type of function generated by linear differential equations with simply-periodic coefficients.

11. Thus it is possible that a differential equation may admit as a solution a sum of terms of the type

$$e^{\theta x} \wp_k(x)\, S_k(x, \theta),$$

which sum is a solution of a linear difference equation of the type (D), when its coefficients are not of a type which embraces the solution.

The linear differential equation with constant coefficients is an example of this peculiarity. This equation admits terms of the type $e^{\theta x} x^k$ as solutions. Such terms satisfy a linear difference equation of the type (B), but it would be absurd to say that some of the constant coefficients of the differential equation belong, in the language of § 1, to a type which embraces the solution $\Sigma e^{\theta x} x^k$.

As another example, we may take the linear differential equation with meromorphic simply-periodic coefficients of period unity. This equation, when the roots of its fundamental equation are all different has simply-periodic functions of the second kind as its independent solutions. These functions are each a solution of a difference equation

$$P(x+1) - P(x) = \chi(x),$$

* See, for instance, a paper by the author, *Messenger of Mathematics*, Vol XXIX., pp. 64–128.

where $\chi(x)$ is a simply-periodic function of the second kind. And the simply-periodic coefficients of the linear differential equation obviously cannot be said to belong to a type which embraces, in the sense we have defined in § 1, the simply-periodic functions of the second kind which are solutions.

12. We have, finally, the theorem :—

When the solution of the difference equation

$$P(x+1)-P(x)=\chi(x), \tag{B}$$

where $\chi(x)$ is a meromorphic function, is not the type of function that can be obtained as a solution of a linear differential equation with uniform simply-periodic coefficients of period unity, it cannot be obtained as the solution of any differential equation of finite order and dimensions with meromorphic coefficients, unless either (1) these coefficients are obtained by differentiation from the function itself, or (2) from these coefficients and $\chi(x)$ and its differentials we can by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, derive the solution itself.

In the more general case when $\chi(x)$ is not meromorphic, but has essential singularities in the finite part of the plane, it must satisfy a linear differential equation with simply-periodic coefficients of period unity if the solution of the difference equation is also to be a solution of a differential equation of finite order and dimensions with uniform coefficients whose coefficients do not some of them belong to a type which embraces the solution itself.

13. We may now extend the previous theorem to the case when the difference equation (B) is of the more general type

$$P(x+1)-\psi(x)\,P(x)=\chi(x). \tag{F}$$

Let $G(x)$ be a particular solution of the equation

$$P(x+1)-\psi(x)\,P(x)=0,$$

and let $H(x)+\phi(x)$, where $\phi(x)$ is simply periodic of period unity, be the general solution of

$$P(x+1)-P(x)=\frac{\chi(x)}{G(x+1)}.$$

Then the complete solution of the equation (F) is obviously

$$P(x) = G(x)\{H(x)+\phi(x)\}.$$

By the theorem just proved $\dfrac{P(x)}{G(x)}$ can, in general, only be a solution of a differential equation some of whose coefficients embrace $H(x)$. That is to say, either the coefficients of the differential equation for $\dfrac{P(x)}{G(x)}$ are obtained by differentiation from $\dfrac{P(x)}{G(x)}$, or from them and $\dfrac{\chi(x)}{G(x+1)}$ and its derivates, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, we can deduce $\dfrac{P(x)}{G(x)}$.

Therefore the coefficients of the differential equation for $P(x)$ are either obtained by the finite combination of a finite number of differentials of $P(x)$ and $G(x)$, or from them and a finite number of successive differentials of $\chi(x)$, $G(x+1)$, and $G(x)$ we can, by the fundamental process of forming finite differences coupled with a finite number of elementary algebraical operations, derive $P(x)$.

Now, in general, $G(x)$, which is the solution of

$$f(x+1)-\psi(x)f(x) = 0,$$

is a much more simple type* of function than $P(x)$, which is the solution of

$$P(x+1)-\psi(x)P(x) = \chi(x). \tag{F}$$

Therefore we may say that, in this case, as in the previous one, the solution of the difference equation cannot, in general, be the solution of a differential equation of finite order and dimensions, unless some of the coefficients of this differential equation are of a type which embraces $P(x)$.

14. The fundamental case of exception occurs when $\dfrac{\chi(x)}{G(x+1)}$ can be expressed as the sum of one or more aggregates of the type $\sum\limits_{k=0}^{n-1} x^k e^{\theta x} \wp_k(x)$.

In this case $$\chi(x) = G(x+1)\Sigma \sum_{k=0}^{n-1} e^{\theta x}x^k \wp_k(x),$$

and the difference equation (F) is resoluble into a sum of others of the type

$$P(x+1)-\psi(x)P(x) = G(x+1)\sum_{k=0}^{m-1} e^{\theta x}x^k\wp_k(x),$$

* The further discussion of this point must be reserved for the investigation referred to in § 1.

whose solution is $G(x)\left\{\sum_{k=0}^{m-1}\left[e^{\theta x}S_k(x,\theta)\wp_k(x)\right]+\wp(x)\right\}$.

Now $\dfrac{G'(x)}{G(x)}$ is a solution of the difference equation

$$P(x+1)-P(x)=\frac{\psi'(x)}{\psi(x)}.$$

Hence the differential equation whose solution is $\dfrac{G'(x)}{G(x)}$ must have as coefficients functions belonging to a type which embraces $\dfrac{G'(x)}{G(x)}$, *i.e.*, which embraces $G(x)$, unless $\dfrac{\psi'(x)}{\psi(x)}$ is typified by $\sum\limits_{l=0} e^{\theta' x}x^l\wp_l(x)$. Therefore, unless $\dfrac{\psi'(x)}{\psi(x)}$ is of this character, the differential equation whose solution is of the form

$$\Sigma\, G(x)\left\{\sum_{k=0}^{m-1} e^{\theta x}S_k(x,\theta)\,\wp_k(x)+\wp(x)\right\}$$

must have as coefficients functions belonging to a type which embraces solutions of the difference equation (F) when $\chi(x)$ is zero.

When, however,
$$\frac{\psi'(x)}{\psi(x)}=\sum_{l=0} e^{\theta' x}x^l\wp_l(x),$$

we have
$$\frac{G'(x)}{G(x)}=\sum_{l=0} e^{\theta' x}S_l(x,\theta')\,\wp_l(x);$$

and therefore
$$G(x)=\exp\left[\int^x \Sigma e^{\theta' x}\,S_l(x,\theta')\wp_l(x)\right].$$

Thus, for the complete case of exception to arise, the difference equation (F) must be resoluble into a sum of others of the type

$$P(x+1)-\exp\left[\int^x \sum_{l=0} e^{\theta' x}x^l\wp_l(x)\right]P(x)$$
$$=\left[\sum_{k=0}^{m-1} e^{\theta x}x^k\wp_k(x)+\wp(x)\right]\exp\left[\int^{x+1}\sum_{l=0} e^{\theta' x}S_l(x,\theta')\,\wp_l(x)\right].$$

NOTE ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS

By H. F. BAKER.

[Communicated May 12th, 1904.—Received June 13th, 1904.]

I SHOULD like to have the opportunity of acknowledging, what I learned in October, 1903, from a paper of Professor Bôcher's (*Amer. Jour.*, Vol. XXIV., p. 311), that the matrizant solution of a system of linear differential equations given in *Proc. London Math. Soc.*, Vol. XXXIV., p. 354 and p. 356, footnote, which grew up naturally in my mind in connection with Schur's series for continuous groups (*Proc. London Math. Soc.*, February 14th, 1901, Vol. XXXIV., p. 97, and Vol. XXXIV., p. 348) had been previously given by Peano, *Math. Ann.*, Vol. XXXII., 1888, pp. 455, 456, with the unimportant limitation, in the statement, to coefficients which are real functions of the variable continuous in an interval for which the convergence of the series is to be proved.

Indeed the idea of using series of repeated integrations, for a single difference equation and a single differential equation regarded as a limit of this, is at least as old as the paper of Caqué, *Liouville's Journal*, 2nd Series, t. IX., 1864, p. 194; while in the paper written by Fuchs in 1870, to give a more general aspect to Caqué's method (*Ann. d. Mat.*, II. Ser., t. IV., p. 36), the convergence of these series for all finite values of the argument other than the singularities is clearly recognized. My ignorance of this paper of Fuchs, at the time of the last note on linear differential equations (*Proc. London Math. Soc.*, Vol. XXXV., p. 333, where, p. 378, I have collected references to papers seeming to be in connection with the method of the paper), is the more inexcusable in that Fuchs's results are expounded in Schlesinger's treatise (Vol. I., pp. 370 and 389), with an application to equations of rank unity. Perhaps, however, the connexion of Fuchs's formula with the matrizant solution is not very obvious; and it may be worth showing that Fuchs's generalized form of Caqué's solution for a single differential equation is a particular case of a general formula given in the note just referred to (*Proc. London Math. Soc.*, Vol. XXXV., p. 339). This is what is proved below.

Let a, β be two matrices of the same number n of rows and columns, of which each element is a function of t, and let σ be a matrix of constants; then

$$\Delta = \Omega(a)\sigma$$

is a matrix whose columns are sets of solutions of the linear system

$$\frac{dx}{dt} = ax.$$

By easily verified formulæ given in *Proc. London Math. Soc.*, Vol. xxxv., 1902, pp. 339, 337, namely,

$$\Omega(a+\beta) = \Omega(a)\,\Omega[\Omega^{-1}(a)\beta\Omega(a)], \qquad \sigma^{-1}\Omega(u)\sigma = \Omega(\sigma^{-1}u\sigma),$$

we have $\Omega(a+\beta)\sigma = \Omega(a)\sigma.\sigma^{-1}\Omega[\Omega^{-1}(a)\beta\Omega(a)]\sigma$

$$= \Delta\Omega[\sigma^{-1}\Omega^{-1}(a)\beta\Omega(a)\sigma] = \Delta\Omega(\Delta^{-1}\beta\Delta);$$

now let $w = \Delta^{-1}\beta\Delta$, or $w\Delta^{-1} = \Delta^{-1}\beta$, and, denoting a row $(h_1 \dots h_n)$ of constants by h, put

$$(u_1^{(0)} \dots u_n^{(0)}) = \Delta h,$$

(A) $$(u_1^{(i)} \dots u_n^{(i)}) = \Delta Q\,[\Delta^{-1}\beta(u_1^{(i-1)} \dots u_n^{(i-1)})] \quad (i = 1, 2, \dots, \infty),$$

so that $u_1^{(0)} \dots u_n^{(0)}$ form a set of solutions of the linear system $dx/dt = ax$, and the successive sets $u_1^{(i)} \dots u_n^{(i)}$ are determined each from the preceding by a single quadrature, denoted by Q. Then we have

$$\Omega(a+\beta)\sigma h = \Delta\Omega(w)h = \Delta h + \Delta Qwh + \Delta Qw\,Qwh + \dots,$$

where

$$\Delta h = (u_1^{(0)} \dots u_n^{(0)}),$$

$$\Delta Qwh = \Delta Q(\Delta^{-1}\beta\Delta h) = \Delta Q[\Delta^{-1}\beta(u_1^{(0)} \dots u_n^{(0)})] = (u_1^{(1)} \dots u_2^{(1)}),$$

$$\Delta Qw\,Qwh = \Delta Q[w\Delta^{-1}(u_1^{(1)} \dots u_n^{(1)})] = \Delta Q[\Delta^{-1}\beta(u_1^{(1)} \dots u_n^{(1)})] = (u_1^{(2)} \dots u_n^{(2)}),$$

and so on. Thus a set of solutions of the linear system $dx/dt = (a+\beta)x$ is given by

$$(x_1 \dots x_n) = (u_1^{(0)} \dots u_n^{(0)}) + (u_1^{(1)} \dots u_n^{(1)}) + (u_1^{(2)} \dots u_n^{(2)}) + \dots,$$

that is, by

(B) $$x_j = u_j^{(0)} + u_j^{(1)} + u_j^{(2)} + \dots.$$

By choosing the matrix σ suitably, with non-vanishing determinant, Δ may be regarded as having for its columns any fundamental set of integrals of the system $dx/dt = ax$, and, by choosing h suitably, the

solution (B), with the law of recurrence (A), represents any set of integrals of the compound system $dx/dt = (a+\beta)x$.

This includes, as we next show, the results of Fuchs, *Ann. d. Mat.*, 1870–71, p. 43, given in Schlesinger, Treatise I., p. 373; we limit ourselves to the case of a homogeneous differential equation.

The single linear equation

$$y^{(n)} = (a_n+b_n)y^{(n-1)}+(a_{n+1}+b_{n-1})y^{(n-2)}+\ldots+(a+b)y$$

becomes, by $x_1 = y,\ x_2 = y',\ \ldots,\ x_n = y^{(n-1)}$,

$$\frac{dx}{dt} = \begin{Bmatrix} 0 & 1 & 0 & . & . \\ 0 & 0 & 1 & 0 & . \\ . & . & . & . & . \\ a+b & a_1+b_1 & . & . & a_n+b_n \end{Bmatrix} x,$$

$$= \left[\begin{Bmatrix} 0 & 1 & 0 & . & . \\ 0 & 0 & 1 & 0 & . \\ . & . & . & . & . \\ a_1 & a_2 & a_3 & . & a_n \end{Bmatrix} + \begin{Bmatrix} 0 & 0 & 0 & . & . \\ 0 & 0 & 0 & 0 & . \\ . & . & . & . & . \\ b_1 & b_2 & b_3 & . & b_n \end{Bmatrix}\right] x = (a+\beta)\,x, \text{ say};$$

let $y_1, \ldots, y_n$ be a set of independent integrals of $y^{(n)} = a_n y^{(n-1)}+\ldots+a_1 y$; for the matrix Δ we can then put

$$\Delta = \begin{Bmatrix} y_1 & \cdots & y_n \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{Bmatrix},$$

and then, if $D_1, \ldots, D_n$ be the determinants of the minors of the elements in the last row of Δ, and $D, = |\,\Delta\,|$, be the determinant of Δ, and the inverse matrix Δ^{-1} be (ϕ_{ij}), so that

$$\phi_{1n} = \frac{D_1}{D},\quad \phi_{2n} = \frac{D_2}{D},\quad \ldots,\quad \phi_{nn} = \frac{D_n}{D},$$

we have, for arbitrary $v_1 \ldots v_n$,

$$\Delta^{-1}\beta(v_1, v_2, \ldots, v_n) = \begin{Bmatrix} \phi_{1n}b_1 \ldots \phi_{1n}b_n \\ \cdots \quad \cdots \quad \cdots \\ \phi_{nn}b_1 \ldots \phi_{nn}b_n \end{Bmatrix} (v_1 \ldots v_n)$$

$$= \left[\frac{D_1}{D}(b_1v_1+\ldots+b_nv_n),\ \ldots,\ \frac{D_n}{D}(b_1v_1+\ldots+b_nv_n)\right];$$

hence the elements of $\Delta Q[\Delta^{-1}\beta(v_1, \ldots, v_n)]$ are respectively

$$y_1 Q\left[\frac{D_1}{D}(b_1v_1+\ldots+b_nv_n)\right]+\ldots+y_n Q\left[\frac{D_n}{D}(b_1v_1+\ldots+b_nv_n)\right],$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$y_1^{(n-1)} Q\left[\frac{D_1}{D}(b_1v_1+\ldots+b_nv_n)\right]+\ldots+y_n^{(n-1)} Q\left[\frac{D_n}{D}(b_1v_1+\ldots+b_nv_n)\right],$$

of which manifestly each is the differential coefficient of the preceding. Put now in particular $(v_1, \ldots, v_n) = (y, y', \ldots, y^{(n-1)})$; the first element of $\Delta Q\,[\Delta^{-1}\beta(v, \ldots, v_n)]$ becomes

$$y_1 Q\left[\frac{D_1}{D}(b_1y+\ldots+b_ny^{(n-1)})\right]+\ldots+y_n Q\left[\frac{D_n}{D}(b_1y+\ldots+b_ny^{(n-1)})\right],$$

or, if $B[y(t)] = b_1y+b_2\dfrac{dy}{dt}+\ldots+b_n\dfrac{d^{n-1}y}{dt^{n-1}}$, it becomes

$$Y_1(t) = \int^t B[y(\xi)]\frac{\begin{vmatrix} y_1(\xi) & \ldots & y_n(\xi) \\ \cdot & \ldots & \cdot \\ y_1^{(n-2)}(\xi) & \ldots & y_n^{(n-2)}(\xi) \\ y_1(t) & \ldots & y_n(t) \end{vmatrix}}{\begin{vmatrix} y_1(\xi) & \ldots & y_n(\xi) \\ \cdot & \ldots & \cdot \\ y_1^{(n-1)}(\xi) & \ldots & y_n^{(n-1)}(\xi) \end{vmatrix}}\,d\xi = \int^t B[y(\xi)]\,\frac{H(\xi, t)}{D(\xi)}\,d\xi, \text{ say.}$$

Put next $(v_1, v_2, \ldots, v_n) = (Y_1, Y_1, \ldots, Y_1^{(n-1)})$; then the first element of $\Delta Q[\Delta^{-1}\beta(v_1, \ldots, v_n)]$ becomes

$$y_1 Q\left[\frac{D_1}{D}(b_1Y_1+\ldots+b_nY_1^{(n-1)})\right]+\ldots+y_n Q\left[\frac{D_n}{D}(b_1Y_1+\ldots+b_nY_1^{(n-1)})\right],$$

or

$$Y_2(t) = \int^t B[Y_1(\xi)]\,\frac{H(\xi, t)}{D(\xi)}\,d\xi,$$

and so on; which are Fuchs's formulæ.

Similarly, Caqué's formula, obtained by taking $a_1 = 0, \ldots, a_n = 0$; $b_1 = p_1, \ldots, b_n = p_n$, and an application to equations of rank unity, obtained by taking $a_1 = \text{const.}, \ldots, a_n = \text{const.}$; $b_1 = Q_1(1/t), \ldots, b_n = Q_n(1/t)$, may be deduced; but, as they are deduced in Schlesinger's treatise (Vol. I., pp. 377 and 389) from Fuchs's formula, they need not be given here.

Another interesting case is that where a is a matrix of constants with linear invariant factors. Taking this in its canonical form, the matrix $\Delta = \Omega(a)$ has only elements of the form $e^{\theta t}$, occurring in its diagonal.

WAVE FRONTS CONSIDERED AS THE CHARACTERISTICS OF PARTIAL DIFFERENTIAL EQUATIONS

By T. H. HAVELOCK.

[Received June 7th, 1904.—Read June 9th, 1904.—Revised August 26th, 1904.]

CONTENTS.

1. *Introduction.*

The partial differential equations which are used to represent the propagation of disturbing effects in physical media have been examined by various methods. If the object is to obtain a direct result, such as the relation of velocity to direction of propagation, the usual limitation consists in dealing only with plane waves. The description of the propagation of an arbitrary disturbance is generally founded upon the solutions of the differential equations given by Poisson and Kirchhoff in the form of surface integrals, or, upon considerations such as those used by Hugoniot.

The application of the theory of characteristics of partial differential equations dates from Riemann's work upon the propagation of waves of finite amplitude in air, and the method has been used by Volterra* to obtain general solutions of equations of motion. The comparison of wave fronts with characteristics appears to have been made separately by Coulon and Hadamard, and has been developed in detail by them recently.† They found their theory of characteristics upon extensions made by Beudon‡ for equations of the second order in n variables and

* Volterra, *Acta Mathematica*, t. XVIII., p. 161, 1894.

† Coulon, Thèse, *Sur l'Integration des équations aux dérivées partielles du second ordre par la méthode des caractéristiques*, Paris, 1902; Hadamard, *Léçons sur la propagation des ondes*, Paris, 1903.

‡ Beudon, *Bulletin de la Soc. Math.*, 1897, p. 108.

for systems of equations with several unknown functions, and application is made of the theory to find the velocity of propagation of wave fronts and the paths of rays in various cases, and other general properties of wave propagation.

In the following paper I have endeavoured to present the subject afresh, and in the introductory sections have followed chiefly an account of characteristic theory given by Hedrick*; previous results will be found to be expressed in a different manner, which also allows for various extensions. In particular, the following may be mentioned.

In all the methods just described the function defining the disturbances is assumed to be continuous along with its first differential coefficients at the boundary of the disturbed portion of the medium; but it has been shown recently by Prof. Love† that there is no physical necessity for these limitations, and consequently the results are not sufficiently general. Now, on the characteristic theory, a wave front is defined as a section of a multiplicity satisfying the principal equation of the characteristics of the partial differential equations: wave fronts with essential discontinuities are discussed both for a single equation and for the system of electrical equations in free æther, and it is shown that owing to the linearity of the equations there is no necessity for the continuity of the first differential coefficients of the function upon the wave front. Provided the function itself is continuous, the same principal equation is satisfied.

In connection with the transformation of differential equations, an invariant property of wave fronts is proved; and use is made thereof in the electrical theory of wave fronts and rays in moving media, and in the case of a theoretically possible medium having electric and magnetic æolotropy with different principal axes.

THE CHARACTERISTICS OF A SINGLE PARTIAL DIFFERENTIAL EQUATION.

2. *Equation of the First Order.*

The theory of the characteristics of partial differential equations may be founded in a consistent manner upon the existence theorems of Cauchy and Kowalevski, being determined, in fact, by the failure of these processes.

Consider a partial differential equation of the first order in one

* Hedrick, *Annals of Mathematics*, Ser. 2, Vol. IV., p. 121.

† Love, *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 37.

dependent variable z and two independent variables x, y,

$$F(x, y, z, p, q) = 0. \tag{1}$$

Suppose this equation solved for p in the form

$$p = f(x, y, z, q). \tag{2}$$

Then there exists one, and only one, analytic solution,

$$z = \nu(x, y), \tag{3}$$

which satisfies the boundary condition

$$z\big]_{x=x_0} = \nu(x_0, y) = \phi(y), \tag{4}$$

where $\phi(y)$ is a preassigned function analytic near $y = y_0$; provided that $f(x, y, z, q)$ is a single-valued analytic function of x, y, z, q in the neighbourhood of the values x_0, y_0, $\phi(y_0)$, $\phi'(y_0)$ of these quantities.

In order that (1) may be put into the form (2) we must have $\partial F/\partial p$ not vanishing. And, if x, y, z be considered as rectangular coordinates, the theorem may be stated thus:—There is one, and only one, analytic integral surface of (1), corresponding to each single-valued solution of the form (2), upon which lies a given analytic curve in a plane parallel to the yz-plane, provided F is analytic and $\partial F/\partial p$ does not vanish for the values of x, y, z along the curve and the corresponding values of p and q given by (1) and the equation of the tangent to the curve.

Conversely, we see that, in general, by a suitable transformation any curve in space determines uniquely a corresponding integral surface. But this process evidently fails if the transformation which puts the given curve into the required form transforms at the same time the differential equation (1) so that it cannot be resolved into the normal form (2); of the curves in space for which the process fails in this way, those which lie on integral surfaces are called the characteristic curves.

It can be shown that through any such characteristic curve there is, in fact, an infinite number of integral surfaces; and these surfaces all touch along this curve, provided they touch at one point of it.

Given an integral surface

$$z = \nu(x, y) \tag{5}$$

and a curve upon it defined by (5) along with

$$y = \lambda(x), \tag{6}$$

then, by the method of the failure of the Cauchy-Kowalevski process,

an ordinary differential equation is obtained for $\lambda(x)$ in the form

$$-F_p(x, y)\frac{dy}{dx}+F_q(x, y) = 0, \tag{7}$$

where F_p and F_q are functions of x, y obtained by using (5) in the values of $\partial F/\partial p$ and $\partial F/\partial q$ respectively.

This may be called the principal equation of the characteristics. Their equations in space can be obtained without assuming an integral surface given, and are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{dz}{pF_p+qF_q} = \frac{-dp}{F_x+pF_z} = \frac{-dq}{F_y+qF_z}, \tag{8}$$

combined with (1), and must be satisfied by

$$y = \lambda(x); \quad z = \nu[x, \lambda(x)]; \quad p = \left[\frac{\partial\nu(x, y)}{\partial x}\right]_{y=\lambda(x)}; \quad q = \left[\frac{\partial\nu(x, y)}{\partial y}\right]_{y=\lambda(x)}.$$

These equations (8) determine, in general, ∞^3 characteristic curves, with one characteristic strip along each curve, *i.e.*, characteristic curve with attached elements of tangent planes. It is important to notice that for a differential equation linear in p and q there are only ∞^2 characteristic curves with ∞^1 characteristic strips along each curve.

3. *Equation of the Second Order.*

In the same way the Cauchy-Kowalevski theorem can be used in partial differential equations of the second order, say in

$$F(x, y, z, p, q, r, s, t) = 0. \tag{9}$$

Characteristic strips are determined by the failure of the existence theorem and satisfy the principal equation

$$R(x, y)\left(\frac{dy}{dx}\right)^2 - S(x, y)\frac{dy}{dx} + T(x, y) = 0, \tag{10}$$

where R, S, T are functions of x, y obtained by putting $z = \nu(x, y)$ in the values of $\partial F/\partial r$, $\partial F/\partial s$, and $\partial F/\partial t$ respectively.

Further, in this case the equations in space determine for a set of solutions a characteristic torsion strip, since r, s, t, as well as p and q, are given at each point of the characteristic curve. Also through every characteristic strip there is an infinite number of integral surfaces. The case of a differential equation linear in r, s, and t is important. For instance, if the equation is

$$A(x, y)\,r+B(x, y)\,s+C(x, y)\,t = D(x, y, z, p, q), \tag{11}$$

then the principal equation (10) does not contain z, p, q, r, s, or t; and along any characteristic curve whose projection on the xy-plane satisfies equation (10) there is in general a one-parameter family of characteristic strips of the equation (11).

These results can be extended to equations with more than two independent variables,* with geometrical interpretation in space of a suitable number of dimensions.

4. *Equations of Higher Order.*

Adopting the same method of finding the principal equation of the characteristics, we may extend this notion to a general linear partial differential equation of order m in n independent variables: *e.g.*,

$$\sum_{(\nu)} P_{\nu_1, \nu_2, \ldots, \nu_n} \frac{\partial^{\nu_1 + \ldots + \nu_n}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \ldots \partial x_n^{\nu_n}} = 0, \tag{12}$$

where the summation (ν) extends over all values such that

$$\nu_1 + \nu_2 + \ldots + \nu_n \leqslant m$$

and the coefficients P are rational integral functions of $x_1, x_2, \ldots, x_n$.

In this case the principal equation of the characteristics can be shown to be

$$\left.\begin{aligned} \sum_{(\mu)} P_{\mu_1, \ldots, \mu_n} \left(\frac{\partial \psi}{\partial x_1}\right)^{\mu_1} \left(\frac{\partial \psi}{\partial x_2}\right)^{\mu_2} \cdots \left(\frac{\partial \psi}{\partial x_n}\right)^{\mu_n} &= 0 \\ \mu_1 + \mu_2 + \ldots + \mu_n &= 0 \end{aligned}\right\}, \tag{13}\dagger$$

with which is to be combined the relation

$$\psi(x_1, x_2, \ldots, x_n) = 0.$$

PROPAGATION OF EFFECTS REPRESENTED BY A SINGLE SCALAR POTENTIAL FUNCTION.

5. *The Wave Front as a Characteristic.*

Consider now the propagation of a disturbance in some medium in which the effects can be represented by a single equation such as

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{14}$$

---

* Beudon, *loc. cit.*

† Cf. E. von Weber, *Encyklopädie der math. Wissenschaften*, II., A. 5, § 55.

In the usual abstract wave theory, such as the Kirchhoff and Poisson solutions, the function ϕ and its first differential coefficients are assumed to be continuous functions in the region under consideration. For instance, in Hugoniot's description of the mode of propagation of a disturbance which is initially local, the first assumption is that the disturbance is propagated at some finite rate; then at any given time the motion of the medium is given by a certain velocity potential ϕ at all points interior to some closed surface S, and by a velocity potential ϕ' at points exterior to S. And on the surface we have

$$\phi = \phi', \quad \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\phi = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}\right)\phi'. \tag{15}$$

The position of the surface

$$S(x, y, z, t) = 0, \tag{16}$$

when t has a given value, is the wave front at the corresponding time.

Now in the given differential equation consider ϕ, x, y, z, t as the coordinates of a point in space of five dimensions. The conditions in (15) show that two given integral surfaces touch along a certain multiplicity, which must consequently be a characteristic strip. The projection of the point-support of this strip upon the plane $\phi = 0$ is the multiplicity given by (16). Hence, from the general theory, S satisfies the principal equation of the characteristics, which for the equation (14) is

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 - \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 = 0. \tag{17}$$

The sections of S by planes $t = \text{constant}$ give the positions of the wave front in ordinary space at the corresponding times; hence equation (17) implies that the wave front thus defined advances with constant normal velocity c at each point. This is a property common to all such wave propagations, whatever be the dimensions of the space considered, and it also belongs to much more general equations.

6. *Generalized Wave Equation.*

Let S be a surface movable with the time and separating space into two regions 1 and 2. Let $\phi(x_1, \ldots, x_n, t)$ be an analytic function defined in region 1 and $\phi_2(x_1, \ldots, x_n, t)$ an analytic function defined in region 1 and region 2. Duhem* has defined S as a wave of order m for the

* Duhem, *Comptes Rendus*, t. CXXXI., p. 1171.

function ϕ, propagating itself in the function ϕ_2, if, on the surface S, ϕ and ϕ_2 are equal to each other, together with all their differential coefficients up to order $m-1$.

Consider the equation

$$\nabla^{2m}\phi - \frac{1}{c^{2m}} \frac{\partial^{2m}\phi}{\partial t^{2m}} = 0, \tag{18}$$

where ∇^{2m} represents the operation $\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ repeated m times. This may be considered as a generalized wave equation of order $2m$ in space of n dimensions. Then a wave front S as defined satisfies the principal equation of the characteristics of equation (18).

Using the general form given in (13), this gives

$$\left[\left(\frac{\partial S}{\partial x_1}\right)^2 + \left(\frac{\partial S}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial S}{\partial x_n}\right)^2\right]^m - \frac{1}{c^{2m}} \left(\frac{\partial S}{\partial t}\right)^{2m} = 0. \tag{19}$$

Hence the wave front moves with constant normal velocity c.

7. *The Principle of Huygens.*

Returning now to the ordinary equations (14) and (17), the surfaces S can be considered as the envelopes of certain cones. Consider for a moment x, y, z, t, ϕ as coordinates of a point; then at any point $(x_0, y_0, z_0, t_0, \phi_0)$ there is a cone T which is enveloped by the planes which are tangent to the integral surfaces of (14) passing through this point, and there is also a corresponding cone of normals N. For a linear equation such as (14) these cones are independent of the value of ϕ and are given as cones in the four-dimensional space represented by x, y, z, t.

Thus for the cone N we obtain in this case

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 - \frac{1}{c^2}(t-t_0)^2 = 0,$$

and consequently for the cone T

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = c^2(t-t_0)^2. \tag{20}$$

The surfaces S are the envelopes of the latter cones when the coordinates of the vertex are subject to given relations. This obviously contains the method of describing the motion of the wave front associated with the principle of Huygens.

8. *The Paths of the Rays as Characteristics.*

Now consider (17) as a partial differential equation of the first order for S; the characteristics of (17), which have been called bicharacteristics of the first equation (14), are obtained according to equation (8) and satisfy

$$\frac{dx}{\frac{\partial S}{\partial x}} = \frac{dy}{\frac{\partial S}{\partial y}} = \frac{dz}{\frac{\partial S}{\partial z}} = \frac{dt}{-\frac{1}{c^2}\frac{\partial S}{\partial t}} = \dots \tag{21}$$

These bicharacteristics can be identified with the paths of the rays. With the usual notation, equation (17) may be written

$$c^2(p_1^2+p_2^2+p_3^2)-p_4^2 = 0.$$

Then the complete integral is given by

$$S = lx+my+nz+Vt+d,$$

where $$V^2 = c^2(l^2+m^2+n^2).$$

The general integral is a surface enveloped by these planes, and is touched by each plane along a characteristic; moreover, there is no singular integral. Then, if we consider sections of these surfaces by planes $t =$ constant, so as to translate into ordinary space with t as the time, we obtain sets of parallel planes with each set touching the wave fronts at corresponding times along a characteristic. But this is the usual definition of the paths of the rays; hence the required identification follows.

9. *Wave Fronts with Essential Discontinuities.*

The preceding work rests upon the usual assumption that ϕ and its first differential coefficients are continuous at wave-fronts, and the comparison of wave fronts with characteristics holds for a general partial differential equation of the second order for ϕ. But the equations which occur in physical problems are generally linear in the differential coefficients of the second order, and for these we may consider wave fronts at which the first differential coefficients of ϕ are discontinuous, while ϕ itself remains continuous. Using ϕ, x, y, z, t as space coordinates, we are now given two integrals which intersect on the cylindrical multiplicity

$$S(x, y, z, t) = 0.$$

But for a linear partial differential equation for ϕ there is a single infinity of characteristic strips passing through each characteristic curve; or, in

other words, a characteristic of zero order is contained in an infinite number of characteristics of the first order. Thus two integral surfaces which intersect do so in a characteristic of zero order, the two strips lying on the integral surfaces along their intersection being characteristics of the first order.

Owing to the linearity of the equation, the projection of the intersection upon $\phi = 0$, namely S, still satisfies the same principal equation of the characteristics; and for the ordinary equation (14) this implies that the wave front advances with constant normal velocity.

If we proceed further and consider cases in which ϕ itself is discontinuous on the wave front, we see from the geometrical analogy that we are given two non-intersecting integral surfaces, and consequently there is nothing to determine S. We must have other data supplied by the actual physical problem. For instance, suppose that ϕ is constant over a wave front, although discontinuous there; then, since $\phi =$ constant is a solution of the differential equation, it follows that each of the given integrals cuts the cylinder S in a characteristic of zero order. Hence we have S satisfying the same principal equation as before.

Further, consider the variation of ϕ' and ϕ'' along their intersection; then we have

$$\left(\frac{\partial\phi'}{\partial x}-\frac{\partial\phi''}{\partial x}\right)dx+\left(\frac{\partial\phi'}{\partial y}-\frac{\partial\phi''}{dy}\right)dy+\left(\frac{\partial\phi'}{\partial z}-\frac{\partial\phi''}{\partial z}\right)dz+\left(\frac{\partial\phi'}{\partial t}-\frac{\partial\phi''}{\partial t}\right)dt=0,$$

where dx, dy, dz, dt are connected by the relation

$$\frac{\partial S}{\partial x}dx+\frac{\partial S}{\partial y}dy+\frac{\partial S}{\partial z}dz+\frac{\partial S}{\partial t}dt=0.$$

Hence, combining these with the principal equation for S, we obtain

$$\frac{\partial\phi'}{\partial\nu}+\frac{1}{c}\frac{\partial\phi'}{\partial t}=\frac{\partial\phi''}{\partial\nu}+\frac{1}{c}\frac{\partial\phi''}{\partial t}, \tag{22}$$ *

where ν is the normal to the wave front drawn in the direction of its motion.

The Characteristics of a System of Partial Differential Equations.— We have considered so far motions involving a single scalar potential function; but, as we wish to examine the propagation of electric effects, it is necessary to extend the theory of characteristics to a system of partial

* Cf. Love, *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 93.

differential equations. We shall indicate the general theory for a system of n equations in two independent variables x, y and n unknown functions $z_1, z_2, \ldots, z_n$.*

10. *Equations of the First Order.*

For a system of equations of the first order, the following is the required extension.

Suppose the equations are given in the form

$$\left.\begin{aligned} F_i(x, y, z_1, z_2, \ldots, z_n, p_1, \ldots, p_n, q_1, \ldots, q_n) = 0 \\ i = 1, 2, \ldots, n; \quad \frac{\partial z_i}{\partial x} = p_i; \quad \frac{\partial z_i}{\partial y} = q_i \end{aligned}\right\}. \tag{23}$$

Then the principal equation of the characteristics corresponding to that given in (7) is now a determinantal equation

$$\Delta = \begin{vmatrix} \frac{\partial F_1}{\partial p_1} dy - \frac{\partial F_1}{\partial q_1} dx, & \frac{\partial F_1}{\partial p_2} dy - \frac{\partial F_1}{\partial q_2} dx, & \ldots, & \frac{\partial F_1}{\partial p_n} dy - \frac{\partial F_1}{\partial q_n} dx \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial F_n}{\partial p_1} dy - \frac{\partial F_n}{\partial q_1} dx, & \frac{\partial F_n}{\partial p_2} dy - \frac{\partial F_n}{\partial q_2} dx, & \ldots, & \frac{\partial F_n}{\partial p_n} dy - \frac{\partial F_n}{\partial q_n} dx \end{vmatrix} = 0, \tag{24}$$

in which z_i, p_i, q_i have been replaced by their values as functions of x and y. The same simplification occurs as before when the equations F_i are linear in the quantities p_i and q_i, so that z_i, p_i, and q_i do not occur in Δ.

11. *Equations of the Second Order.*

For a similar set of equations of the second order, the corresponding principal equation of the characteristics is given by

$$\Delta = 0, \tag{25}$$

where Δ is the determinant

$$\begin{vmatrix} \frac{\partial F_1}{\partial r_1} dy^2 - \frac{\partial F_1}{\partial s_1} dx\,dy + \frac{\partial F_1}{\partial t_1} dx^2, & \ldots, & \frac{\partial F_1}{\partial r_n} dy^2 - \frac{\partial F_1}{\partial s_n} dx\,dy + \frac{\partial F_1}{\partial t_n} dx^2 \\ \ldots & \ldots & \ldots \\ \frac{\partial F_n}{\partial r_1} dy^2 - \frac{\partial F_n}{\partial s_1} dx\,dy + \frac{\partial F_n}{\partial t_1} dx^2, & \ldots, & \frac{\partial F_n}{\partial r_n} dy^2 - \frac{\partial F_n}{\partial s_n} dx\,dy + \frac{\partial F_n}{\partial t_n} dx^2 \end{vmatrix}.$$

* Cf. Goursat, *Equations aux dérivées partielles du second ordre*, t. II., p. 316.

In this determinant z_i, r_i, s_i, t_i have been replaced by their values in terms of x and y. When the equations F_i are linear in the differential coefficients of the second order, so that z_i, r_i, s_i, and t_i do not occur in Δ, then through any characteristic of zero order satisfying equation (25) there passes an infinite number of characteristics of the first order of the system of equations.

PROPAGATION OF ELECTRIC EFFECTS.

12. *Equations for Free Æther.*

If (X, Y, Z) be the electric force measured in electrostatic units, and (α, β, γ) the magnetic force in electromagnetic units, the equations are

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}(X, Y, Z) &= \operatorname{curl}(\alpha, \beta, \gamma); \quad \operatorname{div}(X, Y, Z) = 0 \\ -\frac{1}{c}\frac{\partial}{\partial t}(\alpha, \beta, \gamma) &= \operatorname{curl}(X, Y, Z); \quad \operatorname{div}(\alpha, \beta, \gamma) = 0 \end{aligned}\right\}. \tag{26}$$

Further, a circuital vector (ξ, η, ζ) may be introduced, such that

$$\left.\begin{aligned} (\alpha, \beta, \gamma) &= \frac{\partial}{\partial t}(\xi, \eta, \zeta) \\ (X, Y, Z) &= c \operatorname{curl}(\xi, \eta, \zeta) \end{aligned}\right\}. \tag{27}$$

Then each of the components ξ, η, ζ satisfies independently an equation of the form

$$\nabla^2\phi = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

Consequently the relation between wave fronts and characteristics is the same as in the case of a single equation. Thus, if

$$S(x, y, z, t) = 0$$

is the multiplicity generated by the motion of a wave front, then we have the same principal equation

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 = \frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2. \tag{28}$$

Moreover, since the equations for ξ, η, ζ are linear in the differential coefficients of the second order, it follows that the first differential coefficients of ξ, η, and ζ with respect to x, y, z, and t need not be continuous; but, if ξ, η, and ζ only are continuous on S, then the principal equation (28) is still satisfied. From (27) we see that this allows possible discontinuity

in the electric and magnetic forces at a wave front, and that the wave front has a constant normal velocity.

Assuming then (ξ, η, ζ) to be continuous on S, we can obtain relations between the values of the electric and magnetic forces on different sides of the wave front. For, if (ξ', η', ζ') and (ξ'', η'', ζ'') be the two solutions, we have on S

$$\frac{\partial \xi'}{\partial x} dx + \frac{\partial \xi'}{\partial y} dy + \frac{\partial \xi'}{\partial z} dz + \frac{\partial \xi'}{\partial t} dt = \frac{\partial \xi''}{\partial x} dx + \frac{\partial \xi''}{\partial y} dy + \frac{\partial \xi''}{\partial z} dz + \frac{\partial \xi''}{\partial t} dt,$$

and similar equations in η and ζ.

Combining these with

$$\frac{\partial S}{\partial x} dx + \frac{\partial S}{\partial y} dy + \frac{\partial S}{\partial z} dz + \frac{\partial S}{\partial t} dt = 0$$

and the circuital relation, it is easily seen that they involve the continuity at the wave front of six quantities of which the following two are types, viz.,

$$\left.\begin{aligned} X - \beta \cos(z, N) + \gamma \cos(y, N) \\ \alpha + Y \cos(z, N) - Z \cos(y, N) \end{aligned}\right\}, \tag{29}$$

where N is the normal to the wave front drawn in the direction of motion. These are the boundary conditions given by Prof. Love,* with the interpretation for a disturbance advancing into a region previously at rest, viz., that the electric and magnetic forces are at right angles to each other and are in the wave front.

It is natural to assume the vector (ξ, η, ζ) to be continuous; and, in fact, it appears that in particular cases where this assumption is not made, but the boundary conditions (29) are used instead, the effect is actually to make (ξ, η, ζ) continuous over the wave fronts.

13. *Doubly Refracting Medium.*

Suppose the equations given in the usual form

$$\left.\begin{aligned} \frac{1}{C} \frac{\partial}{\partial t}(\epsilon_1 X, \epsilon_2 Y, \epsilon_3 Z) = \operatorname{curl}(\alpha, \beta, \gamma) \\ -\frac{1}{C} \frac{\partial}{\partial t}(\alpha, \beta, \gamma) = \operatorname{curl}(X, Y, Z) \end{aligned}\right\}. \tag{30}$$

* Love, *loc. cit.*

Then, introducing a circuital vector (ξ, η, ζ) given by

$$\left.\begin{aligned}(\alpha, \beta, \gamma) &= \frac{\partial}{\partial t}(\xi, \eta, \zeta)\\ (\epsilon_1 X, \epsilon_2 Y, \epsilon_3 Z) &= C\,\mathrm{curl}(\xi, \eta, \zeta)\end{aligned}\right\}, \tag{31}$$

the boundary conditions at a wave front could be examined as in the previous section and with similar results. To obtain the properties of the propagation of a wave front of any form it is sufficient to write down the principal equation of the characteristics of the system of equations for ξ, η, ζ; these equations are three of the type

$$c^2\frac{\partial^2\xi}{\partial y^2}+b^2\frac{\partial^2\xi}{\partial z^2}-\frac{\partial^2\xi}{\partial t^2}-c^2\frac{\partial^2\eta}{\partial x\,\partial y}-b^2\frac{\partial^2\zeta}{\partial x\,\partial z}=0, \tag{32}$$

where (a, b, c) are the three principal velocities given by

$$(a, b, c) = (\epsilon_1, \epsilon_2, \epsilon_3)^{-1}C^2.$$

Applying the general theory of § 11 to the set of equations (32), we obtain the principal equation in the form

$$\Delta = \begin{vmatrix} \left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2-\frac{1}{a^2}\left(\frac{\partial S}{\partial t}\right)^2, & -\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}, & -\frac{\partial S}{\partial z}\frac{\partial S}{\partial x} \\ -\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}, & \left(\frac{\partial S}{\partial z}\right)^2+\left(\frac{\partial S}{\partial x}\right)^2-\frac{1}{b^2}\left(\frac{\partial S}{\partial t}\right)^2, & -\frac{\partial S}{\partial y}\frac{\partial S}{\partial z} \\ -\frac{\partial S}{\partial z}\frac{\partial S}{\partial x}, & -\frac{\partial S}{\partial y}\frac{\partial S}{\partial z}, & \left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2 \end{vmatrix}$$

$$= 0. \tag{33}$$

If (l, m, n) be the direction cosines of the normal ν at any point of the wave front, and if we put $\partial\nu/\partial t$ equal to V, we may in (33) replace $\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}, \frac{\partial S}{\partial t}$ by l, m, n, V respectively; then the equation reduces to

$$\frac{l^2}{a^2-V^2}+\frac{m^2}{b^2-V^2}+\frac{n^2}{c^2-V^2}=0. \tag{34}$$

14. *The Paths of the Rays.*

As in the case of a single partial differential equation, the paths of the rays are given as the characteristics of the differential equation of the first order for the wave fronts S; the differential equations defining these characteristics in the case of equation (33) are

$$\frac{dx}{\frac{\partial\Delta}{\partial p_1}}=\frac{dy}{\frac{\partial\Delta}{\partial p_2}}=\frac{dz}{\frac{\partial\Delta}{\partial p_3}}=\frac{dt}{\frac{\partial\Delta}{\partial p_4}}=\ldots, \tag{35}$$

where $(p_1, p_2, p_3, p_4) = \left(\frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, \frac{\partial S}{\partial z}, \frac{\partial S}{\partial t}\right) = (l, m, n, V)\frac{\partial S}{\partial \nu}$.

Differentiating the determinant Δ, we find

$$\frac{\partial \Delta}{\partial p_1} : \frac{\partial \Delta}{\partial p_2} : \ldots = 2p_1\{b^2c^2(p_1^2+p_2^2+p_3^2)+b^2c^2p_1^2+c^2a^2p_2^2+a^2b^2p_3^2\} \\ -2p_1p_4^2(b^2+c^2) : \text{similar expressions.}$$

Hence, if (L, M, N) be the direction cosines of the ray, we have the relation between the ray and the wave normal given by

$$\frac{L}{l\{b^2c^2-V^2(b^2+c^2)+\alpha^2\}} = \frac{M}{m\{c^2a^2-V^2(c^2+a^2)+\alpha^2\}} \\ = \frac{N}{n\{a^2b^2-V^2(a^2+b^2)+\alpha^2\}},$$

where $$\alpha^2 = b^2c^2l^2+c^2a^2m^2+a^2b^2n^2.$$

It can be verified that this is the same as

$$L : M : N = l\left(V^2+\frac{G^4}{V^2-a^2}\right) : m\left(V^2+\frac{G^4}{V^2-b^2}\right) : n\left(V^2+\frac{G^4}{V^2-c^2}\right),$$

where $$G^4\left\{\frac{l^2}{(a^2-V^2)^2}+\frac{m^2}{(b^2-V^2)^2}+\frac{n^2}{(c^2-V^2)^2}\right\} = 1.$$

This is the relation given by Drude,* obtained for plane waves only by defining the ray as the path of the energy.

15. *The Huygens Construction.*

The surfaces S may be considered as the envelopes of cones T passing through points (x_0, y_0, z_0, t_0), when the coordinates of the vertices are subject to given conditions.

The cone T is given as the envelope of the plane

$$(x-x_0)\frac{\partial S}{\partial x_0}+(y-y_0)\frac{\partial S}{\partial y_0}+(z-z_0)\frac{\partial S}{\partial z_0}+(t-t_0)\frac{\partial S}{\partial t_0} = 0,$$

where $\frac{\partial S}{\partial x_0}, \frac{\partial S}{\partial y_0}, \frac{\partial S}{\partial z_0}, \frac{\partial S}{\partial t_0}$ are subject to the relation (33). On working this out, we find that the cone T is given by

$$\frac{a^2(x-x_0)^2}{r^2-a^2(t-t_0)^2}+\frac{b^2(y-y_0)^2}{r^2-b^2(t-t_0)^2}+\frac{c^2(z-z_0)^2}{r^2-c^2(t-t_0)^2} = 0,$$

* Drude, *Lehrbuch der Optik*, p. 302.

where $$r^2 = (x-x_0)^2+(y-y_0)^2+(z-z_0)^2.$$

This evidently contains the Huygens construction for a doubly refracting medium. The Fresnel wave surface is given by taking (x_0, y_0, z_0, t_0) as the origin and taking a section of the cone at time $t = 1$.

16. *Conical Refraction.*

Consider the equation (33), viz.,

$$\Delta = 0,$$

as a partial differential equation for S in four independent variables x, y, z, and t. Then the complete integral is

$$S = lx+my+nz+Vt+d,$$

where
$$\begin{vmatrix} m^2+n^2-V^2/a^2, & -lm, & -nl \\ -lm, & n^2+l^2-V^2/b^2, & -mn \\ -nl, & -mn, & l^2+m^2-V^2/c^2 \end{vmatrix} = 0.$$

The general integral is a surface enveloped by these planes, and is touched by a complete integral along a characteristic. Also in this case there may be considered to be an infinite number of singular integrals; for a singular integral may be defined as an integral surface at each point of which the normal is a double generator of the cone of normals N at that point. Now the cone N at any point (x_0, y_0, z_0, t_0) is given by

$$\begin{vmatrix} (y-y_0)^2+(z-z_0)^2-(t-t_0)^2/a^2, & -(x-x_0)(y-y_0), & -(z-z_0)(x-x_0) \\ -(x-x_0)(y-y_0), & (z-z_0)^2+(x-x_0)^2-(t-t_0)^2/b^2, & -(y-y_0)(z-z_0) \\ -(z-z_0)(x-x_0), & -(y-y_0)(z-z_0), & (x-x_0)^2+(y-y_0)^2-(t-t_0)^2/c^2 \end{vmatrix} = 0.$$

And this cone has for every point two double generators having constant direction cosines given by

$$l : m : n : p = \pm\sqrt{\frac{a^2-b^2}{a^2-c^2}} : 0 : \pm\sqrt{\frac{b^2-c^2}{a^2-c^2}} : b.$$

Hence the singular integrals are given by

$$S = \pm x\sqrt{\frac{a^2-b^2}{a^2-c^2}} \pm z\sqrt{\frac{b^2-c^2}{a^2-c^2}}+bt+d.$$

For this value of S the determinant Δ in (33) not only vanishes, but also all its minors; thus the quantities $\partial\Delta/\partial p_1$, ... vanish, and the characteristics given by (35) are indeterminate.

Consider now $S=0$ as a wave front moving with the time. Then, using the ordinary optical terms, the preceding results evidently express the following fact:—If the wave front S is a plane having its normal in a particular direction, viz., that of an optic axis of the Fresnel wave surface at any point, then there is an infinite number of rays passing through any point on the wave front, and they form a cone of rays.

TRANSFORMATION OF WAVE EQUATIONS.

17. *Invariant Property of Wave Fronts.*

It is known that the characteristics of a partial differential equation are invariant for any contact transformation; for the present purpose we require to show that the determinant Δ which has been used for a set of equations is invariant for any change of the independent variables. It is sufficient to use the form of Δ given in (25) for two independent variables.

Let x', y' be new independent variables defined by

$$x=\phi(x',y'); \qquad y=\psi(x',y').$$

Then the first derivatives of z_κ with respect to the new variables are given by

$$p'_\kappa = p_\kappa \frac{\partial\phi}{\partial x'} + q_\kappa \frac{\partial\psi}{\partial x'},$$

$$q'_\kappa = p_\kappa \frac{\partial\phi}{\partial y'} + q_\kappa \frac{\partial\psi}{\partial y'}.$$

And the second differential coefficients are

$$r'_\kappa = r_\kappa\left(\frac{\partial\phi}{\partial x'}\right)^2 + 2s_\kappa \frac{\partial\phi}{\partial x'}\frac{\partial\psi}{\partial x'} + t_\kappa\left(\frac{\partial\psi}{\partial x'}\right)^2 + p_\kappa\frac{\partial^2\phi}{\partial x'^2} + q_\kappa\frac{\partial^2\psi}{\partial x'^2},$$

with similar expressions for s'_κ and t'_κ.

Now suppose that by the same transformation the differential equation $F_i=0$ is changed into $G_i=0$, a new partial differential equation of the

second order. Then it is easy to show that

$$\frac{\partial F_i}{\partial r_\kappa} dy^2 - \frac{\partial F_i}{\partial s_\kappa} dx\,dy + \frac{\partial F_i}{\partial t_\kappa} dx^2$$

$$= \left\{ \frac{\partial(\phi, \psi)}{\partial(x', y')} \right\}^2 \left\{ \frac{\partial G_i}{\partial r'_\kappa} dy'^2 - \frac{\partial G_i}{\partial s'_\kappa} dx'\,dy' + \frac{\partial G_i}{\partial t'_\kappa} dx'^2 \right\}.$$

Hence, if Δ' be the corresponding determinant for the new set of n equations given by $G_i = 0$, we have

$$\Delta = \left\{ \frac{\partial(\phi, \psi)}{\partial(x', y')} \right\}^{2n} \Delta'.$$

To apply this to the problems we are considering, suppose we have a set of three partial differential equations of the second order in three unknown functions X, Y, Z, and four independent variables x, y, z, t. Also we are given a multiplicity $S(x, y, z, t) = 0$ which satisfies the principal equation of the characteristics of this set, namely, $\Delta = 0$.

Then, if we change to new variables given by

$$x = \lambda_1(x', y', z', t'), \quad y = \lambda_2(x', y', z', t'), \quad z = \lambda_3(x', y', z', t'),$$

$$t = \lambda_4(x', y', z', t'),$$

and such that

$$\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{\partial(x', y', z', t')} \neq 0,$$

the multiplicity $S(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = 0$ will satisfy the principal equation $\Delta' = 0$ of the characteristics of the new set of equations.

Hence, regarding x, y, z as ordinary space coordinates and t as the time, we have the following:—Given a set of wave fronts $S(x, y, z, t) = 0$ corresponding to certain differential equations, then, if the latter equations are transformed in any way by changing the independent variables, the surfaces S when changed by the same transformation are again a set of wave fronts for the new differential equations.

Further, since the paths of the rays are defined as the characteristics of the partial differential equation of the first order for S, it is evident that a similar correspondence will hold for the rays as for the wave fronts. Illustrations of this correspondence are given in the two following sections.

18. *Equations for a Moving Medium.*

Consider the equations of electrical propagation in a moving transparent body. Let (p, q, r) be the velocity and n the refractive index of the isotropic medium; suppose the equations referred to axes at rest to be

given in the form

$$\nabla^2 X - 2\frac{n^2-1}{c^2}\frac{\partial}{\partial t}\left(p\frac{\partial X}{\partial x}+q\frac{\partial X}{\partial y}+r\frac{\partial X}{\partial z}\right)-\frac{n^2}{c^2}\frac{\partial^2 X}{\partial t^2}=0,$$

with similar equations in Y and Z.

Then a wave front $S(x, y, z, t) = 0$ satisfies the equation

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2-2\frac{n^2-1}{c^2}\frac{\partial S}{\partial t}\left(p\frac{\partial S}{\partial x}+q\frac{\partial S}{\partial y}+r\frac{\partial S}{\partial z}\right)-\frac{n^2}{c^2}\left(\frac{\partial S}{\partial t}\right)^2=0.$$

If ω be the normal velocity of S at any point and v the normal component of (p, q, r), this gives

$$n^2\omega^2-2(n^2-1)v\omega=c^2,$$

or, approximately,
$$\omega=\frac{c}{n}+\frac{n^2-1}{n^2}v,$$

which is the usual result obtained from considering plane waves.

Further, the paths of the rays referred to fixed axes are given by

$$\frac{dx}{\dfrac{\partial S}{\partial x}-\dfrac{n^2-1}{c^2}p\dfrac{\partial S}{\partial t}}=\frac{dy}{\dfrac{\partial S}{\partial y}-\dfrac{n^2-1}{c^2}q\dfrac{\partial S}{\partial t}}=\frac{dz}{\dfrac{\partial S}{\partial z}-\dfrac{n^2-1}{c^2}r\dfrac{\partial S}{\partial t}};$$

or, if (L, M, N) be the direction of the ray, and (λ, μ, ν) be the wave normal,

$$L:M:N=\lambda-\frac{n^2-1}{c^2}p\omega:\mu-\frac{n^2-1}{c^2}q\omega:\nu-\frac{n^2-1}{c^2}r\omega.$$

Now consider the linear transformation that is used in the theory of moving media, namely, one of the form

$$x'=a(x-pt),\quad y'=b(y-qt),\quad z'=c(z-rt),\quad t'=t-d(px+qy+rz). \tag{36}$$

It is known that by such a transformation the equations for the moving body can be reduced to equations of the same form as those for the body at rest, up to the second order of small quantities involved. Thus, if $S(x, y, z, t) = 0$ represents wave fronts for the first set of equations, and if they are transformed by (36), they become wave fronts suitable for the same medium at rest. Hence, if the wave fronts S are actually referred to axes moving in the manner indicated in (36) and contracted accordingly, they are then the same as wave fronts for the medium at rest. According to the electrical theory the change of space dimensions involved in (36) is due to a physical change in the moving body; hence the result may be

expressed by saying that relative to axes bound up with the body the wave fronts are the same as if the body were at rest.

Similar considerations apply to the paths of the rays; hence the relative ray paths are the same as if the body were at rest.

19. *The General Electromagnetic Surface.*

Suppose we are given the equations for a medium with electric æolotropy and magnetic isotropy, and we denote by (A) these differential equations when referred to the axes of electrical symmetry. Now it is known that a transformation of the variables x, y, z equivalent to a pure strain will change equations (A) into a set of equations which are those for a medium with both electric and magnetic æolotropy, the two having the same principal axes; also by a rotation of the axes the equations (A) can be transformed into those for electric æolotropy and magnetic isotropy, not referred to the axes of electrical symmetry. Thus, combining the two transformations, we see that a transformation equivalent to a homogeneous strain will alter the equations (A) into a set of equations (B) which are those for a medium with electric and magnetic æolotropy, not having the same principal axes.

Hence, if any set of wave fronts $S(x, y, z, t) = 0$ for the medium (A) be transformed by a homogeneous strain, it becomes a set of wave fronts for the medium (B). A particular case of this is the known theorem that the Fresnel wave surface, if homogeneously strained, becomes the general electromagnetic wave surface for a medium with different principal axes for the electric and magnetic æolotropy.

ON INNER LIMITING SETS OF POINTS IN A LINEAR INTERVAL

By E. W. HOBSON.

[Received and Read March 10th, 1904.]

THE theory of inner limiting sets of points in a linear interval was suggested by a remark made by Borel,* that, if each rational point p/q, of the interval (0, 1), be enclosed in an interval $\left(\frac{p}{q}-\lambda\frac{1}{q^3}, \frac{p}{q}+\lambda\frac{1}{q^3}\right)$, where λ is a positive number, the same for all the rational points, then, as λ is indefinitely diminished, there are, besides the rational points themselves, certain transcendental points (those of Liouville) which remain in the interior of the set of intervals, however small λ may become.

It appears, as in this example, that, if each point x of an open set is enclosed in a series of intervals $\delta_1(x)$, $\delta_2(x)$, ..., $\delta_n(x)$, ... chosen according to some prescribed law, such that, for each point x, $\delta_n(x)$ has a zero limit when n is increased indefinitely, the magnitude and position of $\delta_n(x)$ being assigned for each n and each x, then, in general, some or all of those limiting points of the given set which do not belong to that set remain in the interior of the intervals $\{\delta_n(x)\}$, however great n may be. This consideration led Dr. W. H. Young† to examine the properties of those sets of points for which it is possible, by a proper choice of the system of intervals, to exclude all the limiting points of the set which do not belong to it from the intervals, each such point ceasing to be interior to the set of intervals $\{\delta_n(x)\}$ for some definite value of n, so that the given set of points itself contains all points which are interior to the sets of intervals $\{\delta_n(x)\}$ for every value of n; such a set Dr. Young has named an "inner limiting set" (*innere Grenzmenge*). He has proved that every such set has either the cardinal number a of the rational numbers, or else the cardinal number c of the continuum ; this result will be assumed in the present paper. Dr. Young has also proved that the cardinal number of an inner limiting set is a when the set contains no component that is dense in itself, and is c when it does contain such a component.

In the present communication it is shown that the *necessary* and

---

* *Leçons sur la théorie des fonctions*, p. 44.

† "Zur Lehre der nicht abgeschlossenen Punktmengen," *Leipziger Berichte*, August, 1903.

sufficient condition that an enumerable set may be an inner limiting set is that it contains no component dense in itself. It is further shown that the most general inner limiting set which is unenumerable consists of a set dense in itself and each point of which is of degree c in the set, together with an enumerable set each point of which is of degree a or zero in the set, and which contains no component that is dense in itself.

The theory of inner limiting sets is closely connected with the theory of sets of the first and of the second category, which is of importance in the theory of functions of a real variable. The problem of determining the necessary and sufficient conditions that a given set of points is such that a pointwise discontinuous function can exist such that its points of discontinuity may be the points of the given set, is identical with the problem of determining the conditions under which a given set of points can be exhibited as the limit of a sequence of non-dense closed sets.

The set of points complementary to a set which has this property is an inner limiting set which is everywhere dense and is of cardinal number c in every interval contained in the domain of the set. It would be desirable that the necessary and sufficient conditions that such a set may be an inner limiting set should be obtained in a form which would be applicable as a criterion to the case of an everywhere dense set defined in any manner. This is equivalent to the determination of the necessary and sufficient conditions, in a direct form, that any given set may be of the second category; I have, however, not succeeded in doing this.

1. Let P denote a set of points contained in the linear interval (a, b), and let a point x_1 of the set P be enclosed in a sequence of intervals $\delta_1(x_1), \delta_2(x_1), \ldots, \delta_n(x_1), \ldots$, such that each interval contains the next one, and each contains x_1 in its interior; and let the sequence be such that $\delta_n(x_1)$ converges to the limit zero as n is indefinitely increased. Suppose such a sequence of intervals to be prescribed for each point x of the set P, the length and position of $\delta_n(x)$ being assigned for each value of x and each value of n, subject to the conditions already stated: such a sequence of sets of intervals $\{\delta_1(x)\}, \{\delta_2(x)\}, \ldots, \{\delta_n(x)\}, \ldots$, we shall speak of as a "convergent sequence of sets" of intervals containing the set P. It may be possible to prescribe the law of formation of the intervals so that, if p be any fixed point whatever not belonging to the set P, there exists a number n_1 such that p is not interior to any interval of the set $\{\delta_{n_1}(x)\}$, and is thus exterior to or at an end-point of all the intervals of the set: if the set P is such that it is possible to determine a convergent sequence

of sets of intervals enclosing P, which has the property specified, then P is said to be an "inner limiting set."

If the set P is not an inner limiting set, then, however the intervals of the sequence are constructed, there exist points belonging to the derivative of P, and not to P itself, each of which is interior to some interval of the set $\{\delta_n(x)\}$, however great n may be. If p be such a point of P', then, for any fixed x belonging to P, a value of n exists such that p is not interior to $\delta_n(x)$, but such values of n, when taken for every value of x in P, have no upper limit.

When a point p is not in the interior of the set $\{\delta_n(x)\}$ for the value n_1 of n, but is interior to $\{\delta_{n_1-1}(x)\}$, we shall say that p is *shed* from the sequence of sets of intervals at the index n_1. An inner limiting set is then such that each limiting point of the set which does not belong to it is shed at a definite index from the sequence of sets of intervals, provided the intervals of this sequence are properly chosen.

2. If $P_1, P_2, P_3, \ldots, P_n, \ldots$ denote a sequence of non-dense closed sets of points, the set $M(P_1, P_2, \ldots, P_n, \ldots) = Q$, which contains every point that belongs to any of the given closed sets, and no points which do not belong to some one or more of the given sets, has been termed by Baire* a "set of the first category." If we denote the sets P_1, $M(P_1, P_2)$, $M(P_1, P_2, P_3), \ldots, M(P_1, P_2, \ldots, P_n), \ldots$ by $Q_1, Q_2, Q_3, \ldots, Q_n, \ldots$, then Q is the limit of the sequence $Q_1, Q_2, \ldots, Q_n, \ldots$ of closed sets each one of which contains the preceding one; it is in this form that a set Q of the first category appears as the set of points of discontinuity of a pointwise discontinuous function. The set which is complementary to a set of the first category is said to be "of the second category"; it is known that every set of points of the second category is such that in every interval contained in the interval (a, b) for which the set is defined, there exists a part of the set which has the cardinal number of the continuum.

If P is a set of the second category, the points of P can be enclosed in a set of intervals $\{\delta_n(x)\}$ which is such that those points which are not internal to any of the intervals of $\{\delta_n(x)\}$ form a non-dense closed set Q_n; all the points which do not belong to any of the closed sets $Q_1, Q_2, \ldots, Q_n, \ldots$ are points of P. It thus appears that a set of the second category is an inner limiting set, and thus that an investigation of the nature and properties of inner limiting sets will throw light upon the nature of sets of

---

* *Annali di Mat.* [3], Vol. III., p. 67 (1899).

the second category, and will therefore have an influence upon the theory of pointwise discontinuous functions.

Every enumerable set $(p_1, p_2, \ldots, p_n, \ldots)$ of points forms a set of the first category; for it may be exhibited as the limit of a sequence (p_1), (p_1, p_2), (p_1, p_2, p_3), ..., $(p_1, p_2, \ldots, p_n)$, ... of finite and therefore closed sets; the complementary set is therefore of the second category, and consequently an inner limiting set.

If P is an enumerable set, those of its limiting points which do not belong to P form a set $P'-D(P, P')$ which is an inner limiting set; this set is not identical with the set complementary to P, unless P is everywhere dense. To prove this we observe that the set $P'-D(P, P')$ has no limiting points which belong neither to itself nor to P; hence, if a sequence of sets of intervals can be found enclosing the points of $P'-D(P, P')$ which shed each of the points of P at some definite index, the set $P'-D(P, P')$ is an inner limiting set; the set of intervals $\{\delta_1(x)\}$ enclosing the points of $P'-D(P, P')$ can be so chosen that the point p_1 is exterior to all of them; then $\{\delta_2(x)\}$ can be so chosen that p_2 is exterior to them, and so on; thus each point p_n of P is shed at a definite index, and the theorem is thus established.

The irrational points of (0, 1) form an inner limiting set, but the rational points of (0, 1) do not form an inner limiting set, since, as is well known, the irrational points do not form a set which can be exhibited as the limit of a sequence of non-dense closed sets.

It is easily seen that an inner limiting set remains one, if a finite number of points is added to or subtracted from the set. Also the sum of a finite number of inner limiting sets is itself an inner limiting set; this is not, in general, true of the sum of an indefinitely great number of limiting sets, as may be seen, for example, by considering the case in which each set consists of a single point.

3. It has been shown by Dr. W. H. Young (*loc. cit.*) that an inner limiting set which is enumerable has no component which is dense in itself: it will here be shown that in the case of an enumerable set P this condition is sufficient as well as necessary.

First, let us suppose that the derivative P' of P is also enumerable; in this case P cannot contain a component which is dense in itself; for the derivative of such a component would be perfect and would be a component of the closed set P', which is impossible when P' is enumerable. Divide P into two parts P_1 and P_2; of these let P_1 consist of those points which are not limiting points of the set $P'-D(P, P')$, which

consists of those points of P' which do not belong to P; the other part P_2 consists of those points which are limiting points of $P'-D(P,P')$. The set $P'-D(P,P')$ is such that those of its limiting points which do not belong to the set itself belong to P_2; hence, $P'-D(P,P')$ being enumerable, the set P_2 is, in accordance with what has been proved in § 2, an inner limiting set. Let the points of P_1 be $x_1, x_2, x_3, \ldots, x_n, \ldots$; then, since these points are not limiting points of $P'-D(P,P')$, each of the points x_n can be enclosed in an interval $\delta_1(x_n)$ which contains no points of $P'-D(P,P')$; it follows that the points of $P'-D(P,P')$, not being contained in properly chosen intervals enclosing P_1, and being shed each at a definite index from a properly chosen sequence of sets of intervals enclosing P_2, the set P_1+P_2 or P is an inner limiting set.

A set P whose derivative P' is enumerable is such that a derivative $P^{(\alpha)}$ must vanish, where α is some number of the first or second class; the set is therefore said to be "reducible"; we thus have the following:—

THEOREM I.—*Every reducible set is an inner limiting set.*

4. Next, let us suppose that P', the derivative of the enumerable set P, has the power of the continuum. It can be shown that, if P' is continuous in any interval, P cannot be an inner limiting set; for suppose P' to be continuous in any interval (α, β); then the part of P which is in this interval is everywhere dense in (α, β) and is consequently also dense in itself. We can establish a (1, 1) correspondence between the points of P which are in (α, β) and the rational points of the interval (0, 1), the two sets having necessarily the same order-type; the interval (α, β) can be so chosen that the points α, β are points of P and will correspond to the points 0, 1; the order of the points in (0, 1) may be made the same as in (α, β), the irrational points of (0, 1) corresponding to those points of (α, β) which do not belong to P. To a convergent sequence of sets of intervals in (α, β) there will correspond a similar sequence in (0, 1); now it has been shown in § 2 that the rational points in (0, 1) do not form an inner limiting set; hence the part of P in (α, β) is not an inner limiting set, and therefore P itself is not an inner limiting set. The set P', having thus been shown to be continuous in no interval, can be resolved into two parts G_1 and L_1, of which G_1 is a non-dense perfect set, and L_1 is an enumerable set contained in the interiors of the intervals complementary to G_1.

The set P may be divided into two parts P_1 and Q_1, where P_1 consists of those points which are interior to the free intervals of G_1, and Q_1 of those points which belong to G_1; it may happen that Q_1 does not exist.

Authors of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. d.	s. d.	s. d.
4 pages	Free	3 6	5 0	6 0
8 "	"	4 6	6 6	8 0
12 "	"	7 0	10 6	13 0
16 "	"	8 0	12 0	15 0
32 "	"	14 0	21 0	26 0

It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1*s*. 6*d*. per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s*. 4*d*. per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume*."

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. Francis Hodgson, 89 Farringdon Street, E.C., at the following rates:—

Vol. I.	Nos. 1–8	... Jan. 1865	to Nov. 1866	...	10*s*.
" II.	" 9–20	... Nov. 1866	" 1869	...	16*s*.
" III.	" 21–40	... " 1869	" 1871	...	20*s*.
" IV.	" 41–66	... " 1871	" 1873	...	31*s*. 6*d*.
" V.	" 67–75	... " 1873	" 1874	...	15*s*.
" VI.	" 76–88	... " 1874	" 1875	...	21*s*.
" VII.	" 89–103	... " 1875	" 1876	...	21*s*.
" VIII.	" 104–123	... " 1876	" 1877	...	21*s*.
" IX.	" 124–140	... " 1877	" 1878	...	21*s*.
" X.	" 141–155	... " 1878	" 1879	...	18*s*.
" XI.	" 156–165	... " 1879	" 1880	...	12*s*. 6*d*.
" XII.	" 166–179	... " 1880	" 1881	...	16*s*.
" XIII.	" 180–194	... " 1881	" 1882	...	18*s*.
" XIV.	" 195–214	... " 1882	" 1883	...	20*s*.
" XV.	" 215–232	... " 1883	" 1884	...	20*s*.
" XVI.	" 233–252	... " 1884	" 1885	...	20*s*.
" XVII.	" 253–279	... " 1885	" 1886	...	25*s*.
" XVIII.	" 280–304	... " 1886	" 1887	...	25*s*.
" XIX.	" 305–342	... " 1887	" 1888	...	30*s*.
" XX.	" 343–369	... " 1888	" 1889	...	25*s*.
" XXI.	" 370–398	... " 1889	" 1890	...	25*s*.
" XXII.	" 399–429	... " 1890	" 1891	...	27*s*. 6*d*.
" XXIII.	" 430–449	... " 1891	" 1892	...	20*s*.
" XXIV.	" 450–474	... " 1892	" 1893	...	22*s*.
" XXV.	" 475–499	... " 1893	" 1894	...	22*s*.
" XXVI.	" 500–534	... " 1894	" 1895	...	30*s*.
" XXVII.	" 535–574	... " 1895	" 1896	...	35*s*.
" XXVIII.	" 575–611	... " 1896	" 1897	...	32*s*. 6*d*.
" XXIX.	" 612–657	... " 1897	" 1898		Pt. I. 20*s*. Pt. II. 20*s*.
" XXX.	" 658–678	... " 1898	to Mar. 1899	...	20*s*.
" XXXI.	" 679–709	... April 1899	" Dec. 1899	...	20*s*.
" XXXII.	" 710–737	... Jan. 1900	" June 1900	...	20*s*.
" XXXIII.	" 738–763	... Nov. 1900	" Feb. 1901	...	20*s*.
" XXXIV.	" 764–789	... Mar. 1901	" April 1902	...	20*s*.
" XXXV.	" 790–819	... May 1902	" Jan. 1903	...	20*s*.

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s*. each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25*s*. each Volume.

The price of cases for binding the Volumes of the Second Series is 1*s*. 3*d*.

Vol. 1 of Series 2 (1904) has been published.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 2.—PART 5.

CONTENTS.

Issued January 17, 1905.

LONDON:

PRINTED BY C. F. HODGSON & SON,

2 NEWTON STREET, HIGH HOLBORN, W.C.,

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. LOVE. / J. H. GRACE.

34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

Peterhouse, Cambridge.—J. H. GRACE.

	£.	s.	d.		£.	s.	d.
Annuity (less transfer to Sinking Fund)	34	3	5	Transferred to the General Account of the Society, as above	44	2	7
Dividends on Stock	7	6	3				
Income Tax on Sinking Fund, returned by judgment of the Courts	2	5	2				
Final Distribution on Reorganization of Railway Co.	0	7	9				
	£44	2	7		£44	2	7

II. De Morgan Medal Fund.

	£.	s.	d.		£.	s.	d.
Interest on Capital, from previous account ...	5	17	10	Balance at Bank	8	15	1
Interest during the year	2	17	3				
	£8	15	1		£8	15	1

STATEMENT OF CAPITAL, November, 1904.

	Sum Invested. £.	s.	d.	£.	s.	d.	Description of Investment.
1. Lieut.-Col. Campbell's Fund ...	499	7	5	385	0	0	Guaranteed 4 per Cent. London & North Western Railway Consolidated Stock.
2. Life Composition Fund, from 1902–3	1688	10	0	1758	13	1	New South Wales $3\frac{1}{2}$ per Cent. Inscribed Stock of 1918.
Amount received during 1903–4	42	0	0	10	10	0	In Bank awaiting investment.
3. Invested Surplus Fund	700	0	0	700	7	1	India 3 per Cent. Stock.
Trust Funds.							
1. Lord Rayleigh's Fund	1000	0	0	200	0	0	Guaranteed 3 per Cent Great Indian Peninsula Railway Stock.
							Terminable Annuity of £50. 8*s*. 8*d*. (Class B, deduction for Sinking Fund) of same Railway Co.
2. De Morgan Medal Fund	102	5	9	60	0	0	Great Western Railway 5 per Cent. Preference Stock.

Examined and found correct, all my requirements as Auditor having been complied with.

Dec. 12*th*, 1904. (Signed) J. G. LEATHEM

THE LONDON MATHEMATICAL SOCIETY.

FINANCIAL REPORT BY THE TREASURER FOR THE YEAR 1903–04 (NOV. 12TH, 1903, TO NOV. 10TH, 1904).

Cr. *GENERAL CASH ACCOUNT OF THE SOCIETY.* **Dr.**

		£.	s.	d.
Balance from 1901–02:—				
General Fund … …	£68 13 4			
Life Composition Fund	10 10 0			
		79	3	4
Interest on Capital—				
Lieut.-Col. Campbell's Fund	14 13 7			
Life Composition Fund	58 12 1			
Invested Surplus Fund	20 0 0			
		93	5	8
Interest on Lord Rayleigh's Fund … … …		44	2	7
13 Entrance Fees … … … … …		13	13	0
4 Life Compositions … … … … … …		42	0	0
145 Annual Subscriptions—				
1 for 1900–01 … … …	1 1 0			
27 for 1901–02 … … …	28 7 0			
31 for 1902–03 … … …	32 11 0			
77 for 1903–04 … … …	80 17 0			
9 in advance … … …	9 9 0			
		152	5	0
Sale of *Proceedings*, &c. … … … … … …		159	7	3
		£583	16	10

		£.	s.	d.
Printing *Proceedings*, New Series, &c. … …		470	7	0
Purchase of £42. 9*s*. 10*d*. New South Wales 3½ per Cent. Stock of 1918 for Life Composition Fund … … … … … … …		42	0	0
Purchase of Journals, &c., for Library … …		11	3	8
Postage and Sundries … … … … … …		4	12	8
Rent … … … … … … … … … … …		19	9	0
Attendance … … … … … … … …		8	8	0
Providing Tea at Council … … … … … …		1	11	6
Balance at Bank—				
General Fund … … …	£15 15 0			
Life Composition Fund …	10 10 0			
		26	5	0
		£583	16	10

Examined and found correct, all my requirements as Auditor having been complied with.

Dec. 12th, 1904.

(Signed) J. G. LEATHEM.

We can show that P_1 is an inner limiting set whether Q_1 exists or not; for P_1 consists of a series of sets $P_{11}, P_{12}, \dots, P_{1n}, \dots$ interior to the free intervals $(a_1, b_1), \dots, (a_n, b_n), \dots$, which are complementary to G_1; the set P_{1n} interior to (a_n, b_n) has all its limiting points in that interval, and these since they belong to L_1 are enumerable, and therefore, by Theorem II., $P_{1,n}$ is an inner limiting set. The series of sets of intervals which enclose the points of $P_{1,n}$ may be chosen so that every interval of every set is interior to (a_n, b_n); thus no limiting points of P not belonging to P, except those belonging to $P'_{1,n}$, are ever interior to any interval of the sequence assigned to $P_{1,n}$; as this holds for every n, it follows that P_1 is an inner limiting set, and its points are such that they can be enclosed in a sequence of series of intervals which from the beginning contain no point of G_1.

The set Q_1 consists of points which belong to G_1, and therefore Q_1 has no limiting points in L_1. If every point of G_1 is a limiting point of Q_1, let the points of G_1 be made to correspond in order of magnitude to the points of the continuum (0, 1); the irrational points of (0, 1) correspond to those points of G_1 which do not belong to Q_1; the whole of a complementary interval of G_1, including its two end-points, corresponds to a single point in (0, 1). To an interval in (0, 1) there corresponds an interval which contains in its interior an indefinitely great number of the intervals complementary to G_1; to a convergent sequence of sets in (0, 1) there corresponds a convergent sequence. Since the set of rational points in (0, 1) is dense in itself and not a limiting set, it follows that the set Q_1 is dense in itself and not a limiting set. It has thus been shown that, if Q_1 is dense in G_1, it is not a limiting set, and is dense in itself; in that case P is not a limiting set.

If Q'_1 is not identical with G_1, let Q'_1 be resolved into an enumerable set L_2 and a perfect set G_2; the latter may be absent. The set Q_1 may then be resolved into a component P_2 contained in L_2, and a component Q_2 contained in G_2; thus $P = P_1 + P_2 + Q_2$. The same argument applied to P_2 as was applied to P_1 shows that P_2 is an inner limiting set, and the intervals of the convergent sequence which encloses its points may be taken to be all interior to the intervals complementary to G_2. The set Q_2 contained in G_2 may be treated as Q_1 in G_1 was treated, and we thus have $Q_2 = P_3 + Q_3$, where P_3 is an inner limiting set, and Q_3 is contained in a perfect set G_3; this will be the case unless Q_2 is dense in G_2. Proceeding in this manner, it may happen that, for some integer n, Q_n does not exist, and then P is expressed as the sum of a finite number n of inner limiting sets, and is itself therefore an inner limiting set, and contains no component which is dense in itself; it may, however, happen that, for some number n, Q_n is dense in G_n; in that case, as has been shown above, Q_n

is not an inner limiting set, and is dense in itself; P is in that case not an inner limiting set. If no integer n exists for which either of these things happens, we consider the set $M(P_1, P_2, \ldots, P_n, \ldots)$, where n has every integral value; it may happen that this set contains every point of P, but, if it does not contain every point of P, we take the set

$$P-M(P_1, P_2, \ldots, P_n, \ldots),$$

and resolve it as before into an inner limiting set P_ω and a set Q_ω contained in a perfect set G_ω; we then proceed as before to resolve Q_ω into $P_{\omega+1}$ and a set $Q_{\omega+1}$ contained in a perfect set $G_{\omega+1}$. We proceed further in this manner, and may obtain sets with index any transfinite ordinal number of the second class; in the case of any such number β, we proceed just as in the case of a finite index, provided β is not a limiting number; if β is a limiting number, we obtain the resolution from the preceding numbers as in the case of ω.

We thus have P resolved into $P_1+P_2+\ldots+P_\omega+P_{\omega+1}+\ldots+P_\beta+Q_\beta$, where β is a non-limiting ordinal number of the first or second class, or else P is resolved into $P_1+P_2+\ldots+P_\omega+\ldots+P_\beta+\ldots$ with no last term. Since P is enumerable, this process must come to an end at or before some definite number a of the first or second class; the end of the process comes either (1) when Q_a is dense in G_a, in which case, as we have seen, Q_a is dense in itself and is not an inner limiting set, and thus P is also not an inner limiting set, or (2) when there is no component Q_a in G_a, or (3) when there is no G_a. It has thus been shown that, for P to be an inner limiting set, it is necessary that it should contain no component which is dense in itself, and that, when this condition is satisfied, P is the sum of a finite or infinite number of inner limiting sets, of which there may or may not be a last set.

Let us now assume that P contains no component that is dense in itself; it can then be shown that P must be an inner limiting set. Let P_γ be any one of the components into which P has been resolved, γ denoting an ordinal number of the first or second class; we fix a convergent sequence of sets of intervals enclosing the points of P_γ such that all the intervals are interior to the intervals complementary to G_γ; the set $P_{\gamma+1}+P_{\gamma+2}+\ldots$, which is contained in G_γ, has no limiting points in any of the intervals which enclose the points of P_γ, for all its limiting points must be in G_γ. The convergent sequence of sets of intervals having been determined in the manner described for every P_γ, we can now show that each limiting point p of P which does not belong to P is shed from the whole convergent sequence of sets of intervals, at a definite index. The point p is either a limiting point of P_1 or is contained in G_1; in the former case it is shed

from the intervals enclosing P, at a definite index, and, not being a limiting point of $P_2+P_3+\dots$, is shed from the intervals enclosing the points of that set at a definite index; consequently it is shed from the intervals enclosing P at a definite index, the greater of the two former indices; in the latter case, unless p is in G_2 or in P_2', it is not a limiting point of $P_2+P_3+\dots$, and never comes into any of the intervals enclosing the points of P_1; it is therefore shed at a definite index. If p belongs to $G_1, G_2, \dots$ and to every G before G_α, but is not in G_α, it may be a point of P_α'; in that case it is not a limiting point of the set $P_{\alpha+1}+P_{\alpha+2}+\dots$, and does not come into the interior of any of the intervals which enclose the points of $P_1, P_2, \dots$, or any P with index less than α; it is therefore shed at a definite index, from the sequence of sets of intervals enclosing the points of P. It has thus been shown that every limiting point p of P which does not belong to P is shed at a definite index from the convergent sequence of sets of intervals determined in the manner described above. We have thus—

THEOREM II.—*The necessary and sufficient condition that an enumerable set P may be an inner limiting set is that it contains no component which is dense in itself.*

A corollary from the above proof is that every enumerable set is expressible as the sum of an inner limiting set, and of a set which is dense in itself.

A proof of Theorem II. could no doubt be constructed* which would be independent of the use of transfinite numbers; in many cases, theorems which were first obtained by the employment of transfinite numbers have afterwards been proved by methods which do not involve the use of such numbers. This fact illustrates the high value of the theory of these numbers as an instrument of research; they form the natural language to be employed in dealing with non-finite systems.

5. We proceed to consider the case of unenumerable sets. It has been shown† by G. Cantor that every set of points P can be analysed into four components U, V_a, V_x, V_c which are of the following character:—

* Since this paper was written, a paper has been published by Dr. W. H. Young, "On Sequences of Sets of Intervals containing a given Set of Points," *Proc. London Math. Soc.*, Ser. 2, Vol. 1, p. 262, which contains the following theorem:—"If E contains no component dense in itself, while E' is more than countable, the inner limiting set may be either countably infinite or have the potency c: and we can arrange the intervals so that the inner limiting set consists of E alone." This theorem contains that part of Theorem II. of the present paper which arises when P' is unenumerable, *i.e.*, the second case in the above proof; Dr. Young's proof depends upon the resolution of a set into adherences and coherences.

† See *Acta Mathematica*, Vol. VII.; also Schönflies, *Berichte über die Mengenlehre*, pp. 71–73.

The set U is enumerable and contains no component that is dense in itself; each point of U is of degree zero or of degree a in P, *i.e.*, in a sufficiently small neighbourhood of such a point, there are either no other points of P, or else only an enumerable set of such points.

The set V_a is enumerable and dense in itself, and each point is of degree a in P. The set V_c has the cardinal number c of the continuum, is dense in itself, and each point is of degree c in P, *i.e.*, any neighbourhood, however small, of such a point contains a set of points of P of which the cardinal number is c. The set V_x, if it exists, is dense in itself, and the points of it are all of degree in P, intermediate between a and c; it is not yet definitely known whether any such set as V_x exists, but it is regarded by Cantor as probable that it cannot.

The expression "degree of a point in a set" is here used as equivalent to what has been denominated by Cantor* the *Mächtigkeitsgrad* of the point; an isolated point, *i.e.*, one which is not a limiting point, is of degree zero in the set; a point of any degree λ is one such that in a sufficiently small neighbourhood and in all smaller neighbourhoods of the point the cardinal number of the part of the set is λ.

Taking $P = U + V_a + V_x + V_c$, where one or more of the four components of P may be absent, we observe that, if V_c be absent, the necessary and sufficient conditions that P may be an inner limiting set are (1) that $V_x = 0$; this follows from Dr. Young's theorem that every limiting set has either a or c for its cardinal number, and (2) $V_a = 0$, as has been shown in Theorem II., above.

If V_c be present, we observe that no point of $U + V_a + V_x$ can be a limiting point of V_c, for any limiting point of V_c must be of degree c in the set P. If V_c is everywhere dense in (a, b), it follows that $U + V_a + V_x$ is absent, or $P = V_c$. The set V_c may be non-dense in (a, b), or it may be dense in some parts of (a, b) and non-dense in other parts.

It will be shown that V_c is in general made up of a part which is non-dense in (a, b) and of a finite or indefinitely great number of parts each of which is everywhere dense in the interval in which it lies. Suppose an interval (α, β) can be found in which V_c is everywhere dense; let x be any point in (a, b) such that $x \geqslant \beta$; then those values of x for which V_c is everywhere dense in (α, x), and those values of x for which V_c is not everywhere dense in (α, x), define a section of all the numbers of the continuum (β, b); this section defines a number $\beta_1 \geqslant \beta$. Similarly we may find a

* Dr. W. H. Young has given an investigation of Cantor's theory of adherences and coherences (see *Quarterly Journal* for 1903), which does not involve the use of transfinite ordinal numbers.

number $\alpha_1 \leqslant \alpha$, so that (α_1, β_1) is the greatest interval which can be obtained to contain (α, β) and such that V_c is everywhere dense in it. If in the parts of (a, b) outside (α_1, β_1) the set V_c is anywhere dense, we proceed to fix the greatest interval for which it is everywhere dense. If we proceed in this manner, we obtain a finite or enumerably infinite set of detached intervals contained in (a, b), in each of which V_c is everywhere dense; the remainder of (a, b) may consist of a set of detached intervals, and of a set of points; in this remainder the points of V_c form a non-dense set.

No point of $U+V_a+V_x$ can be in an interval (α_1, β_1) in which V_c is everywhere dense; if $\overline{V}_c$ is the part of V_c which is non-dense, every point of $U+V_a+V_x$ must lie in one of the intervals complementary to the perfect set $\overline{V}'_c$; it is to be observed that in $\overline{V}_c$ we include the end-points of intervals (α_1, β_1), in case those end-points belong to V_c. In order that P may be an inner limiting set it is necessary that the part of $U+V_a+V_x$ which is in each interval complementary to $\overline{V}'_c$ should be an inner limiting set, and this cannot be the case unless $V_a = 0$ and $V_x = 0$. We have thus proved

THEOREM III.—*In order that a set of points may be an inner limiting set it is necessary that the set should contain no points whose degrees in the set are other than* 0, *a, or c, and should contain no component which is dense in itself and of which the points are of degreee a in the set.*

6. The determination of the necessary and sufficient conditions that any given set of points may be an inner limiting set has now been reduced to the problem of determining the requisite criteria in the case of a set which is dense in itself and all the points of which are of degree c in the set. The case in which such a set is non-dense may be reduced by the method of correspondence to the case in which such a set is everywhere dense; thus the question is reducible to that of the determination of the conditions under which a given everywhere dense set of points of degree c in the set may be a set of the second category.

It is possible to find sets which are everywhere dense and of cardinal number c in every interval, which are of the first category. As an example, let a perfect set G_1, of which $\theta(b-a)$ is the greatest complementary interval, be constructed in (a, b); in each of the complementary intervals of G_1 place a set *similar* to G_1; we then have a new perfect set G_2, of which the greatest complementary interval is $\theta^2(b-a)$; proceeding in this way, we form a sequence $\{G_n\}$ of perfect sets such that $\theta^n(b-a)$ is the greatest complementary interval of G_n. The limit G of G_n is a

set of the first category; in every interval contained in (a, b) there is, for a sufficiently great value of n, some interval complementary to G_n, lying inside the interval; therefore G is everywhere dense and has points of cardinal number c in every interval.

It thus appears that there exist sets which are everywhere dense and are of cardinal number c in every interval in (a, b), which are of the first category, and there are others which are of the second category. The question has been raised by Schönflies* whether every such set is necessarily either of the first or else of the second category. This question must certainly be answered in the negative, for, if we divide (a, b) into any finite number of parts, and place in some of them sets of the type indicated which are of the first category, and in the other parts sets which are of the second category, it is clear that the whole set so constituted cannot be either of the first or of the second category.

It is well known that the points which are common to two sets of the second category form a set which is also of the second category. This is true also for non-dense sets contained in a perfect set which are of the second category relatively to the perfect set, as may be seen by the method of correspondence. If we use the results of § 5, we can deduce that the set of points which is the common part of any two inner limiting sets is also an inner limiting set. The theorem is, on using Theorem II., clearly true for inner limiting sets which are enumerable, and the analysis which has been given of limiting sets in general shows that it is true for such limiting sets as are of cardinal number c.

* *Bericht über die Mengenlehre*, p. 110.

NOTE ON THE APPLICATION OF THE METHOD OF IMAGES TO PROBLEMS OF VIBRATIONS

By VITO VOLTERRA.

[Received August 22nd, 1904.—Read November 10th, 1904.]

LORD KELVIN's method of images has yielded the solution of a large number of problems in electrostatics, in the theory of steady electric currents, in hydrodynamics (Stokes, Hicks). It has also been applied with success to problems of elastic equilibrium by Betti, Cerruti, Somigliana, and others. In all these cases the partial differential equations that are involved are of *elliptic* type. Applications of the method have been made also by Betti to problems of the conduction of heat involving differential equations of *parabolic* type. I propose now to explain an application of the method to a question concerning the vibrations of elastic bodies in which, naturally, the differential equations are of *hyperbolic* type.*

The result to which I wish to call especial attention is that in this latter case the application of the method of successive images leads to solutions containing a finite number of terms—not to infinite series—even in the case where the number of images is infinite. This result may seem somewhat paradoxical, but it will be shown to depend upon the fact that it is not necessary to use all the images of the infinite train, but only, in each case, a finite number of them. The reason for this simplification is to be sought in the fact that the characteristic surfaces of the partial differential equations of hyperbolic type are real, and thus the property in question is bound up with the hyperbolic character of the equations, and is, consequently, capable of a very wide application.

In my paper in the *Acta Mathematica*, t. XVIII., there was given the solution of the following problem among others:—

In the differential equation of vibrating membranes, viz.,

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2},$$

* M. Hadamard in a very interesting paper published in the *Bulletin de la Société Mathématique de France*, has also utilised the method of images in a problem of vibrations, but he has not called attention to the point which is chiefly emphasised in the present note.

consider x, y, t as Cartesian coordinates of a point in a space of three dimensions, and let the values of w and its normal derivative $\partial w/\partial n$ be given on a surface σ in this space. It is required to determine, whenever possible, the value of w.

The solution is as follows:—With any point A, (x_1, y_1, t_1) of the space as vertex draw a right circular cone of vertical angle 90° with its axis parallel to the axis of t. Let that sheet of the cone —the "inferior" sheet, say—in which $t < t_1$ cut out on the surface σ a portion σ_a, as in Fig. 1. Let a positive quantity r be determined by the equation

$$r^2 = (x-x_1)^2+(y-y_1)^2.$$

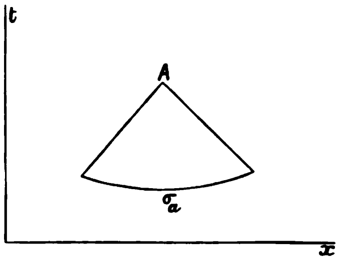


FIG. 1.

Then w is expressed by the formula

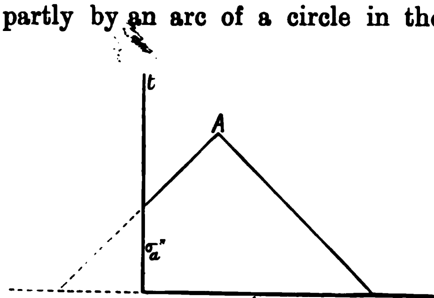
$$w(x_1, y_1, t_1) = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a}\frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\left\{\cos(n, t)-\frac{t_1-t}{r}\cos(n, r)\right\} w\, d\sigma_a$$

$$+\frac{1}{2\pi}\int_{\sigma_a}\frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\left[\frac{\partial w}{\partial t}\cos(n, t)\right.$$

$$\left.-\left\{\frac{\partial w}{\partial x}\cos(n, x)+\frac{\partial w}{\partial y}\cos(n, y)\right\}\right]d\sigma_a. \quad \text{(A)}$$

If the surface σ is the plane of (x, y), this formula reduces to that of Poisson.

Now suppose that the surface σ is formed by that portion of the plane of (x, y) which is defined by $x > 0$ and that portion of the plane of (t, y) which is defined by $t > 0$. Then σ_a may be bounded by a circle in the plane of (x, y), or it may be bounded partly by an arc of a circle in the plane of (x, y) and partly by an arc of an hyperbola in the plane of (t, y). The latter case is represented in Fig. 2, which shows the trace of the cone and planes upon the plane of (x, t). The surface σ consists of a portion σ_a' bounded by a circle, and a portion σ_a'' bounded by an hyperbola. The formula (A) becomes



FGI. 2.

$$w(x_1, y_1, t_1) = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a'} \frac{1}{\sqrt{(t_1^2-r^2)}}\, w\, d\sigma_a' + \frac{1}{2\pi}\int_{\sigma_a'} \frac{1}{\sqrt{(t_1^2-r^2)}}\, \frac{\partial w}{\partial t}\, d\sigma_a'$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a''} \frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\, \frac{t_1-t}{r}\, (\cos n,\, r)\, w\, d\sigma_a''$$

$$-\frac{1}{2\pi}\int_{\sigma_a''} \frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\, \frac{\partial w}{\partial x}\, d\sigma_a''. \qquad \text{(B)}$$

The values of w and $\partial w/\partial t$ on σ_a' are arbitrary, but those of w and $\partial w/\partial x$ on σ_a'' cannot be given arbitrarily; if the values of either w or $\partial w/\partial x$ on σ_a' are given, those of the other are determined thereby. We must therefore seek to eliminate from the formula (B) either the values of w or those of $\partial w/\partial x$ on the surface σ_a'', as is done in analogous cases where the methods of Green are employed. For this purpose we may have recourse to the method of images. Suppose, in fact, that, as shown in Fig. 3, A' is the optical image of the point A in the plane of (y, t).

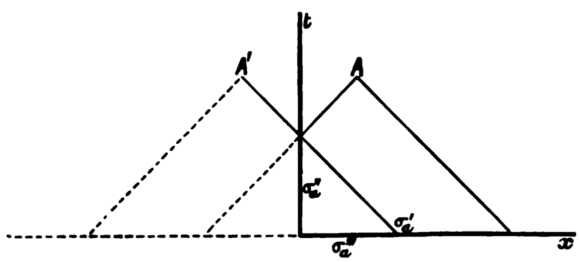


FIG. 3.

The inferior sheet of a cone drawn from A' in the same way as the former cone was drawn from A will cut out on the plane of (y, t) an area σ_a'' bounded by an arc of an hyperbola, and it it will cut out on the plane of (x, y) an area σ_a''' bounded by an arc of a circle, and we shall have the formula

$$0 = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a'''} \frac{1}{\sqrt{(t_1^2-r'^2)}}\, w\, d\sigma_a''' + \frac{1}{2\pi}\int_{\sigma_a'''} \frac{1}{\sqrt{(t_1^2-r'^2)}}\, \frac{\partial w}{\partial t}\, d\sigma$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma_a''} \frac{1}{\sqrt{\{(t_1-t)^2-r'^2\}}}\, \frac{t_1-t}{r}\, \cos(n, r)\, w\, d\sigma_a''$$

$$-\frac{1}{2\pi}\int_{\sigma_a''} \frac{1}{\sqrt{\{(t_1-t)^2-r'^2\}}}\, \frac{\partial w}{\partial t}\, d\sigma_a'', \qquad \text{(C)}$$

where $$r'^2 = (x+x_1)^2+(y-y_1)^2.$$

By adding the formulæ (B) and (C) we eliminate w; by subtracting (C) from (B) we eliminate $\partial w/\partial x$. Thus the method of images leads easily to the desired result.

We proceed to consider successive images. Suppose that the surface σ is formed by the strip σ' of the plane of (x, y) defined by $a > x > 0$, and by the two half planes $(x = 0,\ t > 0)$ and $(x = a,\ t > 0)$, denoted respectively by σ'' and σ'''. It is clear that, if the cone with its vertex at A cuts out on σ' the area σ'_a, and on σ'' and σ''' the areas σ''_a and σ'''_a bounded by arcs of hyperbolas, the formula (A) becomes

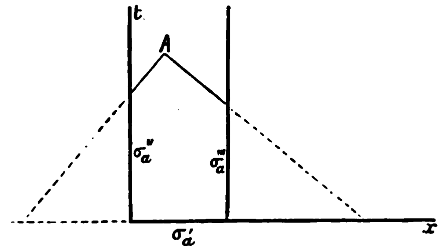


FIG. 4.

$$w(x_1, y_1, t_1) = \frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma'_a}\frac{1}{\sqrt{(t_1^2-r^2)}}\,w\,d\sigma_a + \frac{1}{2\pi}\int_{\sigma'_a}\frac{1}{\sqrt{(t_1^2-r^2)}}\frac{\partial w}{\partial t}\,d\sigma_a$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma''_a}\frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\frac{t_1-t}{r}\cos(n, r)\,w\,d\sigma''_a$$

$$-\frac{1}{2\pi}\int_{\sigma''_a}\frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\frac{\partial w}{\partial x}\,d\sigma''_a$$

$$-\frac{1}{2\pi}\frac{\partial}{\partial t_1}\int_{\sigma'''_a}\frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\frac{t_1-t}{r}\cos(n, r)\,w\,d\sigma'''_a$$

$$+\frac{1}{2\pi}\int_{\sigma'''_a}\frac{1}{\sqrt{\{(t_1-t)^2-r^2\}}}\frac{\partial w}{\partial x}\,d\sigma'''_a.$$

In this formula the values of w and $\partial w/\partial x$ on σ''_a and σ'''_a are not independent, and the method of images can be used to eliminate either of them. For this purpose we must take the images of A with respect to the planes σ'' and σ''', the images of these images with respect to the same planes, and so on indefinitely. The number of images which are obtained in this way is, of course, infinite, but it is important to observe that it is not necessary to use them all. A finite number of them suffices for the solution of the problem.

With each of the images as vertex draw a cone of vertical angle 90° and with its axis parallel to the axis of t, and take the inferior sheets of these cones. If the vertex of any one is at a distance from σ'' or σ''' which exceeds t, that one does not cut these half planes, and it may be omitted.

It is clear that the values of w or of $\partial w/\partial x$ on σ''_a and σ'''_a can be eliminated from the preceding formulæ by taking account of those images only which are at a less distance than t from σ'' and σ'''.

We conclude from this argument that the method of images does not in this case lead to infinite series, but to solutions with a finite number of terms. In place of the three planes σ'', σ', and σ''' we might have five, of which four are perpendicular to σ and cut it in a rectangle. If w vanishes on the four planes perpendicular to σ, we have the well known problem of the vibrations of a rectangular membrane, and we have a solution of this problem by means of definite integrals without having to introduce any infinite series.

ON THE ZEROES OF CERTAIN CLASSES OF INTEGRAL TAYLOR SERIES. PART I.—ON THE INTEGRAL FUNCTION

$$\sum_{n=0}^{\infty} \frac{x^{\phi(n)}}{\{\phi(n)\}!}.$$

By G. H. HARDY.

[Received August 20th, 1904.—Revised October, 1904.—Read November 10th, 1904.]

1. This paper and another which I hope to publish before long are intended as a contribution to the solution of the following problem:—"To find particular (but as large as possible) classes of integral Taylor series, such that the nature of the zeroes associated with the essential singularity at infinity can be determined *with precision*."

This problem is somewhat different from and less ambitious than the classical problem of the theory of integral functions, the object of the works of Poincaré, Hadamard, and their successors in this field. The degree of precision attained by the results proved in these papers is, as will be seen clearly later on, wholly incompatible with initial hypotheses of the same degree of generality as those adopted by the writers referred to. If we start, as they do, by supposing known merely certain limits for the increase (*croissance*) of the moduli of the reciprocals of the coefficients, all that is generally possible is to determine corresponding limits for the increase of the roots. This problem has, of course, up to a certain point, been studied with conspicuous success; and it seems as if all that can be done further in this direction is to impart some additional precision to the known results.*

---

* The original memoirs of Poincaré and Hadamard were: Poincaré, "Sur les fonctions entières" (*Bull. de la Soc. Math. de France*, 1883); Hadamard, "Etude sur les propriétés des fonctions entières" (*Journal de Math.*, 1893, p. 171).

Numerous references to further writings on the subject up to 1900 will be found in M. Borel's *Leçons sur les fonctions entières*, 1900.

Among later publications I may mention: P. Boutroux, "Sur quelques propriétés des fonctions entières" (*Acta Math.*, Vol. XXVIII., p. 97); E. Lindelöf, "Mémoire sur les fonctions entières de genre fini" (*Acta Soc. Fennicae*, Vol. XXXI., p. 1); E. W. Barnes, "A Memoir on Integral Functions" (*Phil. Trans.*, A, Vol. 199, p. 411); A. Wiman, "Sur le cas d'exception dans la théorie des fonctions entières" (*Arkiv for Mat. Ast. och Fysik*, Vol. I., p. 327); Edm. Maillet, "Sur les fonctions entières et quasi-entières" (*Journal de Math.*, 1902, p. 329); J.-L.-W.-V. Jensen, "Sur un nouvel et important théorème de la théorie des fonctions," *Acta Math.*, Vol. XXII.

There are other writings, too numerous to mention, by MM. Lindelöf, Wiman, and others.

My object in these and in some earlier papers* on particular integral functions has been to "find asymptotically" the zeroes of the function under consideration; that is to say, to determine an enumerable sequence of points such that, if σ be an arbitrarily small positive quantity, and circles of radius σ be described with the points as centres, we can choose R so that all the zeroes whose distance from the origin is greater than R lie within the circles, each circle containing one and only one.†

It is evident that to have any hope of obtaining results as precise as this we *must* confine ourselves to "particular cases." In the case of a Taylor series, for instance, we must in general suppose known not merely the increase of the reciprocals of the coefficients, but also their analytical nature. At the same time what is, from the point of view of the general theory, an altogether special case may still include a considerable variety of interesting functions; and, for this reason, I hope that the analysis contained in this and the following paper may be of some interest to those who are engaged in the study of the general theory of integral functions.

The first step in all such investigations must necessarily be the determination of asymptotic representations of the function some one of which is valid in every region of the plane. When this has been done the asymptotic determination of the zeroes is generally easy enough; at any rate, it is generally easy enough to find a series of points in whose neighbourhood the zeroes must be sought for. To find a precise proof that one and only one zero is associated with each point is sometimes a matter of greater difficulty.

It will be generally found, with functions such as we are naturally led to consider, that the whole plane with the exception of certain "barrier regions" E can be divided into a number of regions D, within which the function is given asymptotically by an equation of the form

$$f(x) = \phi_D(x)(1+\rho_x),$$

ρ_x being a function of x which tends uniformly to zero with $1/x$, and $\phi_D(x)$ a function which has no zeroes in D. It then follows that $f(x)$ has no

* "On the Zeroes of the Integral Function $x-\sin x$" (*Messenger*, Vol. XXXI., p. 161); "On the Zeroes of certain Integral Functions" (*Messenger*, Vol. XXXII., p. 36); "The Asymptotic Solution of certain Transcendental Equations" (*Quarterly Journal*, Vol. XXXV., p. 261); "On the Roots of the Equation $\dfrac{1}{\Gamma(x+1)} = c$" (*Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 1); "Note on an Integral Function" (*Messenger*, Vol. XXXIV., p. 1).

† The approximation is not always quite as precise as this.

zeroes except perhaps in the barrier regions E. Suppose, for instance, that there are two regions D separated by a single region E. Then the asymptotic determination of the zeroes will be found to rest upon a proof that within the region E

$$f(x) = \phi_D(x)(1+\rho_x)+\phi_{D'}(x)(1+\rho_x),$$

and that the zeroes are to be found near the points for which

$$0 = \phi_D(x)+\phi_{D'}(x),$$

at any rate asymptotically.

In these two papers I deal with two classes of functions closely connected with the ordinary exponential function. I have no doubt that similar investigations might be carried out for classes of functions similarly formed from other standard functions—from Prof. Mittag-Leffler's function

$$E_a(x) = \sum_0^\infty \frac{x^n}{\Gamma(an+1)},\text{*}$$

for example. I have not attempted to carry out such investigations for the reason that Prof. Mittag-Leffler's long promised memoir on the function $E_a(x)$ has not yet appeared.†

Of the two classes of functions which I consider the second‡ seems to me the more interesting in itself. Theoretically, however, the first is perhaps more so, on account of the extreme simplicity of the method employed, which proceeds directly from the Taylor series, depending merely on a development of the idea, originally due to M. Borel, of the preponderant importance for certain functions of *the numerically greatest term* in the expansion. In dealing with the second class a preliminary transformation of the series into the more easily manipulated form of a definite integral seems to be essential.§

2. The first class of functions is that formed by a suitable selection of terms from the exponential series itself, that is to say the class of functions defined by a series of the form

(1) $$\sum_{n=0}^\infty \frac{x^{\phi(n)}}{\{\phi(n)\}!},$$

* Mittag-Leffler, *Comptes Rendus* (2 Mars, 1903).

† I do not know whether Prof. Mittag-Leffler has asymptotically determined the zeroes of $E_a(x)$.

‡ The functions $\Sigma \frac{x^n}{\Gamma(a+n+1)}$, $\Sigma \frac{x^n}{(n+a)^a n!}$ are typical members of this class.

§ A third class of functions whose zeroes may be investigated by very elementary methods is the class defined by series of the type $\Sigma \frac{x^n}{b_0 b_1 \dots b_n}$, when the increase of $|b_n|$ is rapid.

where $\phi(n)$ is a function of n which is positive and integral for all positive integral values of n, and increases steadily with n. It is very easy to see that the increase of such a function cannot be very far removed from that of e^x. In the first place, if the maximum of the modulus of the function is denoted by $M(r)$, it is obvious that

$$M(r) \leqslant e^r.$$

But, on the other hand, we obtain an inferior limit for $M(r)$ by considering the increase of the greatest term in the expansion of e^r. Now this term is the one for which

$$r/n \geqslant 1 > r/(n+1),$$

i.e., the $[r]$-th. The value of this is

$$r^r/r! \sim e^r/\sqrt{(2\pi r)}.^*$$

Hence it follows that the increase of all the functions with which we are concerned lies between that of e^r and that of $e^r/\sqrt{r}$. In M. Borel's notation† we may say that, if ι is the order of infinity of $M(r)$ for $r = \infty$, then

$$\omega - \tfrac{1}{2} \leqslant \iota \leqslant \omega.$$

It is, of course, only for certain forms of the function ϕ, which we may call *normal* forms, that the precise determination which is desired can be made. Moreover, for it to be practicable it is essential (in general) that the increase of $\phi(n)$ should be sufficiently rapid. I shall first work out in detail what appears to be the simplest case, that in which

$$\phi(n) = n^3,$$

(2)
$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n^3}}{n^3!},$$

and then indicate the various generalisations which can be made.

The Function $f(x) = \sum_{n=0}^{\infty} \frac{x^{n^3}}{n^3!}$.

3. It follows from Stirling's theorem that

$$n^3! = n^{3n^3+\frac{3}{2}} e^{-n^3} \sqrt{(2\pi)}\,(1+\rho),$$

---

* By $f(a) \sim \phi(a)$ I mean, in this paper, $\lim_{a=\infty} f/\phi = 1$; the symbol was introduced by Du Bois-Reymond.

† See his *Leçons sur les séries à termes positifs*, chap. iii.

where $$| \rho | < K/n^3.^*$$

Suppose that $$x = re^{i\theta}, \quad v_n = | x^{n^3}/n^3! | .$$

Then we find by an easy calculation that

$$v_n/v_{n+1} = (n^3/r)^{3n^2+3n+1} \, e^{\frac{9}{2}(n+1)} (1+\rho) \quad (| \rho | < K/n).$$

If $r = N^3$, where N is a large integer,

$$(3) \qquad v_N/v_{N+1} = e^{\frac{9}{2}(N+1)} (1+\rho) \quad (| \rho | < K/N),$$

while, if $n > N$,

$$(3') \qquad v_n/v_{n+1} > v_N/v_{N+1} > e^{\frac{9}{2}(N+1)} (1+\rho).$$

We can fix a large integer N_0, and then take N large in comparison with N_0. Then, if $N_0 \leqslant n < N$, $r = N^3 \geqslant (n+1)^3$, and

$$(3'') \qquad \frac{v_n}{v_{n+1}} < K \left(\frac{n}{n+1}\right)^{9n^2+9n+3} e^{\frac{9}{2}(n+1)} < Ke^{-\frac{9}{2}n}.$$

From (3), (3′), and (3″) it follows that

$$\sum_{N_0}^{\infty} \frac{x^{n^3}}{n^3!} = \frac{x^{N^3}}{N^3!} (1+\rho) \quad (| \rho | < K/N).$$

And thus it is evident that, if N is sufficiently large in comparison with N_0,

$$(4) \qquad f(x) = \sum_0^{\infty} \frac{x^{n^3}}{n^3!} = \frac{x^{N^3}}{N^3!} (1+\rho),$$

when $r = N^3$. Now $\dfrac{x^{N^3}}{N^3!} = \dfrac{N^{3N^3}}{N^3!} \exp (N^3 i\theta)$,

$$\frac{N^{3N^3}}{N^3!} = \frac{e^{N^3}}{\sqrt{(2\pi N^3)}} (1+\rho) = \frac{e^r}{\sqrt{(2\pi r)}} (1+\rho).$$

Thus along the circle of radius $r = N^3$

$$(5) \qquad f(x) = (2\pi r)^{-\frac{1}{2}} e^{r+ri\theta} (1+\rho),$$

where $| \rho | < K/r^{\frac{1}{3}}$. We have thus defined a series of circles round each of which the modulus of $f(x)$ is large, being substantially of the order of $e^r/\sqrt{r}$.

4. From this formula it is evident that when x travels round the circle of radius N^3 the amplitude of $f(x)$ is increased by $2\pi N^3$. Hence it follows

---

* In this paper I use K to denote a number not the same in different inequalities, but always lying between certain fixed limits, say 1,000,000 and 1/1,000,000. I need hardly say that ρ also is not the same in different inequalities or equations.

that, if N is sufficiently large, *there are exactly* N^3 *zeroes of* $f(x)$ *within a circle whose centre is the origin and whose radius is* N^3.

I shall now proceed to determine more precisely the positions of the $3N^2+3N+1$ zeroes which lie between the circles $r = N^3$, $r = (N+1)^3$.

5. Suppose then that $N^3 < r < (N+1)^3$.

If $n \geqslant N+1$, $\quad v_n/v_{n+1} > e^{\frac{1}{2}n}$,

while, if $N_0 \leqslant n \leqslant N-1$, $\quad v_n/v_{n+1} < Ke^{-\frac{1}{2}n}$.

Hence it follows that

$$f(x) = \frac{x^{N^3}}{N^3!}(1+\rho)+\frac{x^{(N+1)^3}}{(N+1)^3!}(1+\rho) \quad (|\rho| < K/N). \tag{6}$$

As r increases from N^3 to $(N+1)^3$ the importance of the second term grows at the expense of the first, and, if x is situated at one of the zeroes, it is evident that

$$x^{3N^2+3N+1} = -\frac{(N+1)^3!}{N^3!}(1+\rho) \tag{7}$$

or

$$r^{3N^2+3N+1} = \frac{(N+1)^3!}{N^3!}(1+\rho) \tag{8}$$

and

$$e^{(3N^2+3N+1)i\theta} = e^{(2k+1)i\pi}(1+\rho), \tag{9}$$

k being an integer. From these two equations we easily deduce

$$r = N^3(1+3/2N+\ldots), \tag{10}$$

and

$$\theta = \frac{2k+1}{3N^2+3N+1}\pi+\rho \quad (k = 0, 1, \ldots, 3N^2+3N). \tag{11}$$

It would be easy to carry the approximation further, but the above formulæ give as much information as is necessary for my present purpose. I may remark in passing that we can always find a lower as well as an upper limit for the number of zeroes of a function $f(x)$ lying within a circle of radius r, *if* $|f(x)|$ *is large for all points on the circle.* These limits may be at once deduced from M. Jensen's well known formula expressing $\int_0^{2\pi} \log |f(x)| \, d\theta$ in terms of the zeroes.

Thus *the* $3N^2+3N+1$ *zeroes of* $f(x)$ *which lie between the circles* $r = N^3$ *and* $r = (N+1)^3$ *are given approximately by some or all of the* $3N^2+3N+1$ *points defined by the equations* (10) *and* (11).

It is natural to suppose that one and only one root of $f(x)$ is thus associated with each of these points; and this is, in fact, the case. I have not thought it worth while to set out the proof of this in detail, as I have given formal proofs of similar points with respect to other functions on several occasions,* and the present case is not one in which any point of particular interest or difficulty arises.

The function $f(x)$ is therefore (roughly speaking) what Mr. Barnes† has called a *ring function*; its zeroes lie (roughly speaking) on circles, separated by circles along which its modulus is everywhere large.

It is easy to see that, if we exclude the zeroes of $f(x)$ from the plane by a series of circles with their centres at the zeroes and of fixed, but arbitrarily small, radius, then throughout the rest of the plane

$$|f(x)| > K\,r^{-\frac{1}{3}}e^{r}.$$

The function has therefore the property that *its modulus tends (in general) to infinity when x approaches infinity in any direction.*

Generalisations.

6. The preceding analysis at once suggests various generalisations. For it clearly depends *only* on the fact that a series of circles can be defined on which the behaviour of $f(x)$ is completely dominated by the behaviour of *one* of its terms, while between any two circles it is completely dominated by the behaviour of *two*. The analysis would therefore be equally practicable for any function of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{\phi(n)}}{\{\phi(n)\}!},$$

provided the increase of $\phi(n)$ is *regular* and *sufficiently rapid;* for instance, if $\phi(n)$ were

$$n^4,\ n^5,\ \ldots,\ [n^p \log n]\ (p \geqslant 3),\ [e^n],\ \ldots.$$

In general we find that the value of $f(x)$ on the circle $r = \phi(n)$ is given by the formula (5), just as in the special case when $\phi(n) = n^3$, so that the number of zeroes within the circle is exactly $\phi(n)$. Moreover, the form

* See the papers referred to above (p. 333).

† *L.c.*, p. 424.

of $f(x)$ may be altered in another way, as the argument applies equally well to

$$F(x) = \sum_{n=0}^{\infty} \frac{\psi(n)\, x^{\phi(n)}}{\{\phi(n)\}!},$$

provided that $\psi(n)$ satisfies certain conditions easily defined. In this case the form of (5) is of course slightly modified. Or, again, we might consider the function

$$F_1(x) = \sum_{n=0}^{\infty} \frac{\psi(n)\, x^{\phi(n)}}{\{\phi(n)\}^{\phi(n)}};$$

but the principle will have been sufficiently illustrated by what precedes. I may, however, remark that the argument will not apply in its present form to

$$f(x) = \sum_{n=0} \frac{x^{n^2}}{n^2!},$$

the increase of n^2 not being sufficiently rapid, as is easily seen on working out the necessary approximations.

[As a matter of fact asymptotic formulæ for $f(x)$ and for its zeroes, analogous to those found in the case when $\phi(n) = n^3$, may be determined; but the investigation is more difficult, and involves the theory of elliptic functions, and it is no longer true that the behaviour of $f(x)$ is dominated by that of one or two terms. I find that, for $r = N^2$,

$$f(x) = \frac{e^{r+ri\theta}}{\sqrt{(2\pi r)}}\, \phi(\theta)(1+\rho),$$

where
$$\phi(\theta) = \sum_{-\infty}^{\infty} e^{\nu^2(i\theta-2)+2\nu N i\theta}.$$

In particular, if $\theta = 0$,
$$f(x) = \frac{Ae^x}{\sqrt{(2\pi x)}},$$

where
$$A = 1+2\sum_{1}^{\infty} e^{-2\nu^2}.$$

There are N^2 zeroes of $f(x)$ within a large circle of radius N^2, and the $2N+1$ which lie between the circles $r = N^2$, $r = (N+1)^2$ are given approximately by

$$r = N^2+N, \qquad \theta = \frac{2q-1}{2N+1}\pi \quad (q = 1, 2, \ldots, 2N+1).$$

—*Added November 4th*, 1904.]

ON THE EXPANSIONS OF THE ELLIPTIC AND ZETA FUNCTIONS OF $\frac{2}{3}K$ IN POWERS OF q

By J. W. L. GLAISHER.

[Received October 13th, 1904.—Read November 10th, 1904.]

1. In a paper in Vol. XXI. of the *Proceedings*,* I have given the expansions of the twelve elliptic and four zeta functions of $\frac{1}{3}K$ in powers of q, the coefficients in the expansions being expressed by means of certain arithmetical functions. Since the publication of that paper I have reduced the number of these arithmetical functions, which are required for the expansions, to two. The new forms may be deduced from those contained in the previous paper, but it seems preferable to give an independent investigation, deriving them from the general expansions of the elliptic and zeta functions. In the previous paper the results were stated without proof, the methods by which they were obtained being merely indicated.

2. The expansions in powers of q of the general elliptic and zeta functions may be written†

$$
\begin{aligned}
k\rho \operatorname{sn} \rho x &= 4\Sigma_1^\infty \Delta(\sin mx)\, q^{\frac{1}{2}m},\\
kk'\rho \operatorname{sd} \rho x &= 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \Delta(\sin mx)\, q^{\frac{1}{2}m},\\
k\rho \operatorname{cd} \rho x &= 4\Sigma_1^\infty E(\cos mx)\, q^{\frac{1}{2}m},\\
k\rho \operatorname{cn} \rho x &= 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} E(\cos mx)\, q^{\frac{1}{2}m};
\end{aligned}
$$

$$
\begin{aligned}
\rho \operatorname{zn} \rho x &= 4\Sigma_1^\infty \Delta'(\sin 2nx)\, q^n,\\
\rho \operatorname{zd} \rho x &= 4\Sigma_1^\infty (-1)^n \Delta'(\sin 2nx)\, q^n,\\
\rho \operatorname{dn} \rho x &= 1 + 4\Sigma_1^\infty E'(\cos 2nx)\, q^n,\\
k'\rho \operatorname{nd} \rho x &= 1 + 4\Sigma_1^\infty (-1)^n E'(\cos 2nx)\, q^n;
\end{aligned}
$$

* "On the q-Series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ and $\frac{1}{4}K$," *Proceedings*, Vol. XXII., 1890, pp. 143–171.

† *Messenger of Mathematics*, Vol. XVIII., 1888, p. 8.

$$
\begin{aligned}
\rho \operatorname{ns} \rho x &= \operatorname{cosec} x + 4\Sigma_1^\infty \Delta(\sin nx)\, q^n, \\
\rho \operatorname{ds} \rho x &= \operatorname{cosec} x + 4\Sigma_1^\infty (-1)^n \Delta(\sin nx)\, q^n, \\
\rho \operatorname{dc} \rho x &= \sec x + 4\Sigma_1^\infty E(\cos nx)\, q^n, \\
k'\rho \operatorname{nc} \rho x &= \sec x + 4\Sigma_1^\infty (-1)^n E(\cos nx)\, q^n;
\end{aligned}
$$

$$
\begin{aligned}
\rho \operatorname{zs} \rho x &= \cot x + 4\Sigma_1^\infty \sigma(\sin 2nx)\, q^{2n}, \\
\rho \operatorname{cs} \rho x &= \cot x - 4\Sigma_1^\infty \zeta'(\sin 2nx)\, q^{2n}, \\
\rho \operatorname{zc} \rho x &= -\tan x - 4\Sigma_1^\infty \zeta(\sin 2nx)\, q^{2n}, \\
k'\rho \operatorname{sc} \rho x &= \tan x - 4\Sigma_1^\infty \xi(\sin 2nx)\, q^{2n};
\end{aligned}
$$

where $\rho = 2K/\pi$, n is any number, m any uneven number, and the arithmetical functions Δ, E, ... are defined as follows.

Let $\delta_1, \delta_2, \ldots$ be all the uneven divisors of any number n, and let $\delta_1', \delta_2', \ldots$ be their conjugates (*i.e.*, so that $\delta_1\delta_1' = \delta_2\delta_2' = \ldots = n$). Let $d_1, d_2, \ldots$ be all the divisors of n and $d_1', d_2', \ldots$ their conjugates.

$$
\begin{aligned}
\text{Let}\quad \Delta\phi(n) &= \Sigma_n \phi(\delta), & \Delta'\phi(n) &= \Sigma_n \phi(\delta'), \\
E\phi(n) &= \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \phi(\delta), & E'\phi(n) &= \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \phi(\delta'), \\
\sigma\phi(n) &= \Sigma_n \phi(d), & \xi\phi(n) &= \Sigma_n (-1)^{d+d'} \phi(d), \\
\zeta\phi(n) &= \Sigma_n (-1)^{d-1} \phi(d), & \zeta'\phi(n) &= \Sigma_n (-1)^{d'-1} \phi(d),
\end{aligned}
$$

where ϕ is any function and n is the number to which the divisors $\delta_1, \delta_2, \ldots, d_1, d_2, \ldots$ relate.* For example,

$$
\Delta\phi(n) = \phi(\delta_1) + \phi(\delta_2) + \ldots, \qquad E\phi(n) = \pm\,\phi(\delta_1) \pm \phi(\delta_2) \pm \ldots,
$$

where in the second equation the sign of the term involving δ_r is positive or negative according as δ_r is of the form $4k+1$ or $4k+3$.

These definitions are also supposed to hold good when n is replaced by any uneven number m. In this case the d's and δ's are the same.

For the expression of the coefficients the functions $E(n)$, $H(n)$, $H_1(n)$ will be used, where

$$
\begin{aligned}
E(n) = {} & \text{the number of divisors of } n \text{ of the form } 4k+1 \\
& - \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 4k+3,\dagger \\
H(n) = {} & \text{the number of divisors of } n \text{ of the form } 3k+1 \\
& - \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 3k+2, \\
H_1(n) = {} & \text{the number of divisors of } n \text{ of the form } 6k+1 \\
& - \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 6k+5.
\end{aligned}
$$

---

* Thus, when the argument is $2nx$, it is with the divisors of n, not $2n$, that we are concerned. For example, $\sigma(\sin 2nx) = \sin 2d_1x + \sin 2d_2x + \ldots$, where $d_1, d_2, \ldots$ are the divisors of n.

† This definition of $E(n)$ is inconsistent with the general definition of $E\phi(n)$, but, as $\Sigma_n(-1)^{\frac{1}{2}(\delta-1)}\delta$ does not occur, it is convenient to use $E(n)$ to denote $\Sigma_n(-1)^{\frac{1}{2}(\delta-1)}$.

3. Putting $x = \frac{1}{3}\pi$, the first group of expansions becomes

$$k\rho \operatorname{sn} \tfrac{2}{3}K = 4\Sigma_1^\infty . \Sigma_m \sin \tfrac{1}{3}\delta\pi . q^{\frac{1}{2}m},$$
$$kk'\rho \operatorname{sd} \tfrac{2}{3}K = 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} . \Sigma_m \sin \tfrac{1}{3}\delta\pi . q^{\frac{1}{2}m},$$
$$k\rho \operatorname{cd} \tfrac{2}{3}K = 4\Sigma_1 . \Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi . q^{\frac{1}{2}m},$$
$$k\rho \operatorname{cn} \tfrac{2}{3}K = 4\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} . \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi . q^{\frac{1}{2}m}.$$

Now $\sin \frac{1}{3}\delta\pi = \frac{1}{2}\sqrt{3}$ if δ is of the form $6k+1$, and $= -\frac{1}{2}\sqrt{3}$ if δ is of the form $6k+5$. If δ is divisible by 3, it is zero. Therefore

$$\Sigma_m \sin \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\sqrt{3}\, H_1(m).$$

To calculate the value of $\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \frac{1}{3}\delta\pi$ we notice that, if δ is not divisible by 3, $\cos \frac{1}{3}\delta\pi = \frac{1}{2}$, and that, if δ is divisible by 3,

$$\cos \tfrac{1}{3}\delta\pi = -1 = \tfrac{1}{2} - \tfrac{3}{2}.$$

Thus $$\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} - \tfrac{3}{2}\Sigma_m (-1)^{\frac{1}{2}(\epsilon-1)},$$

where the second term occurs only when m is divisible by 3, in which case ϵ is any divisor of m which is divisible by 3.

The first term $= \frac{1}{2}E(m)$. To express the second term, let $m = 3\mu$; then $\epsilon_r = 3\eta_r$, where η_r is any divisor of μ; therefore

$$\Sigma_m (-1)^{\frac{1}{2}(\epsilon-1)} = \Sigma_\mu (-1)^{\frac{1}{2}(3\eta-1)} = -\Sigma_\mu (-1)^{\frac{1}{2}(\eta-1)} = -E(\mu) = -E(\tfrac{1}{3}m).$$

We thus find

$$\Sigma_m (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi = \tfrac{1}{2}E(m) + \tfrac{3}{2}E(\tfrac{1}{3}m).$$

This equation also holds good when m is not divisible by 3, if we define $E(r)$ to be zero when r is fractional.

The group of expansions therefore becomes

$$k\rho \operatorname{sn} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty H_1(m)\, q^{\frac{1}{2}m},$$
$$kk'\rho \operatorname{sd} \tfrac{2}{3}K = 2\sqrt{3}\, \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H_1(m)\, q^{\frac{1}{2}m},$$
$$k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty \{E(m) + E(\tfrac{1}{3}m)\}\, q^{\frac{1}{2}m}.$$
$$k\rho \operatorname{cn} \tfrac{2}{3}K = 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \{E(m) + E(\tfrac{1}{3}m)\}\, q^{\frac{1}{2}m}.$$

4. Putting $x = \frac{1}{3}\pi$, the second group is

$$\rho \operatorname{zn} \tfrac{2}{3}K = 4\Sigma_1^\infty . \Sigma_n \sin \tfrac{2}{3}\delta'\pi . q^n,$$
$$\rho \operatorname{zd} \tfrac{2}{3}K = 4\Sigma_1^\infty . (-1)^n\, \Sigma_n \sin \tfrac{2}{3}\delta'\pi . q^n,$$
$$\rho \operatorname{dn} \tfrac{2}{3}K = 1 + 4\Sigma_1^\infty . \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi . q^n,$$
$$k'\rho \operatorname{nd} \tfrac{2}{3}K = 1 + 4\Sigma_1^\infty . (-1)^n\, \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi . q^n.$$

Consider the value of $\quad A = \Sigma_m \sin \frac{2}{3}\delta'\pi.$

If n is uneven, the system of numbers δ' consists of $\delta_1, \delta_2, \ldots$, and $A = \frac{1}{2}\surd 3\, H_1(n)$. If $n = 2m$ (m being uneven), the system δ' consists of $2\delta_1, 2\delta_2, \ldots$ and $A = -\frac{1}{2}\surd 3\, H_1(n)$; if $n = 4m$, the system δ' consists of $4\delta_1, 4\delta_2, \ldots$ and $A = \frac{1}{2}\surd 3\, H_1(n)$; and so on. Thus we find, if $n = 2^i m$,

$$\Sigma_n \sin \tfrac{2}{3}\delta'\pi = (-1)^i\, \tfrac{1}{2}\surd 3\, H_1(n).$$

Consider now the value of

$$A = \Sigma_n (-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi.$$

We have $\quad \cos \frac{2}{3}\delta'\pi = -\frac{1}{2}$, except when δ is divisible by 3,

and $\quad = -\frac{1}{2}+\frac{3}{2}$ when δ is divisible by 3.

Therefore $\quad A = -\frac{1}{2}\Sigma_n(-1)^{\frac{1}{2}(\delta-1)}+\frac{3}{2}\Sigma_n(-1)^{\frac{1}{2}(\epsilon-1)},$

where in the second term (which occurs only when n is divisible by 3) ϵ is any uneven divisor of n whose conjugate is divisible by 3.

The first term $= -\frac{1}{2}E(n)$; and to evaluate the second term we notice that, if $n = 2^i . 3\mu$, where μ is uneven, then ϵ is any divisor of μ. Thus the second term $= \frac{3}{2}E(\frac{1}{3}n)$, and we find

$$\Sigma_n(-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{2}{3}\delta'\pi = -\tfrac{1}{2}E(n)+\tfrac{3}{2}E(\tfrac{1}{3}n).$$

The second group of expansions therefore becomes

$$\begin{aligned}
\rho \operatorname{zn} \tfrac{2}{3}K &= 2\surd 3\, \Sigma_1^\infty (-1)^i H_1(n) q^n,\\
\rho \operatorname{zd} \tfrac{2}{3}K &= 2\surd 3\, \Sigma_1^\infty (-1)^n (-1)^i H_1(n) q^n,\\
\rho \operatorname{dn} \tfrac{2}{3}K &= 1-2\Sigma_1^\infty \{E(n)-3E(\tfrac{1}{3}n)\}\, q^n,\\
k'\rho \operatorname{nd} \tfrac{2}{3}K &= 1-2\Sigma_1^\infty (-1)^n \{E(n)-3E(\tfrac{1}{3}n)\}\, q^n.
\end{aligned}$$

5. The third group is

$$\begin{aligned}
\rho \operatorname{ns} \tfrac{2}{3}K &= \operatorname{cosec} \tfrac{1}{3}\pi+4\Sigma_1^\infty . \Sigma_n \sin \tfrac{1}{3}\delta\pi . q^n,\\
\rho \operatorname{ds} \tfrac{2}{3}K &= \operatorname{cosec} \tfrac{1}{3}\pi+4\Sigma_1^\infty . (-1)^n\, \Sigma_n \sin \tfrac{1}{3}\delta\pi . q^n,\\
\rho \operatorname{dc} \tfrac{2}{3}K &= \sec \tfrac{1}{3}\pi+4\Sigma_1^\infty . \Sigma_n(-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi . q^n,\\
k'\rho \operatorname{nc} \tfrac{2}{3}K &= \sec \tfrac{1}{3}\pi+4\Sigma_1^\infty . (-1)^n \Sigma_n(-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi . q^n.
\end{aligned}$$

If $n = 2^i m$,

$$\begin{aligned}
\Sigma_n \sin \tfrac{1}{3}\delta\pi &= \Sigma_m \sin \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\surd 3\, H_1(m) = \tfrac{1}{2}\surd 3\, H_1(n) \quad (\S\ 3),\\
\Sigma_n(-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi &= \Sigma_n(-1)^{\frac{1}{2}(\delta-1)} \cos \tfrac{1}{3}\delta\pi = \tfrac{1}{2}E(m)+\tfrac{3}{2}E(\tfrac{1}{3}m)\\
&= \tfrac{1}{2}E(n)+\tfrac{3}{2}E(\tfrac{1}{3}n) \quad (\S\ 3);
\end{aligned}$$

and therefore

$$\rho\,\mathrm{ns}\,\tfrac{2}{3}K = 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty H_1(n)q^n,$$

$$\rho\,\mathrm{ds}\,\tfrac{2}{3}K = 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty (-1)^n H_1(n)q^n,$$

$$\rho\,\mathrm{dc}\,\tfrac{2}{3}K = 2+2\Sigma_1^\infty \{E(n)+3E(\tfrac{1}{3}n)\}\,q^n,$$

$$k'\rho\,\mathrm{nc}\,\tfrac{2}{3}K = 2+2\Sigma_1^\infty (-1)^n \{E(n)+3E(\tfrac{1}{3}n)\}\,q^n.$$

6. In the fourth group the four coefficients are all different in form and depend upon all the divisors, instead of only upon the uneven divisors, of n.

The expansions are

$$\rho\,\mathrm{zs}\,\tfrac{2}{3}K = \cot\tfrac{1}{3}\pi+4\Sigma_1^\infty.\,\Sigma_n \sin\tfrac{2}{3}d\pi\,.\,q^{2n},$$

$$\rho\,\mathrm{cs}\,\tfrac{2}{3}K = \cot\tfrac{1}{3}\pi+4\Sigma_1^\infty.\,\Sigma_n (-1)^{d'} \sin\tfrac{2}{3}d\pi\,.\,q^{2n},$$

$$\rho\,\mathrm{zc}\,\tfrac{2}{3}K = -\tan\tfrac{1}{3}\pi+4\Sigma_1^\infty.\,\Sigma_n (-1)^{d} \sin\tfrac{2}{3}d\pi\,.\,q^{2n}.$$

$$k'\rho\,\mathrm{sc}\,\tfrac{2}{3}K = \tan\tfrac{1}{3}\pi-4\Sigma_1^\infty.\,\Sigma_n (-1)^{d+d'} \sin\tfrac{2}{3}d\pi\,.\,q^{2n}.$$

It is evident that

$$\sin\tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3} \text{ when } d \text{ is of the form } 3k+1,$$

and $$= -\tfrac{1}{2}\sqrt{3} \quad ,, \quad ,, \quad 3k+2.$$

Therefore $$\Sigma_n \sin\tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,H(n),$$

and $$\rho\,\mathrm{zs}\,\tfrac{2}{3}K = 1/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty H(n)q^{2n}.$$

7. A different form of the value of $\Sigma_n \sin\tfrac{2}{3}d\pi$ will now be obtained in connection with the evaluation of

$$\Sigma_n(-1)^{d'}\sin\tfrac{2}{3}d\pi, \qquad \Sigma_n(-1)^{d}\sin\tfrac{2}{3}d\pi, \qquad \Sigma_n(-1)^{d+d'}\sin\tfrac{2}{3}d\pi.$$

Let $n = 2^i m\ (i > 0)$, and let $\delta_1, \delta_2, \dots$ be the divisors of m (which of course are all uneven). The system of divisors of n is therefore $\delta_1, \delta_2, \dots, 2\delta_1, 2\delta_2, \dots, \dots, 2^i\delta_1, 2^i\delta_2, \dots$. Now

$\sin\tfrac{2}{3}\delta\pi = \tfrac{1}{2}\sqrt{3}$ or $-\tfrac{1}{2}\sqrt{3}$ according as δ is of the form $6k+1$ or $6k+5$,

$\sin\tfrac{4}{3}\delta\pi = -\tfrac{1}{2}\sqrt{3}$ or $\tfrac{1}{2}\sqrt{3}$,, ,, ,, ,,

and, in general,

$\sin\tfrac{2}{3}2^r\delta\pi = (-1)^r\,\tfrac{1}{2}\sqrt{3}$ or $-(-1)^r\,\tfrac{1}{2}\sqrt{3}$

according as δ is of the form $6k+1$ or $6k+5$.

Thus
$$\Sigma_n \sin \tfrac{2}{3}\delta\pi = \tfrac{1}{2}\sqrt{3}\, H_1(n),$$
$$\Sigma_n \sin \tfrac{2}{3}2\delta\pi = -\tfrac{1}{2}\sqrt{3}\, H_1(n),$$
$$\dots \quad \dots \quad \dots \quad \dots$$
$$\Sigma_n \sin \tfrac{2}{3}2^i\delta\pi = (-1)^i \tfrac{1}{2}\sqrt{3}\, H_1(n).$$

The even or uneven characters of d, d', $d+d'$, according to the different forms of d, are shown in the following table, in which $n = 2^i m$ and $i > 0$.

d	d'	$d+d'$
$\delta_1, \delta_2, \dots$ (uneven)	even	uneven
$2\delta_1, 2\delta_2, \dots$ (even)	even	even
$2^2\delta_1, 2^2\delta_2, \dots$ (even)	even	even
... ...	... ...	... ...
$2^i\delta_1, 2^i\delta_2, \dots$ (even)	uneven	uneven

It follows therefore that, if $i > 0$,
$$\Sigma_n \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{1+(-1)+(-1)^2+\dots+(-1)^i\}\, H_1(n)$$
$$= \tfrac{1}{2}\sqrt{3}\,\frac{1+(-1)^i}{2} H_1(n),$$
$$\Sigma_n(-1)^{d'} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{1+(-1)+(-1)^2+\dots-(-1)^i\}\, H_1(n)$$
$$= \tfrac{1}{2}\sqrt{3}\,\frac{1-3(-1)^i}{2} H_1(n),$$
$$\Sigma_n(-1)^{d} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{-1+(-1)+(-1)^2+\dots+(-1)^i\}\, H_1(n)$$
$$= \tfrac{1}{2}\sqrt{3}\,\frac{-3+(-1)^i}{2} H_1(n),$$
$$\Sigma_n(-1)^{d+d'} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\,\{-1+(-1)+(-1)^2+\dots-(-1)^i\}\, H_1(n)$$
$$= -\tfrac{1}{2}\sqrt{3}\,\frac{3+3(-1)^i}{2} H_1(n).$$

When n is uneven, *i.e.* when $i = 0$, the first three formulæ still hold good, but in place of the last we have
$$\Sigma_n(-1)^{d+d} \sin \tfrac{2}{3}d\pi = \tfrac{1}{2}\sqrt{3}\, H_1(n).$$

The four expansion formulæ therefore become, if $n = 2^i m$,

$$\rho \operatorname{zs} \tfrac{2}{3}K = 1/\sqrt{3} + \sqrt{3}\,\Sigma_1^\infty \{1+(-1)^i\} H_1(n) q^{2n},$$

$$\rho \operatorname{cs} \tfrac{2}{3}K = 1/\sqrt{3} + \sqrt{3}\,\Sigma_1^\infty \{1-3(-1)^i\} H_1(n) q^{2n},$$

$$\rho \operatorname{zc} \tfrac{2}{3}K = -\sqrt{3} - \sqrt{3}\,\Sigma_1 \{3-(-1)^i\} H_1(n) q^{2n},$$

$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3}+3\sqrt{3}\,\Sigma_1^\infty \{1+(-1)^i\} H_1(n) q^{2n} \quad (\text{if } i > 0$$

$$= \sqrt{3}-2\sqrt{3}\,\Sigma_1^\infty H_1(n) q^{2n} \quad (\text{if } i = 0).$$

8. The coefficients in the expansions of the sixteen functions have therefore been expressed by means of two arithmetical functions $E(n)$ and $H_1(n)$, but in connection with the latter the factor $(-1)^i$, depending upon the structure of n, occurs. It will now be shown that this factor may be got rid of, and that all the coefficients can be expressed by means of the two functions $E(n)$ and $H(n)$.

It was shown in § 6 that

$$\Sigma_n \sin \tfrac{2}{3} d\pi = \tfrac{1}{2}\sqrt{3}\, H(n),$$

and in § 7 that, if $n = 2^i m$,

$$\Sigma_n \sin \tfrac{2}{3} d\pi = \tfrac{1}{2}\sqrt{3}\, \frac{1+(-1)^i}{2} H_1(n).$$

Comparing these two results, we see that

$$\{1+(-1)^i\} H_1(n) = 2H(n).$$

Now, if n is even,

$H(n) =$ the number of divisors of n of the forms $6k+1$ and $6k+4$

$-$ " " " $6k+5$ and $6k+2$

$= H_1(n) - H(\tfrac{1}{2}n)$;

for the divisors of n of the forms $6k+4$, $6k+2$ are the doubles of the divisors of $\tfrac{1}{2}n$ of the forms $3k+2$, $3k+1$ respectively.

We have thus obtained the formulæ

$$H_1(n) = H(n) + H(\tfrac{1}{2}n),$$

$$(-1)^i H_1(n) = 2H(n) - H_1(n) = H(n) - H(\tfrac{1}{2}n),^*$$

which still hold good when n is uneven if we define $H(r)$ to be zero when r is fractional.

* These equations show that $H_1(n) = H(n)$ or $H(\tfrac{1}{2}n)$ according as i is even or uneven.

9. Expressing in terms of H the coefficients which have been obtained in terms of $H_1(n)$, we have

$$\Sigma_n \sin \tfrac{2}{3}\delta'\pi = \tfrac{1}{2}\surd 3(-1)^i H_1(n) = \tfrac{1}{2}\surd 3 \{H(n) - H(\tfrac{1}{2}n)\} \quad (\S\ 4),$$

$$\Sigma_n \sin \tfrac{1}{3}\delta\pi = \tfrac{1}{2}\surd 3\, H_1(n) = \tfrac{1}{2}\surd 3 \{H(n) + H(\tfrac{1}{2}n)\} \quad (\S\ 5),$$

$$\Sigma_n \sin \tfrac{2}{3}d\pi = \frac{\surd 3}{2}\,\frac{1+(-1)^i}{2} H_1(n) = \tfrac{1}{2}\surd 3\, H(n),$$

$$\Sigma_n (-1)^{d'} \sin \tfrac{2}{3}d\pi = \frac{\surd 3}{2}\,\frac{1-3(-1)^i}{2} H_1(n) = -\frac{\surd 3}{2}\{H(n) - 2H(\tfrac{1}{2}n)\},$$

$$\Sigma_n (-1)^{d} \sin \tfrac{2}{3}d\pi = \frac{\surd 3}{2}\,\frac{-3+(-1)^i}{2} H_1(n) = -\frac{\surd 3}{2}\{H(n) + 2H(\tfrac{1}{2}n)\},$$

$$\Sigma_n (-1)^{d+d'} \sin \tfrac{2}{3}d\pi = -\frac{\surd 3}{2}\,\frac{3+3(-1)^i}{2} H_1(n) = -\frac{3\surd 3}{2} H(n) \quad (\text{if } i > 0),$$

and

$$= \tfrac{1}{2}\surd 3\, H(n) \quad (\text{if } i = 0).$$

10. Collecting the expansions, the six which depend upon E are

$$k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^\infty \{E(m) + 3E(\tfrac{1}{3}m)\}\, q^{\frac{1}{2}m},$$

$$k\rho \operatorname{cn} \tfrac{2}{3}K = 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \{E(m) + 3E(\tfrac{1}{3}m)\}\, q^{\frac{1}{2}m},$$

$$\rho \operatorname{dn} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty \{E(n) - 3E(\tfrac{1}{3}n)\}\, q^n,$$

$$k'\rho \operatorname{nd} \tfrac{2}{3}K = 1 - 2\Sigma_1^\infty (-1)^n \{E(n) - 3E(\tfrac{1}{3}n)\}\, q^n,$$

$$\rho \operatorname{dc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty \{E(n) + 3E(\tfrac{1}{3}n)\}\, q^n,$$

$$k'\rho \operatorname{nc} \tfrac{2}{3}K = 2 + 2\Sigma_1^\infty (-1)^n \{E(n) + 3E(\tfrac{1}{3}n)\}\, q^n,$$

and the ten which depend upon H are

$$k\rho \operatorname{sn} \tfrac{2}{3}K = 2\surd 3\, \Sigma_1^\infty H(m) q^{\frac{1}{2}m},$$

$$kk'\rho \operatorname{sd} \tfrac{2}{3}K = 2\surd 3\, \Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} H(m) q^{\frac{1}{2}m},$$

$$\rho \operatorname{zn} \tfrac{2}{3}K = 2\surd 3\, \Sigma_1^\infty \{H(n) - H(\tfrac{1}{2}n)\}\, q^n,$$

$$\rho \operatorname{zd} \tfrac{2}{3}K = 2\surd 3\, \Sigma_1^\infty (-1)^n \{H(n) - H(\tfrac{1}{2}n)\}\, q^n,$$

$$\rho \operatorname{ns} \tfrac{2}{3}K = 2/\surd 3 + 2\surd 3\, \Sigma_1^\infty \{H(n) + H(\tfrac{1}{2}n)\}\, q^n,$$

$$\rho \operatorname{ds} \tfrac{2}{3}K = 2/\surd 3 + 2\surd 3\, \Sigma_1^\infty (-1)^n \{H(n) + H(\tfrac{1}{2}n)\}\, q^n,$$

$$\rho \operatorname{zs} \tfrac{2}{3}K = 1/\surd 3 + 2\surd 3\, \Sigma_1^\infty H(n) q^{2n},$$

$$\rho \operatorname{cs} \tfrac{2}{3}K = 1/\surd 3 - 2\surd 3\, \Sigma_1^\infty \{H(n) - 2H(\tfrac{1}{2}n)\}\, q^{2n},$$

$$\rho \operatorname{zc} \tfrac{2}{3}K = -\surd 3 - 2\surd 3\, \Sigma_1^\infty \{H(n) + 2H(\tfrac{1}{2}n)\}\, q^{2n},$$

$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \surd 3 + 2\surd 3\, \Sigma_1^\infty \{1 + 2(-1)^n\} H(n) q^{2n}.$$

11. The formulæ in the E-group, which contains the expansions of the six even functions of $\frac{2}{3}K$, are the same as those given on p. 144 of the previous paper, except that by the use of the symbol $E(\frac{1}{3}n)$ two series are combined into one, *e.g.*, in the previous paper the first series was written

$$k\rho \operatorname{cd} \tfrac{2}{3}K = 2\Sigma_1^{\infty} E(m) q^{\frac{1}{3}m} + 6\Sigma_1^{\infty} E(m) q^{m}.$$

The coefficients in the ten expansions forming the H-group, which represent the uneven functions of $\frac{2}{3}K$, were originally expressed in the previous paper (pp. 144, 145) by means of six arithmetical functions $H(n)$, $H'(n)$, $H''(n)$, $H_1(n)$,* $I(n)$, $\mathrm{i}(n)$. These six functions were subsequently (p. 148) expressed in terms of $H(m)$ and $(-1)^i H(m)$, where $n = 2^i m$; and it was pointed out (p. 150) that the six functions could also be expressed in terms of H and H_1, so that the expansions of the sixteen functions involved only the three arithmetical functions E, H, H_1.

At that time I failed to notice the very simple formula

$$H_1(n) = H(n) + H(\tfrac{1}{2}n),$$

by means of which H_1 can be expressed in terms of H, so that (as shown in this paper) the ten expansions involve only a single function H, and can be expressed each by a single series if we adopt the convention that $H(r)$ is zero when r is fractional.

12. The following equations express in terms of the function H the arithmetical functions which were defined and used in the previous paper, and which on p. 148 of that paper were expressed in terms of $H(m)$ and $(-1)^i H(m)$,

$$\begin{aligned}
H_1(n) &= H(n) + H(\tfrac{1}{2}n),\\
H'(n) &= H(n) - H(\tfrac{1}{2}n),\\
H''(n) &= H(\tfrac{1}{2}n),\\
\mathrm{h}(n) &= H(n) - 2H(\tfrac{1}{2}n),\\
I(n) &= H(n) + 2H(\tfrac{1}{2}n),\\
I'(n) &= (-1)^{n-1} H(n) + H(\tfrac{1}{2}n),\\
I''(n) &= \{1 + (-1)^n\} H(n) + H(\tfrac{1}{2}n),\\
\mathrm{i}(n) &= -\{1 + 2(-1)^n\} H(n).
\end{aligned}$$

* In the previous paper $H_1(n)$ was denoted by $J(n)$. I have changed the notation because in subsequent papers I have used $J(n)$ to denote the excess of the number of divisors of n of the forms $8k+1$ and $8k+3$ over the number of those of the forms $8k+5$ and $8k+7$. This function in the previous paper (p. 163) was denoted by $T(n)$.

13. The quantity $H(\frac{1}{2}n)$ may be replaced by $H(2n)$ in all the formulæ, for, if n is uneven, both are zero, and, if n is even,

$$H(\tfrac{1}{2}n) = H(2^2 . \tfrac{1}{2}n) = H(2n).$$

The last eight of the H-expansions may therefore be written

$$\begin{aligned}
\rho \operatorname{zn} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty \{H(n) - H(2n)\}\, q^n, \\
\rho \operatorname{zd} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty (-1)^n \{H(n) - H(2n)\}\, q^n, \\
\rho \operatorname{ns} \tfrac{2}{3}K &= 2/\sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty \{H(n) + H(2n)\}\, q^n, \\
\rho \operatorname{ds} \tfrac{2}{3}K &= 2/\sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty (-1)^n \{H(n) + H(2n)\}\, q^n, \\
\rho \operatorname{zs} \tfrac{2}{3}K &= 1/\sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^{2n}, \\
\rho \operatorname{cs} \tfrac{2}{3}K &= 1/\sqrt{3} - 2\sqrt{3}\, \Sigma_1^\infty \{H(n) - 2H(2n)\}\, q^{2n}, \\
\rho \operatorname{zc} \tfrac{2}{3}K &= -\sqrt{3} - 2\sqrt{3}\, \Sigma_1^\infty \{H(n) + 2H(2n)\}\, q^{2n}, \\
k'\rho \operatorname{sc} \tfrac{2}{3}K &= \sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty \{1 + 2(-1)^n\}\, H(n)\, q^{2n}.
\end{aligned}$$

Whatever the value of n, either $H(n)$ or $H(2n)$ must be zero. Of course both may be zero.

Similarly, $E(\frac{1}{3}n)$ may be replaced by $E(3n)$, for, if n is not divisible by 3, both are zero, and, if n is divisible by 3,

$$E(\tfrac{1}{3}n) = E(3^2 . \tfrac{1}{3}n) = E(3n).$$

Thus the E-expansions may be written

$$\begin{aligned}
k\rho \operatorname{cd} \tfrac{2}{3}K &= 2\Sigma_1^\infty \{E(m) + 3E(3m)\}\, q^{\frac{1}{2}m}, \\
k\rho \operatorname{cn} \tfrac{2}{3}K &= 2\Sigma_1^\infty (-1)^{\frac{1}{2}(m-1)} \{E(m) + 3E(3m)\}\, q^{\frac{1}{2}m}, \\
\rho \operatorname{dn} \tfrac{2}{3}K &= 1 - 2\Sigma_1^\infty \{E(n) - 3E(3n)\}\, q^n, \\
k'\rho \operatorname{nd} \tfrac{2}{3}K &= 1 - 2\Sigma_1^\infty (-1)^n \{E(n) - 3E(3n)\}\, q^n, \\
\rho \operatorname{dc} \tfrac{2}{3}K &= 2 + 2\Sigma_1^\infty \{E(n) + 3E(3n)\}\, q^n, \\
k'\rho \operatorname{nc} \tfrac{2}{3}K &= 2 + 2\Sigma_1^\infty (-1)^n \{E(n) + 3E(3n)\}\, q^n.
\end{aligned}$$

Whatever the value of n, either $E(n)$ or $E(3n)$ must be zero. Of course both may be zero.

14. If only $H(n)$ be used, *i.e.* not $H(\frac{1}{2}n)$ or $H(2n)$, the last eight of the H-expansions may be expressed as follows:—

$$\begin{aligned}
\rho \operatorname{zn} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^n - 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^{2n}, \\
\rho \operatorname{zd} \tfrac{2}{3}K &= 2\sqrt{3}\, \Sigma_1^\infty (-1)^n H(n)\, q^n - 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^{2n}, \\
\rho \operatorname{ns} \tfrac{2}{3}K &= 2/\sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^n + 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^{2n}, \\
\rho \operatorname{ds} \tfrac{2}{3}K &= 2/\sqrt{3} + 2\sqrt{3}\, \Sigma_1^\infty (-1)^n H(n)\, q^n + 2\sqrt{3}\, \Sigma_1^\infty H(n)\, q^{2n},
\end{aligned}$$

$$\rho \operatorname{zs} \tfrac{2}{3}K = 1/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty H(n)q^{2n},$$
$$\rho \operatorname{cs} \tfrac{2}{3}K = 1/\sqrt{3}-2\sqrt{3}\,\Sigma_1^\infty H(n)q^{2n}+4\sqrt{3}\,\Sigma_1^\infty H(n)q^{4n},$$
$$\rho \operatorname{zc} \tfrac{2}{3}K = -\sqrt{3}-2\sqrt{3}\,\Sigma_1^\infty H(n)q^{2n}-4\sqrt{3}\,\Sigma_1^\infty H(n)q^{4n},$$
$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3}+6\sqrt{3}\,\Sigma_1^\infty H(n)q^{2n}-8\sqrt{3}\,\Sigma_1^\infty H(m)q^{2m}.$$

The last formula may also be written

$$k'\rho \operatorname{sc} \tfrac{2}{3}K = \sqrt{3}+6\sqrt{3}\,\Sigma_1^\infty H(n)q^{8n}-2\sqrt{3}\,\Sigma_1^\infty H(m)q^{2m}.$$

The following mode of expressing the first group may be noticed, as the even and uneven powers of q are separated:

$$\rho \operatorname{zn} \tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty \{H(2n)-H(n)\}\,q^{2n}+2\sqrt{3}\,\Sigma_1^\infty H(m)q^{m},$$
$$\rho \operatorname{zd} \tfrac{2}{3}K = 2\sqrt{3}\,\Sigma_1^\infty \{H(2n)-H(n)\}\,q^{2n}-2\sqrt{3}\,\Sigma_1^\infty H(m)q^{m},$$
$$\rho \operatorname{ns} \tfrac{2}{3}K = 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty \{H(2n)+H(n)\}\,q^{2n}+2\sqrt{3}\,\Sigma_1^\infty H(m)q^{m},$$
$$\rho \operatorname{ds} \tfrac{2}{3}K = 2/\sqrt{3}+2\sqrt{3}\,\Sigma_1^\infty \{H(2n)+H(n)\}\,q^{2n}-2\sqrt{3}\,\Sigma_1^\infty H(m)q^{m}.$$

15. The values of the elliptic and zeta functions for the argument $\tfrac{1}{3}K$ are deducible at once from those for the argument $\tfrac{2}{3}K$ by the formula

$$\operatorname{cd} \tfrac{2}{3}K = \operatorname{sn} \tfrac{1}{3}K, \quad \operatorname{cn} \tfrac{2}{3}K = k' \operatorname{sd} \tfrac{1}{3}K, \quad \operatorname{zc} \tfrac{2}{3}K = -\operatorname{zs} \tfrac{1}{3}K, \quad \ldots,$$

but in this paper I have preferred to express the results by means of the argument $\tfrac{2}{3}K$, instead of $\tfrac{1}{3}K$ as in the previous paper, because with the former argument the groups of formulæ are more regular, *e.g.*, when so expressed the six E-formulæ represent the even functions and the ten H-formulæ the uneven functions.

Many of the formulæ in the previous paper are improved by the change from $\tfrac{1}{3}K$ to $\tfrac{2}{3}K$, *e.g.*, the last three relations in § 22 (p. 152) become

$$\operatorname{cd} \tfrac{2}{3}K+\operatorname{cn} \tfrac{2}{3}K = 1,$$
$$\operatorname{nc} \tfrac{2}{3}K-\operatorname{nd} \tfrac{2}{3}K = 1,$$
$$\operatorname{dc} \tfrac{2}{3}K-\operatorname{dn} \tfrac{2}{3}K = 1,$$

and the six formulæ in § 23 (pp. 152, 153) represent the six even functions of $\tfrac{2}{3}K$. Also the three formulæ at the top of p. 151 represent $\operatorname{sn}^2 \tfrac{2}{3}K$, $k'^2 \operatorname{sd}^2 \tfrac{2}{3}K$, $k'^2 \operatorname{sc}^2 \tfrac{2}{3}K$.

16. By extending the convention that the function is zero when the argument is fractional from E and H to the arithmetical functions Δ', ζ, σ, ... we may combine into one the two series which occur in the expansions of the squared elliptic and zeta functions on p. 158 of the previous paper.

Selecting from each of the first three groups the expansions in which the constant term can be combined with the first of the two series, we have

$$k^2\rho^2 \operatorname{sn}^2 \tfrac{2}{3}K = 12\Sigma_1^\infty \{\Delta'(n)-3\Delta'(\tfrac{1}{3}n)\}\, q^n,$$

$$k^2k'^2\rho^2 \operatorname{sd}^2 \tfrac{2}{3}K = 12\Sigma_1^\infty (-1)^{n-1}\{\Delta'(n)-3\Delta'(\tfrac{1}{3}n)\}\, q^n,$$

$$k'^2\rho^2 \operatorname{sc}^2 \tfrac{2}{3}K = 3-12\Sigma_1^\infty \{\zeta(n)-3\zeta(\tfrac{1}{3}n)\}\, q^n.$$

In the fourth group a term in ρ^2 occurs in each of the expansions, *e.g.*,

$$\rho^2 \operatorname{ds}^2 \tfrac{2}{3}K+\tfrac{1}{3}(k^2-k'^2)\rho^2 = 1+12\Sigma_1^\infty \{\sigma(n)-3\sigma(\tfrac{1}{3}n)\}\, q^{2n}.$$

17. A table of the values of $E(n)$ up to $n = 1000$ was given in the *Proceedings* of this Society, Vol. xv., 1884, p. 106,* and tables of the same extent of $H(n)$, and of $J(n)$, *i.e.*, of the $T(n)$ of the previous paper, have been given in the *Messenger*, Vol. XXXI., 1901, pp. 64–72 and 82–91. The introductions prefixed to the latter two tables contain references to other papers in which the functions $H(n)$ and $J(n)$ are considered.

* Two errors in this table are pointed out in the *Messenger*, Vol. XXXI., p. 66, viz., the arguments 802 and 922 should not be omitted, for the values of $E(802)$ and $E(922)$ are each 2.

ON THE REDUCIBILITY OF COVARIANTS OF BINARY QUANTICS OF INFINITE ORDER

By P. W. WOOD.

[Received October 28th, 1904.—Read November 10th, 1904.]

THE present paper deals with the reducibility of covariants of unit degree in the coefficients of each of any number of binary quantics of infinite order: here the word "reducibility" is used, not in any conventional sense, but to imply the possibility of expressing the covariant as a sum of products of covariants of lower total degrees. The forms considered are, of course, all of weight at least equal to the minimum weight for irreducibility.

The problem for any covariant of degree $(\delta+1)$ is made to depend on the evaluation of an algebraical expression, which is the product of $(2^\delta-1)$ linear factors, and the investigation is ultimately independent of any considerations connected with invariant algebra: theoretically it is possible to determine, by the expansion of this product of order $(2^\delta-1)$, if any covariant is reducible; practically the expansion involves considerable labour. The method employed depends on the use of the linear partial differential operators introduced by the author in a previous paper.*

A complete investigation is given of all covariants of degree 4 and of all Jacobians of degree 5, the covariants being written in the form

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta};$$

and certain reducible covariants of this form of degree $(\delta+1)$ and weight $\geqslant (2^\delta-1)$, the maximum weight for irreducibility, are also determined.

The reducibility of individual covariants such as

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_{\delta+1})^{\lambda_\delta}$$

is completely investigated: it is shown that the necessary and sufficient condition that any such covariant is reducible is that the sum of κ of the indices $\lambda_1, \lambda_2, \dots, \lambda_\delta$ is less than $(2^\kappa-1)$, κ being any one of the numbers $1, 2, 3, \dots, \delta$.

A list of certain reducible forms is given at the end of the paper.

---

* Wood, "On the Irreducibility of Perpetuant Types," *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

1.

All the results in the present paper depend on the following known theorem in partial differential equations:—

Let $O_1, O_2, \ldots, O_\lambda$ be λ distinct linear partial differential operators in δ variables with constant coefficients, and let the most general solution of the equation $O_r(u) = 0$ be given by $u = f_r(u_r)$, where $r = 1, 2, 3, \ldots, \lambda$; then the most general solution of the equation

$$\{O_1 O_2 \ldots O_\lambda\} u = 0$$

is given by
$$u = \sum_{r=1}^{r=\lambda} f_r(u_r).$$

If two of the operators, say O_λ and $O_{\lambda-1}$, are identical (so that they differ only by a constant multiplier), then the most general solution of

$$\{O_1 O_2 \ldots O_{\lambda-1}^2\} u = 0$$

is given by
$$u = \sum_{r=1}^{r=\lambda-1} f_r(u_r) + \nu_{\lambda-1} f_{\lambda-1}(u_{\lambda-1}),$$

where $u = f_{\lambda-1}(u_{\lambda-1})$ is a solution of $O_{\lambda-1}(u) = 0$ and $\nu_{\lambda-1}$ is any linear function of the δ variables which is not itself a solution of $O_{\lambda-1}(u) = 0$.

If any number of operators become identical in sets, the result requires modification in a similar way.

2.

This result has an immediate application to covariants of binary forms of infinite order. It is known* that any quantic in the variables $x_1, x_2, \ldots, x_\delta$ which is annihilated by the operator

$$O \equiv \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}}$$

is, on substituting $x_s = \dfrac{(a_s a_{s+1})}{a_{s_2} a_{s+1_2}} \Pi$ $(s = 1, 2, \ldots, \delta)$, equal to a linear function of terms each of which is a product of covariants involving symbolical letters with the suffixes of the two sets

$$1, 2, 3, \ldots, r^{(1)},\ r^{(2)}+1, \ldots, r^{(3)},\ r^{(4)}+1, \ldots, r^{(5)},\ \ r^{(2\theta)}+1, \ldots, r^{(2\theta+1)}, \ldots;$$

$$r^{(1)}+1, \ldots, r^{(2)},\ r^{(3)}+1, \ldots, r^{(4)},\ r^{(5)}+1, \ldots, r^{(6)},\ r^{(2\theta+1)}+1, \ldots, r^{(2\theta+2)}, \ldots$$

* Wood, *loc. cit.*

respectively. Here $r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ are any ϵ of the suffixes $1, 2, 3, \ldots, \delta$, such that $r^{(1)} < r^{(2)} < \ldots < r^{(\epsilon)}$; $\Pi \equiv a_{1_z} a_{2_z} \ldots a_{\delta+1_z}$, $a_1, a_2, \ldots, a_{\delta+1}$ being the symbols of $(\delta+1)$ binary quantics of infinite order in the variables z_1, z_2; and the dotted lines ... in the two sets above are to be replaced by consecutive numbers, so that there is a break in the sequence whenever we arrive at a suffix number $r^{(\theta)}$ involved in the operator O. The operator O is completely determined by the suffixes $r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ involved; the number of such operators is therefore $(2^\delta - 1)$, and corresponding to each operator the suffixes of the symbols are divided into two sets as above.

From the result of § 1, it follows that any quantic in $x_1, x_2, \ldots, x_\delta$, which is annihilated by the successive application of the $(2^\delta - 1)$ operators like

$$O \equiv \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}},$$

is expressible as a sum of quantics every term of which, on substituting $\dfrac{(a_r a_{r+1})}{a_{r_z} a_{r+1_z}} \Pi$ for x_r $(r = 1, 2, \ldots, \delta)$, represents a covariant of total degree $(\delta+1)$, which is a product of covariants of lower total degrees. We therefore deduce that—

The necessary and sufficient condition that any covariant $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \ldots (a_\delta a_{\delta+1})^{\lambda_\delta}$ of total degree $(\delta+1)$ is reducible (*i.e.*, is expressible as a sum of covariants, each of which is a product of covariants of lower total degrees) is that the expression $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_\delta^{\lambda_\delta}$ should be annihilated by the successive application of the $(2^\delta - 1)$ operators like

$$O \equiv \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}},$$

$r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ being any ϵ of the suffixes $1, 2, \ldots, \delta$, such that $r^{(1)} < r^{(2)} < \ldots < r^{(\epsilon)}$.

3.

A necessary consequence of this is that, since any quantic in $x_1, x_2, \ldots, x_\delta$ of order less than $(2^\delta - 1)$ is annihilated by the successive application of these operators, any covariant of degree $(\delta+1)$ and weight less than $(2^\delta - 1)$ is reducible.

More generally, if any expression is to be regarded as "reducible," when it can be expressed in terms of forms of λ other classes, such

that to each of these λ classes corresponds a single linear differential operator (in the sense that any form annihilated by an operator necessarily belongs to the corresponding class, and conversely), then λ is the minimum weight of the expression for irreducibility. This point of view simplifies considerably much of the work involved in an earlier paper "On Perpetuant Syzygies."* The following example may be cited :—

We use the symbol $[a_{r_1} a_{r_2} \dots a_{r_l}]$ to denote any covariant involving all the symbolical letters $a_{r_1}, a_{r_2}, \dots, a_{r_l}$, and the symbol $[a_{r_1} a_{r_2} \dots a_{r_l}]'$ to denote any covariant involving all the symbolical letters $a_1, a_2, \dots, a_{\delta+1}$ except $a_{r_1}, a_{r_2}, \dots, a_{r_l}$; and we define any transvectant

$$\left([a_{r_1} a_{r_2} \dots a_{r_l}],\ [a_{r_1} a_{r_2} \dots a_{r_l}]'\right)^{\lambda}$$

as reducible, if $\lambda < \mu_{r_1, r_2, \dots, r_l}$, where $\mu_{r_1, r_2, \dots, r_l}$ is a number depending on our definition of reducibility: further, we suppose that a_1 always occurs in the covariant $[a_{r_1} a_{r_2} \dots a_{r_l}]'$, and that $r_1 < r_2 < \dots < r_l$. It is clear that the number of quantities μ is $(2^{\delta}-1)$. If we use σ_r to denote the sum of all the μ's whose first suffix is r, r being any one of the suffixes $2, 3, \dots, \delta$ (so that $\sigma_{\delta+1} = \mu_{\delta+1}$, $\sigma_{\delta} = \mu_{\delta}+\mu_{\delta, \delta+1}$), then an extension of Grace's perpetuant type theorem† was there obtained in the following form :—

The conditions that $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta} a_{\delta+1})^{\lambda_{\delta}}$ is irreducible, according to the definition which determines the quantities μ, are

$$\lambda_{\delta} \geqslant \sigma_{\delta+1}, \quad \lambda_{\delta-1} \geqslant \sigma_{\delta}, \quad \dots, \quad \lambda_1 \geqslant \sigma_2.$$

By means of § 1 we proceed to find a single condition for the irreducibility of

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_{\delta} a_{\delta+1})^{\lambda_{\delta}}$$

in the less general form

$$\lambda_1+\lambda_2+\dots+\lambda_{\delta} \geqslant \Sigma\mu.$$

---

* Young and Wood, *supra*.

It is shown, for instance, at some length that, in considering the minimum weight of a product $C_{\kappa} C_m$, which is to be irreducible as there defined, corresponding to every product $C_{\kappa'} C_{m'}$ following $C_{\kappa} C_m$ in a certain sequence, the minimum weight of $C_{\kappa} C_m$ for irreducibility is increased by unity; this follows at once from the statement above, since to every such product $C_{\kappa'} C_{m'}$ corresponds a definite operator such as O.

The method used in the earlier paper consisted in the actual application of Stroh's syzygies of degree 4, and required a demonstration that such syzygies as were applied were all independent.

† *Proc. London Math. Soc.*, Vol. XXXV.

For let the operator $\dfrac{\partial}{\partial x_{r^{(1)}}} - \dfrac{\partial}{\partial x_{r^{(2)}}} + \ldots + (-)^{s-1}\dfrac{\partial}{\partial x_{r^{(s)}}}$ be denoted by

$$O(r^{(2\theta)}+1 \ldots r^{(2\theta+1)}),$$

the reason for this notation being that any quantic in $x_1, x_2, \ldots, x_\delta$ annihilated by this operator is (by § 2) a product of covariants (writing the suffixes in place of the symbols a)

$$[1, 2, \ldots, r^{(1)}, r^{(2)}+1, \ldots, r^{(3)}, \ldots] \text{ and } [1, 2, \ldots, r^{(1)}, r^{(2)}+1, \ldots, r^{(3)}, \ldots]'.$$

Any quantic in $x_1, x_2, \ldots, x_\delta$ of order less than $\Sigma\mu$ is annihilated by repeated application of the $(2^\delta-1)$ operators O in such a way that the operator $O(r^{(2\theta)}+1 \ldots r^{(2\theta+1)})$ is applied exactly $\mu_{1, 2, \ldots, r^{(1)}, r^{(2)}+1, \ldots, r^{(3)}, \ldots}$ times, and the other $(2^\delta-2)$ operators are each applied the corresponding number of times, so that in all $\Sigma\mu$ linear operators are successively applied. Any such quantic is therefore expressible in the form

$$\Sigma\left\{\sum_m p_x^m f(u')\right\},$$

where the second Σ refers to the values

$$0, 1, 2, 3, \ldots, (\mu_{1, 2, \ldots, r^{(1)}, r^{(2)}, \ldots, r^{(3)}, \ldots}-1)$$

of m, $u = f(u')$ is the most general solution of the equation

$$\left\{O(r^{(2\theta)}+1 \ldots r^{(2\theta+1)})\right\} u = 0,$$

p_x^m is any quantic of order m in $x_1, x_2, \ldots, x_\delta$, which is not itself a solution of this equation, and the first Σ refers to the summation of terms derived similarly from each of the $(2^\delta-1)$ operators such as O.

It is clear that $p_x^m f(u')$ can, on substituting for the variables x, be replaced by transvectants

$$\left([1, 2, \ldots, r^{(1)}, r^{(2)}+1, \ldots, r^{(3)}, \ldots], [1, 2, \ldots, r^{(1)}, r^{(2)}+1, \ldots, r^{(3)}, \ldots]'\right)^{m'},$$
$$m' \leqslant m,$$

and that therefore all the terms included under the second Σ above are, in accordance with definition, reducible. The same argument applies to the terms corresponding to each of the other operators, and therefore any covariant of weight less than $\Sigma\mu$ is expressible in terms of forms defined as reducible.

The present method gives no information as to the actual mode of expressing any reducible form, and only indicates the minimum weight of the whole covariant without any reference to restrictions on individual indices; in a large number of cases, however, it is the determination of the total weight which is important.

4.

An immediate consequence of the result of § 2 is a short proof of the irreducibility of the type form

$$(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta} \equiv x_1^{\lambda_1} x_2^{\lambda_2} \dots x_\delta^{\lambda_\delta},$$

where $\lambda_1 \geqslant 2^{\delta-1}$, $\lambda_2 \geqslant 2^{\delta-2}$, ..., $\lambda_\delta \geqslant 1$.

If we apply the $(2^\delta - 1)$ operators successively to $x_1^{2^{\delta-1}} x_2^{2^{\delta-2}} \dots x_{\delta-1}^2 x_\delta$, the result is $(2^{\delta-1})!\,(2^{\delta-2})! \dots 2!\,1!$, as we can see by operating first with $\partial/\partial x_\delta$, then with $\partial/\partial x_{\delta-1}$ and $\partial/\partial x_{\delta-1} - \partial/\partial x_\delta$, that is, with $\partial^2/\partial x_{\delta-1}^2$, next with $\partial/\partial x_{\delta-2}$, $\partial/\partial x_{\delta-2} - \partial/\partial x_{\delta-1}$, $\partial/\partial x_{\delta-2} - \partial/\partial x_\delta$ and $\partial/\partial x_{\delta-2} - \partial/\partial x_{\delta-1} + \partial/\partial x_\delta$, that is, with $\partial^4/\partial x_{\delta-2}^4$, and so on. The result of operating on

$$x_1^{\lambda_1 - 2^{\delta-1}} x_2^{\lambda_2 - 2^{\delta-2}} \dots x_\delta^{\lambda_\delta - 1}\, x_1^{2^{\delta-1}} x_2^{2^{\delta-2}} \dots x_\delta$$

successively in this way must therefore be different from zero, for a linear partial differential operator with constant coefficients annihilates the product of two quantics only if it annihilates each of these quantics. It follows therefore that all type forms are actually irreducible.

5.

In general, the determination of the reducibility of any form of degree $(\delta+1)$ requires the evaluation of the product of operators

$$P_\delta \equiv \prod_{(2^\delta - 1)} \left\{ \frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \dots + (-)^{\epsilon-1} \frac{\partial}{\partial x_{r^{(\epsilon)}}} \right\};$$

any form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots a_\delta a_{\delta+1})^{\lambda_\delta}$ is clearly reducible if, and only if, P_δ contains *no* term

$$\left(\frac{\partial}{\partial x_1}\right)^{\mu_1} \left(\frac{\partial}{\partial x_2}\right)^{\mu_2} \dots \left(\frac{\partial}{\partial x_\delta}\right)^{\mu_\delta},$$

for which all the following inequalities hold:—

$$\lambda_1 \geqslant \mu_1, \quad \lambda_2 \geqslant \mu_2, \quad \dots, \quad \lambda_\delta \geqslant \mu_\delta.$$

The direct evaluation of P_δ for large values of δ involves considerable labour. The important point for our purpose is to determine what terms of the expression are absent.

Certain special cases of reducible forms are easily found. Thus the number of operators involving κ of the variables $x_1, x_2, \dots, x_\delta$ and no others is $(2^\kappa - 1)$, and therefore:—

If in the form $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ the sum of any κ indices is less than $(2^\kappa - 1)$ for any value of κ, then the form is reducible.

This result has been previously given by Grace,* whose proof depends on the use of generalized transvectants.

* "Extension of Two Theorems on Covariants," *Proc. London Math. Soc.*, Ser. 2, Vol. 1.

6. *Covariants of Degree* 4.

We have, putting $z_s = \partial/\partial x_s$, $s = 1, 2, 3$,

$$P_3 \equiv \prod_7 \left\{\frac{\partial}{\partial x_{r(1)}} - \frac{\partial}{\partial x_{r(2)}} + \ldots + (-)^{s-1}\frac{\partial}{\partial x_{r(s)}}\right\}$$

$$\equiv z_1 z_2 z_3 (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)(z_1 - z_2 + z_3)$$

$$\equiv z_1^4 z_2^2 z_3 - z_1^4 z_2 z_3^2 - 2z_1^3 z_2^3 z_3 + 2z_1^3 z_2^2 z_3^2 + z_1^2 z_2^4 z_3 - 2z_1^2 z_2^2 z_3^3 + z_1^2 z_2 z_3^4$$
$$- z_1 z_2^4 z_3^2 + 2z_1 z_2^3 z_3^3 - z_1 z_2^2 z_3^4.$$

Hence the only reducible covariants of degree 4, excluding those found in § 5, are the following:—

$$(a_1 a_2)^3 (a_2 a_3)(a_3 a_4)^3,$$
$$(a_1 a_2)^2 (a_2 a_3)^3 (a_3 a_4)^2,$$

each of which is of weight 7, the minimum weight for irreducibility.

7.

We shall next discuss completely Jacobians of degree 5. The general case of degree 5, involving as it does the evaluation of the continued product of 15 linear factors, requires considerable labour for a complete discussion. The Jacobians to be considered are

(i.) $(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu (a_4 a_5)$, (iii.) $(a_1 a_2)^\lambda (a_2 a_3)(a_3 a_4)^\mu (a_4 a_5)^\nu$,

(ii.) $(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)(a_4 a_5)^\nu$, (iv.) $(a_1 a_2)(a_2 a_3)^\lambda (a_3 a_4)^\mu (a_4 a_5)^\nu$;

and the forms (i.) and (iv.), (ii.) and (iii.) are for our purpose indistinguishable.

8. *Jacobians* $(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu (a_4 a_5) \equiv x_1^\lambda x_2^\mu x_3^\nu x_4$.

Operate first with $\partial/\partial x_4$; then the form is or is not reducible according as $x_1^\lambda x_2^\mu x_3^\nu$ is or is not annihilated by

$$\{z_1 z_2 z_3 (z_1 - z_2)(z_1 - z_3)(z_2 - z_3)(z_1 - z_2 + z_3)\}^2;$$

for the variable z_4 may be omitted, since x_4 is absent.

The expansion of this expression is given by the scheme:—

Indices	8.4.2	8.3.3	8.2.4	7.5.2	7.4.3	7.3.4	7.2.5	6.6.2	6.5.3	6.4.4	6.3.5	6.2.6
Coefficients...	1	−2	1	−4	8	−4	0	6	−10	0	6	−2
Indices	5.7.2	5.6.3	5.5.4	5.4.5	5.3.6	5.2.7	4.8.2	4.7.3	4.6.4	4.5.5	4.4.6	4.3.7
Coefficients...	−4	2	14	−18	6	0	1	4	−16	14	0	−4
Indices	4.2.8	3.8.3	3.7.4	3.6.5	3.5.6	3.4.7	3.3.8	2.8.4	2.7.5	2.6.6	2.5.7	2.4.8
Coefficients...	1	−2	4	2	−10	8	−2	1	−4	6	−4	1

Here the upper rows give the indices of z_1, z_2, z_3 respectively, and the lower rows give the corresponding coefficients.

From this expansion we see that the only reducible Jacobians

$$\left((a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu,\ a_5\right)$$

are, excluding those found in § 5, the four following :—

$\lambda =$	7	5	6	4
$\mu =$	2	2	4	4
$\nu =$	5	7	4	6

These are all of weight 15, the minimum weight for irreducibility : reducible Jacobian forms of this nature of higher weight are included among those of § 5.

9. *Jacobians* $(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)(a_4 a_5)^\nu \equiv x_1^\lambda x_2^\mu x_3 x_4^\nu$.

Operate first with $\partial/\partial x_3$; then the form is or is not reducible according as $x_1^\lambda x_2^\mu x_4^\nu$ is or is not annihilated by

$$z_1^2 z_2^2 z_4^2 (z_1 - z_2)^2 (z_1^2 - z_4^2)(z_2^2 - z_4^2)\left\{(z_1 - z_2)^2 - z_4^2\right\}.$$

The expansion of this expression is given by the scheme :—

Indices	8.4.2	8.3.3	8.2.4	7.5.2	7.4.3	7.3.4	7.2.5	6.6.2	6.5.3	6.4.4	6.3.5	6.2.6
Coefficients...	1	0	−1	−4	0	4	0	6	0	−8	0	2
Indices	5.7.2	5.6.3	5.5.4	5.4.5	5.3.6	5.2.7	4.8.2	4.7.3	4.6.4	4.5.5	4.4.6	4.3.7
Coefficients...	−4	0	10	0	−6	0	1	0	−8	0	8	0
Indices	4.2.8	3.8.3	3.7.4	3.6.5	3.5.6	3.4.7	3.3.8	2.8.4	2.7.5	2.6.6	2.5.7	2.4.8
Coefficients...	−1	0	4	0	6	0	2	−1	0	2	0	−1

Here the upper rows give the indices of z_1, z_2, z_4 respectively, and the lower rows give the corresponding coefficients.

From this expansion we see that the only reducible Jacobians

$$\left((a_1 a_2)^\lambda (a_2 a_3)^\mu,\ (a_4 a_5)^\nu\right)$$

are, excluding those defined in § 5, those for which the total weight is 15

For in this case the expression

$$x_{r(1)}^{\lambda_{r(1)}} x_{r(2)}^{\lambda_{r(2)}} \dots x_{r(\epsilon)}^{\lambda_{r(\epsilon)}}$$

is annihilated by the successive application of the $(2^{\epsilon}-1)$ operators involving only the letters $x_{r(1)}, x_{r(2)}, \dots, x_{r(\epsilon)}$, and therefore the expression

$$x_1^{\lambda_1} x_2^{\lambda_2} \dots x_\delta^{\lambda_\delta}$$

must be annihilated by the successive application of all the $(2^{\delta}-1)$ operators. This result enables us to construct any number of reducible forms of any degree, which are not included among the forms of §§ 5, 11. Thus we know that $(a_1 a_2)^2 (a_2 a_3)^3 (a_3 a_4)^2$ is reducible, and therefore each of the forms

$$(a_1 a_2)^{\lambda} (a_2 a_3)^2 (a_3 a_4)^3 (a_4 a_5)^2,$$

$$(a_1 a_2)^2 (a_2 a_3)^{\lambda} (a_3 a_4)^3 (a_4 a_5)^2$$

of degree 5 is reducible whatever value λ may have: and from any reducible form given in § 11 we can construct reducible forms of higher degrees by interpolating symbolical determinants.

13.

IV. We shall next consider covariants $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ of degree $(\delta+1)$ and weight $(2^{\delta}-1)$, which are such that the sum of κ of the indices is equal to $(2^{\kappa}-1)$.

Take first the case where κ is unity, and the single index λ_ϵ is also unity. Operation with $\partial/\partial x_\epsilon$ removes the variable x_ϵ from $x_1^{\lambda_1} \dots x_\epsilon \dots x_\delta^{\lambda_\delta}$. We are therefore only concerned with the effect of the remaining operators on

$$x_1^{\lambda_1} \dots x_{\epsilon-1}^{\lambda_{\epsilon-1}} x_{\epsilon+1}^{\lambda_{\epsilon+1}} \dots x_\delta^{\lambda_\delta}.$$

Denoting by O any operator involving only the variables $x_1, x_2, \dots, x_{\epsilon-1}$, and by O' any operator involving only the variables $x_{\epsilon+1}, x_{\epsilon+2}, \dots, x_\delta$, we are concerned with operators of the types

$$O,\ O-\frac{\partial}{\partial x_\epsilon},\ \frac{\partial}{\partial x_\epsilon}-O',\ O',\ O-O',\ O-\frac{\partial}{\partial x_\epsilon}+O';$$

wherein we may put $\partial/\partial x_\epsilon$ zero, since the variable x_ϵ is absent from the expression we are operating upon; the remaining operators therefore combine in pairs, and their product is

$$\Pi\left\{O^2 . O'^2 (O^2-O'^2)\right\};$$

since there are $(2^{\epsilon-1}-1)$ operators O, and $(2^{\delta-\epsilon}-1)$ operators O', the order of this product is

$$2\{2^{\epsilon-1}-1+2^{\delta-\epsilon}-1+(2^{\epsilon-1}-1)(2^{\delta-\epsilon}-1)\} = 2\{2^{\epsilon-1}.2^{\delta-\epsilon}-1\} = 2^{\delta}-2,$$

as it should be.

Now $(\lambda_1+\lambda_2+\ldots+\lambda_{\epsilon-1})+(\lambda_{\epsilon+1}+\lambda_{\epsilon+2}+\ldots+\lambda_\delta) = (2^\delta-2)$,

and it is at once obvious that, if either, and therefore each, of $(\lambda_1+\lambda_2+\ldots+\lambda_{\epsilon-1})$ and $(\lambda_{\epsilon+1}+\lambda_{\epsilon+2}+\ldots+\lambda_\delta)$ is odd, then, since the operators O and O' affecting the respective portions occur only in the forms O^2 and O'^2, $x_1^{\lambda_1}\ldots x_{\epsilon-1}^{\lambda_{\epsilon-1}}x_{\epsilon+1}^{\lambda_{\epsilon+1}}\ldots x_\delta^{\lambda_\delta}$ is annihilated by the operators, and therefore the original form is reducible. Hence*

> Any Jacobian $(C_\epsilon, C_{\delta-\epsilon})$ of degree δ and weight $(2^{\delta-1}-1)$ is reducible, if either of C_ϵ, $C_{\delta-\epsilon}$, and therefore the other, is of odd weight.

The reducible Jacobians of degree 5, found in § 9, are a special case of this more general result.

14.

V. Covariants $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\ldots(a_\delta a_{\delta+1})^{\lambda}$ of degree $(\delta+1)$ and weight $(2^\delta-1)$ having the sum of κ indices equal to $(2^\kappa-1)$ may be treated similarly. We first operate with the $(2^\kappa-1)$ operators involving only those variables the sum of whose indices is equal to $(2^\kappa-1)$; if the result vanishes then, as in § 12, the whole covariant is reducible. If the result does not vanish, let the remaining variables consist of $(\theta+1)$ sets of consecutive variables (where $\theta \leqslant \epsilon$), and suppose $O_1, O_2, \ldots, O_{\theta+1}$ are typical operators involving the variables of those sets alone, respectively; thus, any operator O_1 involves only the variables preceding the first of the variables already removed. Any operator may, since we may neglect the variables already removed, be written in the form

$$O_{r^{(1)}} \pm O_{r^{(2)}} \pm \ldots \pm O_{r^{(\epsilon)}},$$

where $r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ are any ϵ of the suffixes $1, 2, \ldots, (\theta+1)$. Now between the sets of variables corresponding to the operators $O_{r^{(1)}}$ and $O_{r^{(2)}}$ suppose the variables $x_u, x_{u+1}, \ldots, x_{v-1}, x_v$ have been removed. The

* The Jacobian (C_κ, C'_κ), where each of C_κ, C'_κ is of weight $2^{2\kappa-2}-1$, mentioned by Young and Wood (*loc. cit.*, § 25), is a special case of the above, and is, as was supposed, actually reducible.

original operators were all of the nature

$$O_{r^{(1)}}-O_{r^{(2)}}, \qquad O_{r^{(1)}}-\frac{\partial}{\partial x_u}+O_{r^{(2)}},$$

$$O_{r^{(1)}}-\frac{\partial}{\partial x_u}+\frac{\partial}{\partial x_{u+1}}-O_{r^{(2)}}, \qquad O_{r^{(1)}}-\frac{\partial}{\partial x_u}+\frac{\partial}{\partial x_{u+1}}-\frac{\partial}{\partial x_{u+2}}+O_{r^{(2)}},$$

$$\ldots \quad \ldots \quad \ldots \qquad \ldots \quad \ldots \quad \ldots \quad \ldots$$

Hence, in this way we get the operator $O_{r^{(1)}}-O_{r^{(2)}}$ occurring $(2^{v-u}-1)$ times, and also the operator $O_{r^{(1)}}+O_{r^{(2)}}$ occurring $(2^{v-u}-1)$ times; in the same way it is easily seen that each of the $2^{\epsilon-1}$ operators

$$O_{r^{(1)}} \pm O_{r^{(2)}} \pm O_{r^{(3)}} \pm \ldots \pm O_{r^{(\epsilon)}}$$

occurs the same number of times, whatever be the arrangement of the signs $+$ and $-$; also the product

$$\Pi\,(O_{r^{(1)}} \pm O_{r^{(2)}} \pm \ldots \pm O_{r^{(\epsilon)}}),$$

where we take all the $2^{\epsilon-1}$ possible arrangements of sign, involves only

$$O^2_{r^{(1)}},\ O^2_{r^{(2)}},\ \ldots,\ O^2_{r^{(\epsilon)}}.$$

Hence in general the product of operators operating on the $(\theta+1)$ sets of letters remaining involves only $O_1^2, O_2^2, \ldots, O_{\theta+1}^2$: the order of this product is of course the same as the sum of the indices remaining; if therefore the sum of the indices in any set is odd, then the original form is annihilated by the successive application of the $(2^\delta-1)$ operators, and must therefore be reducible. Hence

> If in the covariant $(a_1a_2)^{\lambda_1}(a_2a_3)^{\lambda_2}\ldots(a_\delta a_{\delta+1})^{\lambda_\delta}$ of degree $(\delta+1)$ and weight $(2^\delta-1)$ the sum of κ of the indices is $(2^\kappa-1)$ and the removal of the κ corresponding determinantal factors leaves us with groups of consecutive letters, then, if the weight of the factors involving the letters of any single group is odd, the covariant is reducible: this condition is sufficient but not necessary.

Thus the covariant $(a_1a_2)^9(a_2a_3)^4(a_3a_4)^2(a_4a_5)^7(a_5a_6)$ is reducible; for the removal of the three factors $(a_2a_3)^4$, $(a_3a_4)^2$, (a_5a_6), the sum of whose indices is 7, leaves us with the groups a_1a_2 and a_4a_5 of weights 9 and 7 respectively.

The determination of the necessary and sufficient conditions for the reducibility of forms of this kind (*i.e.*, those for which the sum of κ indices is equal to $2^\kappa-1$) requires the evaluation of products, which are much simpler than the corresponding products P_δ, but, nevertheless, offer considerable difficulty.

15.

The criteria hitherto found for the reducibility of any form all apply, with the exception of those of §§ 5, 12, only to forms of degree $(\delta+1)$ and of weight $(2^\delta-1)$, the minimum weight for irreducibility, the covariant being always written in the form

$$(a_1 a_2)^{\lambda_1}(a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}.$$

It should be remarked that (by § 5) a form $(a_1 a_2)^{\lambda_1}(a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ is certainly irreducible if there is any irreducible covariant

$$(a_1 a_2)^{\mu_1}(a_2 a_3)^{\mu_2} \dots (a_\delta a_{\delta+1})^{\mu_\delta}$$

such that $$\lambda_1 \geqslant \mu_1, \quad \lambda_2 \geqslant \mu_2, \quad \dots, \quad \lambda_\delta \geqslant \mu_\delta.$$

If we are given any covariant $(a_1 a_2)^{\lambda_1}(a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ of degree $(\delta+1)$ and of weight $(2^\delta-1+\phi)$, which is irreducible, we can theoretically determine the linear function of type forms, to which the covariant is, neglecting product forms, equivalent. The number of type forms of this weight is $\binom{\phi+\delta-1}{\delta-1}$.

We assume $x_1^{\lambda_1}x_2^{\lambda_2}\dots x_\delta^{\lambda_\delta} = x_1^{2^{\delta-1}}x_2^{2^{\delta-2}}\dots x_{\delta-1}^2 x_\delta \cdot p_x^\phi + R,$

where p_x^ϕ is a quantic of order ϕ in the variables $x_1, x_2, \dots, x_\delta$, and R is a sum of terms giving rise to product forms. Operate on both sides with the $(2^\delta-1)$ operators like

$$\frac{\partial}{\partial x_{r^{(1)}}} - \frac{\partial}{\partial x_{r^{(2)}}} + \dots + (-)^{\epsilon-1}\frac{\partial}{\partial x_{r^{(\epsilon)}}}:$$

the terms R disappear, and in the resulting expression of order ϕ we can, by equating coefficients, obtain the $\binom{\phi+\delta-1}{\delta-1}$ equations necessary to determine the coefficients of p_x^ϕ, and so find the type forms in terms of which the covariant is expressible: these equations for determining the constants cannot be inconsistent, since we know by Grace's perpetuant type theorem that all covariants admit of expression in this way; if, conversely, we could prove in any manner that these equations for the $\binom{\phi+\delta-1}{\delta-1}$ constants are not inconsistent, we should have an alternative proof of Grace's theorem.

Thus consider the covariant of degree 4 $(a_1 a_2)^3(a_2 a_3)^2(a_3 a_4)^3$; assume

$$x_1^3 x_2^2 x_3^3 \equiv x_1^4 x_2^2 x_3 (p_1 x_1 + p_2 x_2 + p_3 x_3) + R;$$

operating on both sides by means of the expansion of § 6, we have on

reduction $3(x_3-x_1) = 5p_1x_1+p_2(3x_2-6x_1)+p_3(2x_3-2x_2+4x_1)$,

which gives us $\quad p_1 = -\frac{3}{5}, \quad p_2 = 1, \quad p_3 = \frac{3}{2}$.

Hence

$$(a_1a_2)^3(a_2a_3)^2(a_3a_4)^3 \equiv -\tfrac{3}{5}(a_1a_2)^5(a_2a_3)^2(a_3a_4)+(a_1a_2)^4(a_2a_3)^3(a_3a_4)$$
$$+\tfrac{3}{2}(a_1a_2)^4(a_2a_3)^2(a_3a_4)^2+ \text{ product forms.}$$

16.

The remainder of the paper is devoted to the discussion of the reducibility of covariants

$$(a_1a_2)^{\lambda_1}(a_1a_3)^{\lambda_2}\dots(a_1a_{\delta+1})^{\lambda_\delta},$$

wherein a particular letter a_1 occurs in every determinantal factor. The results, though simpler, are of less value, since the previous notation is more in conformity with a possible treatment of forms of finite order.

Covariants of degree $(\delta+1)$ written in the form
$(a_1a_2)^{\lambda_1}(a_1a_3)^{\lambda_2}\dots(a_1a_{\delta+1})^{\lambda_\delta}$.

It follows from § 2 that $(a_1a_2)^{\lambda_1}(a_1a_3)^{\lambda_2}\dots(a_1a_{\delta+1})^{\lambda_\delta}$ is reducible, if, and only if,

$$x_1^{\lambda_1}(x_1+x_2)^{\lambda_2}\dots(x_1+x_2+\dots+x_\delta)^{\lambda_\delta}$$

is annihilated by the successive application of the $(2^\delta-1)$ operators like

$$\frac{\partial}{\partial x_{r^{(1)}}}-\frac{\partial}{\partial x_{r^{(2)}}}+\dots+(-)^{\epsilon-1}\frac{\partial}{\partial x_{r^{(\epsilon)}}};$$

but our investigations are considerably simplified if we choose new variables y, given by

$$y_r = \frac{(a_1a_{r+1})}{a_{1_z}a_{r+1_z}}\Pi, \quad r = 1, 2, \dots, \delta;$$

so that $\quad y_r = x_1+x_2+\dots+x_r,$

and therefore $\quad \dfrac{\partial}{\partial x_r} = \dfrac{\partial}{\partial y_r}+\dfrac{\partial}{\partial y_{r+1}}+\dots+\dfrac{\partial}{\partial y_\delta}$ $\Bigg\}\ r = 1, 2, \dots, \delta.$

The typical operator $\dfrac{\partial}{\partial x_{r^{(1)}}}-\dfrac{\partial}{\partial x_{r^{(2)}}}+\dots+(-)^{\epsilon-1}\dfrac{\partial}{\partial x_{r^{(\epsilon)}}}$ becomes

$$\sum_{s=r^{(1)}}^{s=\delta}\frac{\partial}{\partial y_s}-\sum_{s=r^{(2)}}^{s=\delta}\frac{\partial}{\partial y_s}+\dots+(-)^{\epsilon-1}\sum_{s=r^{(\epsilon)}}^{s=\delta}\frac{\partial}{\partial y_s}$$

$$=\sum_{s=r^{(1)}}^{s=r^{(2)}-1}\frac{\partial}{\partial y_s}+\sum_{s=r^{(3)}}^{s=r^{(4)}-1}\frac{\partial}{\partial y_s}+\dots+\sum_{s=r^{(\epsilon-1)}}^{s=r^{(\epsilon)}-1}\frac{\partial}{\partial y_s}, \qquad \text{if } \epsilon \text{ is even,}$$

or

$$\sum_{s=r^{(1)}}^{s=r^{(2)}-1}\frac{\partial}{\partial y_s}+\sum_{s=r^{(3)}}^{s=r^{(4)}-1}\frac{\partial}{\partial y_s}+\dots+\sum_{s=r^{(\epsilon-2)}}^{s=r^{(\epsilon-1)}-1}\frac{\partial}{\partial y_s}+\sum_{s=r^{(\epsilon)}}^{s=\delta}\frac{\partial}{\partial y_s}, \quad \text{if } \epsilon \text{ is odd.}$$

Hence the $(2^\delta-1)$ operators are all transformed into operators like

$$\frac{\partial}{\partial y_{r^{(1)}}}+\frac{\partial}{\partial y_{r^{(2)}}}+\ldots+\frac{\partial}{\partial y_{r^{(\epsilon)}}},$$

where $r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ are any ϵ of the suffixes $1, 2, 3, \ldots, \delta$. Therefore

The necessary and sufficient condition that the covariant $(a_1 a_2)^{\lambda_1}(a_1 a_3)^{\lambda_2}\ldots(a_1 a_{\delta+1})^{\lambda_\delta}$ should be reducible is that $x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_\delta^{\lambda_\delta}$ should be annihilated by the successive application of the $(2^\delta-1)$ operators like

$$\frac{\partial}{\partial x_{r^{(1)}}}+\frac{\partial}{\partial x_{r^{(2)}}}+\ldots+\frac{\partial}{\partial x_{r^{(\epsilon)}}};$$

$r^{(1)}, r^{(2)}, \ldots, r^{(\epsilon)}$ being any ϵ of the suffixes $1, 2, \ldots, \delta$.

The continued product of the operators is symmetrical in each of $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_\delta$, as we should expect, since the sequence of the letters $a_1, a_2, \ldots, a_{\delta+1}$ is arbitrary.

17.

It follows at once that

(1) Any perpetuant of degree $(\delta+1)$ and weight less than $(2^\delta-1)$ is reducible;

(2) The type form $(a_1 a_2)^{\lambda_1}(a_1 a_3)^{\lambda_2}\ldots(a_1 a_{\delta+1})^{\lambda_\delta}$, where

$$\lambda_1 \geqslant 2^{\delta-1},\quad \lambda_2 \geqslant 2^{\delta-2},\quad \ldots,\quad \lambda_\delta \geqslant 1,$$

is irreducible;

(3) The form $(a_1 a_2)^{\lambda_1}(a_1 a_3)^{\lambda_2}\ldots(a_1 a_{\delta+1})^{\lambda_\delta}$ is reducible, if the sum of κ indices is less than $(2^\kappa-1)$ for any value of κ.

(4) In general a form $(a_1 a_2)^{\lambda_1}(a_1 a_3)^{\lambda_2}\ldots(a_1 a_{\delta+1})^{\lambda_\delta}$ is reducible if, and only if, $\prod_{(2^\delta-1)}\left(\frac{\partial}{\partial x_{r^{(1)}}}+\frac{\partial}{\partial x_{r^{(2)}}}+\ldots+\frac{\partial}{\partial x_{r^{(\epsilon)}}}\right)$ contains *no* term

$$\left(\frac{\partial}{\partial x_1}\right)^{\mu_1}\left(\frac{\partial}{\partial x_2}\right)^{\mu_2}\ldots\left(\frac{\partial}{\partial x_\delta}\right)^{\mu_\delta},$$

for which the following inequalities hold:—

$$\lambda_1 \geqslant \mu_1,\quad \lambda_2 \geqslant \mu_2,\quad \ldots,\quad \lambda_\delta \geqslant \mu_\delta;$$

and a form $(a_1 a_2)^{\lambda_1}(a_1 a_3)^{\lambda_2}\ldots(a_1 a_{\delta+1})^{\lambda_{\delta+1}}$ is certainly irreducible, if there is any irreducible covariant $(a_1 a_2)^{\mu_1}(a_1 a_3)^{\mu_2}\ldots(a_1 a_{\delta+1})^{\mu_\delta}$, such that

$$\lambda_1 \geqslant \mu_1,\quad \lambda_2 \geqslant \mu_2,\quad \ldots,\quad \lambda_\delta \geqslant \mu_\delta.$$

18.

To determine if the expression $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_\delta^{\lambda_\delta}$ is annihilated by

$$P_\delta \equiv \prod_{(2^\delta-1)} \left(\frac{\partial}{\partial x_{r^{(1)}}} + \frac{\partial}{\partial x_{r^{(2)}}} + \dots + \frac{\partial}{\partial x_{r^{(\delta)}}}\right) \equiv \prod_{(2^\delta-1)} (z_{r^{(1)}} + z_{r^{(2)}} + \dots + z_{r^{(\delta)}}),$$

it is only necessary to find the *indices* of the variables $z_1, z_2, \dots, z_\delta$ in the terms of this expansion: the coefficients of the terms will not affect the result if we are investigating the reducibility of a single covariant: since P_δ is symmetrical in $z_1, z_2, \dots, z_\delta$, it is obvious that, if P_δ contains any term $z_1^{\mu_1} z_2^{\mu_2} \dots z_\delta^{\mu_\delta}$, then it also contains all terms derived from this by permuting the indices $\mu_1, \mu_2, \dots, \mu_\delta$. It is easily verified that

$$P_3 \equiv z_1 z_2 z_3 (z_2+z_3)(z_3+z_1)(z_1+z_2)(z_1+z_2+z_3)$$

contains the terms $z_1^4 z_2^2 z_3$, $z_1^3 z_2^3 z_3$, $z_1^3 z_2^2 z_3^2$, and therefore contains all possible terms of order 7, such that the sum of κ indices $\geqslant (2^\kappa - 1)$, where $\kappa = 1, 2$. We proceed to show that in general

P_δ contains all terms of order $(2^\delta - 1)$, such that the sum of any κ indices $\geqslant (2^\kappa-1)$, for the values $1, 2, \dots, \delta$ of κ.

To prove this we assume the result for P_δ, and thence prove it for $P_{\delta+1}$: the indices of the terms arising in $P_{\delta+1}$ may be found from those arising n P_δ by the following considerations. If O is a typical operator of P_δ, we have

$$P_\delta = \prod_{(2^\delta-1)} (O)$$

and

$$P_{\delta+1} = z_{\delta+1} . P_\delta . \prod_{(2^\delta-1)} (O + z_{\delta+1});$$

now, since we are not concerned with the numerical coefficients, the terms in $\Pi(O+z_{\delta+1})$ are the same as those which will arise by increasing by $z_{\delta+1}$ each of the variables $z_1, z_2, \dots, z_\delta$ in the expression for P_δ: of course, since all the variables z occur with positive signs, there can be no cancelling of terms.

We assume that, neglecting numerical coefficients,

$$P_\delta \equiv \Sigma z_1^{\lambda_1} z_2^{\lambda_2} \dots z_\delta^{\lambda_\delta},$$

where the λ's are chosen in all possible ways, such that their sum is $(2^\delta - 1)$ and the sum of any κ of them $\geqslant (2^\kappa - 1)$. A typical set of terms of the product $\Pi(O+z_{\delta+1})$ will be, on expanding by Taylor's theorem,

$$z_{\delta+1}^r \left(\sum_{t=1}^{t=\delta} \frac{\partial}{\partial z_t}\right)^r P_\delta,$$

where

$$r = 0, 1, 2, \dots, (2^\delta - 1);$$

hence $z_{\delta+1}\Pi(O+z_{\delta+1})$ will, by hypothesis, contain *all* possible terms

$$z_{\delta+1}^{r}\, z_1^{\lambda'_1} z_2^{\lambda'_2} \dots z_\delta^{\lambda'_\delta}, \quad r = 1, 2, \dots, 2^\delta,$$

where the quantities λ' are any positive integers or zeros, such that their sum is $(2^\delta - r)$, and the sum of any κ of them $\geqslant (2^\kappa - r)$.

The terms of $P_{\delta+1}$ are therefore found by multiplying together every term $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_\delta^{\lambda_\delta}$ with every term $z_1^{\lambda'_1} z_2^{\lambda'_2} \dots z_\delta^{\lambda'_\delta} z_{\delta+1}^{r}$, where the quantities λ and λ' are any integers whatever satisfying these respective conditions: the resulting product will therefore contain all terms

$$z_1^{\mu_1} z_2^{\mu_2} \dots z_\delta^{\mu_\delta} z_{\delta+1}^{r}, \quad r = 1, 2, \dots, 2^\delta,$$

where the μ's are any positive integers whatever whose sum is $\{2^{\delta+1} - (r+1)\}$, such that the sum of any κ of them $\geqslant (2^\kappa - 1)$, and also $\geqslant \{2^{\kappa+1} - (r+1)\}$: that is $P_{\delta+1}$ contains all possible terms

$$z_1^{\mu_1} z_2^{\mu_2} \dots z_\delta^{\mu_\delta} z_{\delta+1}^{\mu_{\delta+1}},$$

where the μ's are such that the sum of any κ of them $\geqslant (2^\kappa - 1)$.

Hence, if the result is true for P_δ, it is also true for $P_{\delta+1}$, but it is known to be true when $\delta = 2, 3$, and it is therefore true in general.

19.

From this result it follows that

The necessary and sufficient condition that the covariant

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_{\delta+1})^{\lambda_\delta}$$

should be reducible is that, for some value of κ, the sum of κ of the indices $\lambda_1, \lambda_2, \dots, \lambda_\delta$ is less than $(2^\kappa - 1)$.

Any reducible covariant must satisfy these conditions, and this result determines completely the reducibility of any covariant written in the form $(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_{\delta+1})^{\lambda_\delta}$. The investigation of the reducibility of a transvectant or of the sum of any number of such covariants requires the determination of the numerical coefficients in the expansion of P_δ.

20. *Reducible Forms.*

I. Any covariant $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ is reducible, if

(i.) The sum of κ of the indices $\lambda_1, \lambda_2, \dots, \lambda_\delta$ is less than $(2^\kappa - 1)$ for any value of κ. (§ 5.)

(ii.) The covariant $(b_1 b_2)^{\lambda_{r^{(1)}}} (b_2 b_3)^{\lambda_{r^{(2)}}} \dots (b_\epsilon b_{\epsilon+1})^{\lambda_{r^{(\epsilon)}}}$ is reducible, where $r^{(1)}, r^{(2)}, \dots, r^{(\epsilon)}$ are any ϵ of the suffixes $1, 2, \dots, \delta$, such that $r^{(1)} < r^{(2)} < \dots < r^{(\epsilon)}$. (§ 12.)

[This includes (i.) as a special case.]

II. Any covariant $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ of weight $(2^\delta - 1)$ is reducible, if

(i.) $\delta = 2\epsilon - 1$, and $\lambda_1 = \lambda_{2\epsilon-1}$, $\lambda_2 = \lambda_{2\epsilon-2}$, $\dots$, $\lambda_{\epsilon-1} = \lambda_{\epsilon+1}$. (§ 11.)

(ii.) The sum of κ indices is $(2^\kappa - 1)$, and the sum of the indices is odd in any of the groups into which the δ indices are divided by removing the κ indices whose sum is $(2^\kappa - 1)$. (§ 14.)

III. The *only* covariants $(a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu$ of degree 4 which are reducible are the covariants included among the classes I. (i.) and II. (i.). (§ 6.)

IV. (i.) The *only* reducible Jacobians $\left((a_1 a_2)^\lambda (a_2 a_3)^\mu (a_3 a_4)^\nu,\ a_5\right)$ of degree 5, other than those forms included in I. (i.), are those for which

$\lambda =$	7	5	6	4
$\mu =$	2	2	4	4
$\nu =$	5	7	4	6

. (§ 8.)

(ii.) The *only* reducible Jacobians $\left((a_1 a_2)^\lambda (a_2 a_3)^\mu,\ (a_4 a_5)^\nu\right)$ of degree 5 are included in the covariants of the classes I. (i.) and II. (i.). (§ 9.)

V. The necessary and sufficient condition that any covariant

$$(a_1 a_2)^{\lambda_1} (a_1 a_3)^{\lambda_2} \dots (a_1 a_{\delta+1})^{\lambda_\delta}$$

is reducible is that, for some value of κ, the sum of κ of the indices $\lambda_1, \lambda_2, \dots, \lambda_\delta$ is less than $(2^\kappa - 1)$. (§ 19.)

[*Note added December 12th.*—Reference should be made to a paper "On an Integration Theorem as to Rational Integral Functions with bearing on the Theory of Forms," by Professor Elliott, in the current volume of the *Messenger of Mathematics*, where the general conditions of reducibility are determined from a different point of view, and the application of differential operators similar to those above is further extended.]

ON DEEP-WATER WAVES.

By HORACE LAMB.

Presidential Address.

[Delivered at the Annual General Meeting, November 10th, 1904.]

Introduction.

MR. PRESIDENT,

The records of the Society show that great latitude has been allowed to the retiring President as to the manner in which he shall discharge the obligation of giving a valedictory address. It is unnecessary, therefore, and might even be improper, to offer any apology for inviting attention on this occasion to a somewhat special problem. I propose to review the theory of the waves produced on deep water by a local disturbance of the surface. In spite of its special character, the subject is, I think, in many ways an attractive one. Apart from its great intrinsic interest, the problem is important historically. It was the first, or all but the first, hydrodynamical question to be attacked systematically from the basis of the general equations; and it offered to Poisson and Cauchy a field in which they could test the efficiency of analytical methods which were at that time new and unfamiliar. From another, and a more general, point of view, the problem has an interest in that it deals with the most conspicuous mechanical analogue of a dispersive medium, *i.e.*, one in which wave-velocity varies with wave-length. It shows, for example, how widely the effects of a single initial impulse may differ from what takes place in the case of sound, or of the vibrations of an elastic solid. Again, the theory of group-velocity, which has so many interesting applications, had its origin in the present connection.

The problem in question was proposed as a prize subject by the French Academy for the year 1816. At the expiry of the statutory period, Poisson, who as a member of the Academy could not compete, handed in the first part of a memoir* on the subject, in order to secure the independent character of the results which he had already obtained. This was read on

* "Mémoire sur la théorie des ondes," *Mém. de l'Acad. Roy. d. Sc.*, t. I., 1816. This is quoted as "P." in the sequel, with a reference to the page.

October 2nd, 1815, and was followed by a second part on December 18th of the same year; and the memoir as a whole was published in the volume dated 1818. Meantime, the prize was awarded to Cauchy, whose essay had been presented in September, 1815. For some reason which does not appear, this essay was not published until 1827.*

Cauchy's memoir begins with a general hydrodynamical introduction, which contains his well-known proof that rotational motion cannot be generated in a liquid by the action of ordinary forces. When he comes to the special problem, only the principal steps in the calculation are indicated in the text, but the memoir is equipped with an elaborate apparatus of notes, in which the missing details are supplied, and many cognate analytical points are discussed. Considerable additions to the notes were made when the paper was at length published in 1827; these were prompted in some measure by a study of Poisson's investigation, which, as already mentioned, had appeared in 1818.

Poisson's investigation is much shorter, and, on the whole, more to the immediate point. In the main results both writers agree very closely, although their methods, involving elaborate processes of approximation, are often different.†

It is not possible to examine the work of these writers at all carefully without a feeling of deep admiration for the analytical skill which was brought to bear on a problem which is even now difficult in some of its branches, and for the success with which they attained a solution for most purposes practically complete. Yet, notwithstanding the labour and skill bestowed upon it, this solution has generally been regarded as obscure, if not doubtful, and it has certainly never been quite adequately interpreted. At the present date there should, I think, be no difficulty in disengaging the essential results from the clouds of analysis in which they have been involved, and in putting the whole matter, so far at least as the two-dimensional form of the problem is concerned, into a simple and easily intelligible shape.

This is, at all events, what I have ventured to attempt in §§ 1–5 of this communication. In the remaining §§ 6, 7, I have discussed the waves produced by a periodic application of force to the surface; the results are simple, and may present, I hope, some features of interest.

* "Théorie de la propagation des ondes à la surface d'un fluide pesant d'une profondeur indéfinie," *Mém. prés. par div. Savans à l'Acad. Roy. d. Sc.*, t. I., 1827; reprinted in Cauchy, *Œuvres*, t. I. The memoir is cited in the sequel as "C.," with a reference to the original paging.

† An analysis of the two memoirs from the mathematical point of view is given by H. Burkhardt in his valuable report on "Entwickelungen nach oscillirenden Functionen," Leipzig, 1901 ..., pp. 429, 439.

Two-Dimensional Problems.

1. If the origin be in the undisturbed surface, and the axis of y be drawn vertically upwards, we have, as usual,

$$p = \rho\left(\frac{\partial\phi}{\partial t} - gy\right), \tag{1}$$

where p is the pressure, ρ the density, and ϕ must satisfy

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0. \tag{2}$$

The elevation η of the disturbed surface, supposed free, is accordingly given by

$$\eta = \frac{1}{g}\frac{\partial\phi_0}{\partial t}, \tag{3}$$

where the zero suffix indicates surface-value ($y = 0$). We have also the kinematical condition

$$\frac{\partial\eta}{\partial t} = -\left(\frac{\partial\phi}{\partial y}\right)_0. \tag{4}$$

The typical solution of our equations, for the case of initial rest, is

$$\eta = \cos\sigma t \cos kx, \tag{5}$$

$$\phi = g\frac{\sin\sigma t}{\sigma} e^{ky} \cos kx, \tag{6}$$

provided

$$\sigma^2 = gk; \tag{7}$$

this is of course the ordinary theory of "standing" waves of simple-harmonic profile.*

If we generalize this by Fourier's theorem, then, corresponding to the initial conditions

$$\eta = f(x), \qquad \phi_0 = 0, \tag{8}$$

we have

$$\eta = \frac{1}{\pi}\int_0^\infty \cos\sigma t\, dk \int_{-\infty}^{\infty} f(a) \cos k(x-a)\, da, \tag{9}$$

$$\phi = \frac{g}{\pi}\int_0^\infty \frac{\sin\sigma t}{\sigma} e^{ky}\, dk \int_{-\infty}^{\infty} f(a) \cos k(x-a)\, da. \tag{10}$$

If the initial elevation be confined to the immediate neighbourhood of

---

* Lamb, *Hydrodynamics*, § 217.

the origin, so that $f(a)$ vanishes for all but infinitesimal values of a, we have, assuming

$$\int_{-\infty}^{\infty} f(a)\,da = 1, \tag{11}$$

$$\phi = \frac{g}{\pi}\int_0^{\infty} \frac{\sin\sigma t}{\sigma} e^{ky} \cos kx\,dk. \tag{12}$$

This may be expanded in the form

$$\phi = \frac{gt}{\pi}\int_0^{\infty} \left\{1 - \frac{gt^2}{3!}k + \frac{(gt^2)^2}{5!}k^2 - \dots\right\} e^{ky}\cos kx\,dk, \tag{13}$$

where use is made of (7). If we write*

$$-y = r\cos\theta, \qquad x = r\sin\theta, \tag{14}$$

we have, y being negative,

$$\int_0^{\infty} e^{ky}\cos kx\, k^n\,dk = \frac{n!}{r^{n+1}}\cos(n+1)\theta, \tag{15}$$

so that (13) becomes

$$\phi = \frac{gt}{\pi}\left\{\frac{\cos\theta}{r} - \frac{1}{3}(\tfrac{1}{2}gt^2)\frac{\cos 2\theta}{r^2} + \frac{1}{3.5}(\tfrac{1}{2}gt^2)^2\frac{\cos 3\theta}{r^3} - \dots\right\}, \tag{16}$$

a result which is easily verified. From this the value of η is obtained by (3), putting $\theta = \frac{1}{2}\pi$. Hence†

$$\eta = \frac{1}{\pi x}\left\{\frac{gt^2}{2x} - \frac{1}{3.5}\left(\frac{gt^2}{2x}\right)^3 + \frac{1}{3.5.7.9}\left(\frac{gt^2}{2x}\right)^5 - \dots\right\}. \tag{17}$$

It may be noted that the value of η as given by (9) is in the present case indeterminate.

It is evident at once that any particular phase of the surface disturbance, *e.g.*, a zero or a maximum or a minimum of η, is associated with a definite value of $gt^2/2x$, and therefore that the phase in question travels over the surface with a constant *acceleration*. The meaning of this somewhat remarkable result will appear later (§ 3).

For purposes of actual numerical computation, the series in (17) is convenient only when we are dealing with the first stages of the disturbance at any point; it converges very slowly when $gt^2/2x$ is no longer small.

---

* The investigation might be simplified still further. It is only necessary to calculate the value of ϕ for points on the axis of y, which is a line of symmetry. Its value at other points can then be written down at once, by a property of harmonic functions. Cf. Thomson and Tait, § 498.

† "C.," p. 93; "P.," p. 112 (with some clerical errors in the numerical denominators).

The methods employed by Cauchy and Poisson respectively for meeting this difficulty are historically very interesting. The series is virtually the same as one (usually designated by M*) which occurs in the theory of Fresnel's diffraction integrals; and, in fact, the memoirs under review contain implicitly a good deal relating to this theory which is commonly assigned to other writers and to a much later date.

For the present purpose we may establish the connection by summing the series as follows. Writing, with a slight change from the usual convention,†

$$\left.\begin{aligned} M &= \omega - \frac{\omega^3}{3.5} + \frac{\omega^5}{3.5.7.9} - \ldots, \\ N &= \frac{\omega^2}{1.3} - \frac{\omega^4}{1.3.5.7} + \frac{\omega^6}{1.3.5.7.9.11} - \ldots, \end{aligned}\right\} \tag{18}$$

and $$\chi = iM - N = \frac{i\omega}{1} + \frac{i^2\omega^2}{1.3} + \frac{i^3\omega^3}{1.3.5} + \frac{i^4\omega^4}{1.3.5.7} + \ldots, \tag{19}$$

we find $$2\omega\frac{d\chi}{d\omega} = \chi + i\omega(1+\chi),$$

or $$\frac{d}{d\omega}\left(\frac{\chi}{\sqrt{\omega}}\right) - \tfrac{1}{2}i\left(\frac{\chi}{\sqrt{\omega}}\right) = \frac{i}{2\sqrt{\omega}}. \tag{20}$$

The solution of this equation, subject to the condition that χ must ultimately vary as ω, when ω is small, is

$$\chi = \tfrac{1}{2}i\sqrt{\omega}\, e^{\frac{1}{2}i\omega}\int_0^\omega \frac{e^{-\frac{1}{2}iu}}{\sqrt{u}}\,du. \tag{21}$$

Hence, equating separately imaginary and real parts,

$$\left.\begin{aligned} M &= \tfrac{1}{2}\sqrt{\omega}\left\{\cos\tfrac{1}{2}\omega\int_0^\omega \cos\tfrac{1}{2}u\frac{du}{\sqrt{u}} + \sin\tfrac{1}{2}\omega\int_0^\omega \sin\tfrac{1}{2}u\frac{du}{\sqrt{u}}\right\}, \\ N &= \tfrac{1}{2}\sqrt{\omega}\left\{\sin\tfrac{1}{2}\omega\int_0^\omega \cos\tfrac{1}{2}u\frac{du}{\sqrt{u}} - \cos\tfrac{1}{2}\omega\int_0^\omega \sin\tfrac{1}{2}u\frac{du}{\sqrt{u}}\right\}. \end{aligned}\right\} \tag{22‡}$$

Hence (17) may be written

$$\eta = \frac{\sqrt{\omega}}{2\pi x}\left\{\cos\tfrac{1}{2}\omega\int_0^\omega \cos\tfrac{1}{2}u\frac{du}{\sqrt{u}} + \sin\tfrac{1}{2}\omega\int_0^\omega \sin\tfrac{1}{2}u\frac{du}{\sqrt{u}}\right\}, \tag{23}$$

---

* Cf. Rayleigh, *Scientific Papers*, Vol. III., p. 129.

† Cf. Rayleigh, *loc. cit.* The functions M and $dM/d\omega$ are tabulated on pp. 393–6, *infra*.

‡ Cf. Rayleigh, *loc. cit.*

where $$\omega = \frac{gt^2}{2x}. \tag{24}$$

This is equivalent to a result given by Poisson.* The definite integrals are practically of Fresnel's forms, and may be considered as known functions; so that the present problem may be regarded as completely solved. Moreover, Lommel, in his researches on Diffraction, has given a table of the function†

$$1 - \frac{\omega^2}{3.5} + \frac{\omega^4}{3.5.7.9} - \dots$$

or M/ω in our notation; from this the values of M in Table I. at the end of this paper are derived. We are thus enabled to delineate the first nine or ten waves with great ease. Fig. 1 shows the variation of η with the time, at a particular place; for different places the intervals between assigned phases vary as $\sqrt{x}$, whilst the corresponding elevations vary inversely as x. Figs. 2A and 2B, on the other hand, show the wave-profile at a particular instant; at different times, the horizontal distances between corresponding points vary as the square of the time that has elapsed since the beginning of the disturbance, whilst corresponding elevations vary inversely as the square of this time.‡

When $gt^2/2x$ is large, the definite integrals in (22) approximate to the limit $\sqrt{\pi}$, and we then have

$$\eta = \frac{g^{\frac{1}{2}} t}{2^{\frac{3}{2}} \pi^{\frac{1}{2}} x^{\frac{3}{2}}} \left(\cos \frac{gt^2}{4x} + \sin \frac{gt^2}{4x} \right), \tag{25}$$

in agreement with the result of Poisson and Cauchy.

Expressions for the remainder are also given by these writers. Thus Poisson obtains (virtually) the semi-convergent expansion§

$$M = \tfrac{1}{2}\sqrt{(\pi\omega)}(\cos \tfrac{1}{2}\omega + \sin \tfrac{1}{2}\omega) - \left\{ \frac{1}{\omega} - \frac{1.3.5}{\omega^3} + \frac{1.3.5.7.9}{\omega^5} - \dots \right\}, \tag{26}$$

* "P.," p. 110.

† Denoted by $\sqrt{\frac{\pi}{2\omega}}\, U_{\frac{1}{2}}(\omega, 0)$ in his paper, *Abh. d. k. Bayer. Akad. d. Wiss.*, II. Cl., XV. Bd., 2. Abth., p. 124 (1886).

‡ In Fig. 1 the abscissæ are proportional to $\sqrt{\omega}$, and the ordinates to M. In Figs. 2A, 2B the abscissæ are proportional to $1/\omega$, and the ordinates to ωM. See Table I.

§ "P.," p. 116, where, however, there are some mistakes of sign.

whilst Cauchy's formula* is equivalent to

$$M = \tfrac{1}{2}\surd(\pi\omega)(\cos\tfrac{1}{2}\omega + \sin\tfrac{1}{2}\omega) - \int_0^\infty e^{-\surd(2\omega v)}\cos v\,dv. \tag{27}$$

These expressions are readily identified.

2. In the case of initial *impulses* applied to the surface, supposed horizontal, the typical solution is

$$\rho\phi = \cos\sigma t\, e^{ky}\cos kx, \tag{28}$$

$$\eta = -\frac{1}{g}\sigma\sin\sigma t\cos kx, \tag{29}$$

with $\sigma^2 = gk$ as before. Hence, if the initial conditions be

$$\rho\phi_0 = F(x), \qquad \eta = 0, \tag{30}$$

we have $$\phi = \frac{1}{\pi\rho}\int_0^\infty \cos\sigma t\, e^{ky}\,dk\int_0^\infty F(a)\cos k(x-a)\,da, \tag{31}$$

$$\eta = -\frac{1}{\pi g\rho}\int_0^\infty \sigma\sin\sigma t\,dk\int_0^\infty F(a)\cos k(x-a)\,da. \tag{32}$$

For a concentrated impulse acting at the point $x = 0$ of the surface, we have, putting

$$\int_{-\infty}^{\infty} F(a)\,(da) = 1, \tag{33}$$

$$\phi = \frac{1}{\pi\rho}\int_0^\infty \cos\sigma t\, e^{ky}\cos kx\,dk. \tag{34}$$

This integral may be treated in the same manner as (12); but it is evident that the results may be obtained immediately by performing the operation $1/g\rho\,.\,\partial/\partial t$ upon those of § 1. Thus, from (16) and (17), we derive

$$\phi = \frac{1}{\pi\rho}\left\{\frac{\cos\theta}{r} - \tfrac{1}{2}gt^2\frac{\cos 2\theta}{r^2} + \frac{1}{1.3}(\tfrac{1}{2}gt^2)^2\frac{\cos 3\theta}{r^3} - \ldots\right\}, \tag{35}$$

$$\eta = \frac{t}{\pi\rho x^2}\left\{\frac{1}{1} - \frac{3}{1.3.5}\left(\frac{gt^2}{2x}\right)^2 + \frac{5}{1.3.5.7.9}\left(\frac{gt^2}{2x}\right)^4 - \ldots\right\}. \tag{36†}$$

* "C.," pp. 93, 129. In the course of a long additional note (XVI.) appended in 1827, Cauchy transforms the latter integral into

$$\sqrt{\frac{\omega}{2\pi}}\int_0^\infty \frac{e^{-\frac{1}{2}\omega x}\sqrt{x}\,dx}{1+x^2},$$

thus virtually anticipating Gilbert, *Mém. de l'Acad. de Bruxelles*, t. XXXI., 1862.

† Cf. "C.," pp. 106, 108.

The latter formula may also be written

$$\eta = \frac{t}{\pi\rho x^2}\frac{dM}{d\omega} = \frac{t}{2\pi\rho x^2}\left(1+\frac{M}{\omega}-N\right), \tag{37}$$

where M and N are defined by (18), and $\omega = gt^2/2x$, as before. The function

$$\frac{\omega}{1.3}-\frac{\omega^3}{1.3.5.7}+\frac{\omega^5}{1.3.5.7.9.11}-\ldots,$$

which is equivalent to N/ω in our notation, has also been tabulated* by Lommel, so that the forms of the first few waves can be traced without difficulty. Fig. 3 shows the rise and fall of the surface at a particular place; for different places the intervals between assigned phases vary as $\sqrt{x}$, as in the former case, but the corresponding elevations now vary inversely as $x^{\frac{3}{2}}$. In Figs. 4A and 4B, which give an instantaneous view of the wave-profile, the horizontal distances between corresponding points vary as the square of the time, whilst corresponding ordinates vary inversely as the cube of the time.†

For large values of $gt^2/2x$, we find, performing the operation $1/g\rho\,.\,\partial/\partial t$ upon (25),

$$\eta = \frac{g^{\frac{1}{2}}t^2}{2^{\frac{3}{2}}\pi^{\frac{1}{2}}\rho x^{\frac{5}{2}}}\left(\cos\frac{gt^2}{4x}-\sin\frac{gt^2}{4x}\right), \tag{38}‡$$

approximately.

3. It remains to examine the meaning and the consequences of the results above obtained. It will be sufficient to consider, chiefly, the case of § 1, where an initial elevation is supposed to be concentrated in a line of the surface.

At any subsequent time t the surface is occupied by a wave-system whose advanced portions are delineated in Figs. 2A, 2B. For sufficiently small values of x the form of the waves is given by (25); hence as we approach the origin the waves are found to diminish continually in length, and to increase continually in height, in both respects without limit.

As t increases, the wave-system is stretched out horizontally, propor-

---

* Under the name $\sqrt{\frac{\pi}{2\omega}}\,U_{\frac{3}{2}}(\omega,\,0)$.

† In Fig. 3 the abscissæ are proportional to $\sqrt{\omega}$, and the ordinates to $\sqrt{\omega}\,dM/d\omega$. In Figs. 4A, 4B the abscissæ are proportional to $1/\omega$, and the ordinates to $\omega^2 dM/d\omega$. See Table II. Both tables have been constructed by Mr. H. J. Woodall.

‡ Cf. "C.," p. 108. The case of a primitive impulse is passed over by Poisson: "pour éviter quelques difficultés que présente le cas des vîtesses initiales, nous nous bornerons à considérer celui où le fluide part du repos."

tionally to the square of the time, whilst the vertical ordinates are correspondingly diminished, so that the area

$$\int \eta \, dx$$

included between the wave-profile, the axis of x, and the ordinates corresponding to any two assigned phases (*i.e.*, two assigned values of ω) is constant.* The latter statement may be verified immediately from the mere form of (17) or (23).

The oscillations of level, on the other hand, at any particular place, are represented in Fig. 1. These follow one another more and more rapidly, with ever increasing amplitude. For sufficiently great values of t, the course of these oscillations is given by (25).

In the region where this formula holds, at any assigned epoch, the changes in length and height from wave to wave are very gradual, so that a considerable number of consecutive waves may be represented approximately by a curve of sines. The circumstances are, in fact, all approximately reproduced when

$$\Delta \frac{gt^2}{4x} = 2\pi. \tag{39}$$

Hence, if we vary t alone, we have, putting $\Delta t = \tau$, the period of oscillation,

$$\tau = \frac{4\pi x}{gt}; \tag{40}$$

whilst, if we vary x alone, putting $\Delta x = -\lambda$, where λ is the wave-length, we find

$$\lambda = \frac{8\pi x^2}{gt^2}. \tag{41}$$

The wave-velocity is to be found from

$$\Delta \frac{gt^2}{4x} = 0; \tag{42}$$

this gives

$$\frac{\Delta x}{\Delta t} = \frac{2x}{t} = \sqrt{\frac{g\lambda}{2\pi}}, \tag{43}$$ †

* This statement does not apply to the case of an initial *impulse*. The corresponding proposition then is that

$$\int \phi_0 dx,$$

taken between assigned values of ω, is constant. This appears from (35).

† "P.," p. 120, gives a formula equivalent to $\tau = \sqrt{(2\pi\lambda/g)}$, but not the formula for the wave-velocity.

as in the case of an infinitely long train of simple harmonic waves of length λ.

We can now see something of a reason why each wave should be continually accelerated. The waves in front are longer than those behind, and are accordingly moving faster. The consequence is that all the waves are continually being drawn out in length, so that their velocities of propagation continually increase as they advance. But the higher the rank of a wave in the sequence, the smaller is its acceleration.

So far, we have been considering the progress of individual waves. But, if we fix our attention on a *group* of waves, characterized as having (approximately) a given wave-length λ, the position of this group is regulated according to (43) by the formula

$$\frac{x}{t} = \tfrac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}; \tag{44}$$

i.e., the group advances with a constant velocity equal to *half* that of the component waves.* The group does not, however, maintain a constant amplitude as it proceeds; it is easily seen from (25) that for a given value of λ the amplitude varies inversely as $\sqrt{x}$.

It appears that the region in the immediate neighbourhood of the origin may be regarded as a kind of source, emitting on each side an endless succession of waves of continually increasing amplitude and frequency, whose subsequent careers are governed by the laws above explained. This persistent activity of the source is not paradoxical; for our assumed initial accumulation of a finite volume of elevated fluid on an infinitely narrow base implies an unlimited store of energy.

In any practical case, however, the initial elevation is distributed over a band of finite breadth—we will denote this breadth by l. The disturbance at any point P is made up of parts due to the various elements, δa, say, of the breadth l; these are to be calculated by the preceding formulæ, and integrated over the breadth of the band. In the result the mathematical infinity, and other perplexing peculiarities, which we meet with in the case of a concentrated line-source, disappear. It would be easy to write down the requisite formulæ,† but, as they are not very tractable, and contain nothing not implied in the preceding state-

* "C.," p. 90, remarks: "Puisque le mouvement des ondes est uniformément accéleré, la vitesse de chaque onde (mésurée à son sommet) est nécessairement égale à deux fois l'abscisse de ce sommet divisée par le temps." If Cauchy had only introduced the wave-length explicitly into his formulæ, he would have been on the verge of the theory of group velocity.

† "C.," pp. 99, 185.

ment, they may be passed over. It is more instructive to examine, in a general way, how the previous results will be modified.

The initial stages of the disturbance at a distance x, such that l/x is small, will evidently be much the same as on the former hypothesis; the parts due to the various elements δa will simply reinforce one another, and the result will be sufficiently expressed by (17) or (25) provided we multiply by

$$\int_{-a}^{a} f(a)\, da,$$

i.e., by the sectional area of the initially elevated fluid. The formula (25), in particular, will hold when $gt^2/2x$ is large, so long as the wave-length λ at the point considered is large compared with l, *i.e.*, by (41), so long as $gt^2/2x \,.\, l/x$ is small. But when, as t increases, the length of the waves at x becomes comparable with or smaller than l, the contributions from the different parts of l are no longer sensibly in the same phase, and we have something analogous to "interference" in the optical sense. The result will, of course, depend on the special character of the initial distribution of the values of $f(a)$ over the space l,* but it is plain that the increase of amplitude must at length be arrested, and that ultimately we shall have a gradual dying out of the disturbance.

There is one feature generally characteristic of the later stages which must be more particularly adverted to, as it has been the cause of some perplexity: I mean a fluctuation in the amplitude of the waves. This is readily accounted for on "interference" principles. As a sufficient illustration, let us suppose that the initial elevation is uniform over the breadth l, and that we are considering a stage of the disturbance so late that the value of λ in the neighbourhood of the point x under consideration has become small compared with l. We shall evidently have a series of groups of waves separated by bands of comparatively smooth water, the centres of these bands occurring whenever l is an exact multiple of λ, say $l = n\lambda$. Substituting in (41), we find

$$\frac{x}{t} = \tfrac{1}{2}\sqrt{\frac{gl}{2n\pi}}, \tag{45}$$

i.e., the bands in question move forward with a constant velocity, which is, in fact, the group-velocity corresponding to the average wave-length in the neighbourhood.

This fluctuation was first pointed out by Poisson, in the particular

* Cf. Burnside, "On Deep-water Waves resulting from a Limited Original Disturbance," *Proc. London Math. Soc.*, Vol. xx., p. 22 (1888).

case where the initial elevation (or rather, depression) has a parabolic outline; but the whole matter was, I think, needlessly confused by the distinction which he draws* between "la théorie des ondes qui se propagent d'un mouvement uniformément acceleré" and "la théorie des ondes qui se propagent avec une vîtesse constante." The latter phenomenon is more accurately described in a later passage†:—"Lorsque on a $K = 0$, l'amplitude des oscillations verticales est nulle; par conséquent les racines de cette équation détermineront, à chaque instant, sur la surface fluide des points qui n'auront aucun mouvement vertical, et qu'on pourra regarder comme des espèces de *nœuds*, mobiles à cette surface: l'espace compris entre deux nœuds consécutifs forme un groupe d'ondes, que l'on peut considérer comme une seule onde, *dentelée* dans toute son étendue, laquelle paraît se mouvoir à la surface en s'élargissant à raison de la différence de vîtesse des deux nœuds qui la terminent." No objection can be raised to this, except that the use of the term "onde dentelée," instead of "un groupe d'ondes," when used apart from this context, is apt to be misleading. The same criticism applies to Cauchy's "ondes sillonnées."‡ Both terms rather suggest to the mind the idea of a wave whose profile is affected with a large number of relatively slight undulations, as when capillary waves are superposed on waves two or three feet in length. In the long note (XVI.) appended to his paper in 1827,§ Cauchy, referring to Poisson's memoir, states that he had been led independently to the discovery of the same circumstance, which he also unfortunately describes as "l'existence d'une série d'ondes propagées avec des vîtesses constantes," and proceeds to work out results in great, and even excessive, detail on a number of special hypotheses as to the distribution of the initial surface-elevation. He shows, moreover, by examples, that in some cases the minima of wave-amplitude are non-existent.

One or two remarks may be added before we leave this part of the subject. The ideal solution of § 3 necessarily fails to give any information as to what takes place at the source itself. If, to elucidate this point in a special case, we assume

$$f(a) = \frac{Q}{\pi} \frac{b}{b^2+a^2}, \tag{46}$$

the formula (9) gives

$$\phi = \frac{gQ}{\pi} \int_0^\infty \frac{\sin \sigma t}{\sigma} e^{k(y-b)} \cos kx \, dk. \tag{47}$$

* "P.," p. 76.
† "P.," p. 120.
‡ "C.," p. 193.
§ "C.," p. 188.

Hence the surface-elevation at the origin is

$$\eta = \frac{Q}{\pi}\int_0^\infty \cos\sigma t\, e^{-kb}\,dk = \frac{2Q}{\pi g}\int_0^\infty \cos\sigma t\, e^{-\sigma^2 b/g}\sigma\, d\sigma. \tag{48}$$

The definite integral cannot be expressed in finite terms, but the form shows that η is always less than its initial value $Q/\pi b$, and that it tends with increasing t to the limit 0. It may be proved without difficulty that η passes once only through the value zero, and that its asymptotic value is

$$-\frac{2Q}{gt^2},$$

approximately.

Finally, it is to be recalled that Lord Kelvin has recently obtained, by a most ingenious method, a number of special cases of free water-waves.* In the one most fully examined, the initial elevation of the free surface is of the type

$$\eta \propto \frac{\sqrt{[\sqrt{(b^2+x^2)}+b]}}{\sqrt{(b^2+x^2)}}.$$

Since this makes $\int_0^x \eta\, dx$ increase indefinitely with x, the circumstances are not those of a *localized* initial disturbance, and the consequent motion accordingly does not come under the general description above given.

Three-Dimensional Problems.

4. If the axis of z be drawn vertically upwards, we have

$$p = \rho\left(\frac{\partial\phi}{\partial t} - gz\right), \tag{49}$$

where ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{50}$$

or, in the case of symmetry about Oz,

$$\frac{\partial^2\phi}{\partial\varpi^2} + \frac{1}{\varpi}\frac{\partial\phi}{\partial\varpi} + \frac{\partial^2\phi}{\partial z^2} = 0, \tag{51}$$

where

$$\varpi = \sqrt{(x^2+y^2)}. \tag{52}$$

The surface-elevation ζ is given by

$$\zeta = \frac{1}{g}\frac{\partial\phi_0}{\partial t}, \tag{53}$$

and is subject to the condition

$$\frac{\partial\zeta}{\partial t} = -\left(\frac{\partial\phi}{\partial z}\right)_0. \tag{54}$$

* *Proc. R.S. Edin.*, Vol. XXV., p. 185 (1904); *Phil. Mag.* (6), Vol. VII., p. 609.

The typical solution for the case of initial rest is

$$\phi = g\frac{\sin\sigma t}{\sigma}e^{kz}J_0(k\varpi), \tag{55}$$

$$\zeta = \cos\sigma t\, J_0(k\varpi), \tag{56}$$

provided $\sigma^2 = gk$, as before.

To generalize this, subject to the condition of symmetry, we have recourse to the theorem

$$f(\varpi) = \int_0^\infty J_0(k\varpi)\,k\,dk\int_0^\infty f(a)\,J_0(ka)\,a\,da, \tag{57}$$

which may be recognized as a degenerate form of the expansion of an arbitrary function of latitude on a sphere in a series of zonal harmonics.* Thus, corresponding to the initial conditions,

$$\zeta = f(\varpi), \qquad \phi_0 = 0, \tag{58}$$

we have
$$\phi = g\int_0^\infty \frac{\sin\sigma t}{\sigma}e^{kz}J_0(k\varpi)\,k\,dk\int_0^\infty f(a)\,J_0(ka)\,a\,da, \tag{59}$$

$$\zeta = \int_0^\infty \cos\sigma t\, J_0(k\varpi)\,k\,dk\int_0^\infty f(a)\,J_0(ka)\,a\,da. \tag{60}$$

If the initial elevation be concentrated in the immediate neighbourhood of the origin, then, assuming

$$\int_0^\infty f(a)\,2\pi a\,da = 1, \tag{61}$$

---

* Viz., if $\mu = \cos\theta$, we have

$$F(\mu) = \Sigma(n+\tfrac{1}{2})P_n(\mu)\int_{-1}^1 F(\mu')P_n(\mu')\,d\mu',$$

where
$$P_n(\mu) = 1 - \frac{n(n+1)}{1^2}\sin^2\tfrac{1}{2}\theta + \frac{(n-1)n(n+1)(n+2)}{1^2.2^2}\sin^4\tfrac{1}{2}\theta - \ldots.$$

Denoting by ϖ the length of the chord drawn to the variable point from the pole ($\theta = 0$) of the sphere, we have
$$\varpi = 2a\sin\tfrac{1}{2}\theta, \qquad \varpi\,d\varpi = -a^2 d\mu,$$

where a is the radius; so that the formula may be written

$$f(\varpi) = \frac{1}{a^2}\Sigma(n+\tfrac{1}{2})H_n(\varpi)\int_0^{2a} f(\varpi')H_n(\varpi')\varpi'd\varpi',$$

where
$$H_n(\varpi) = 1 - \frac{n(n+1)}{2^2}\frac{\varpi^2}{a^2} + \frac{(n-1)n(n+1)(n+2)}{2^2.4^2}\frac{\varpi^4}{a^4} - \ldots.$$

Putting
$$k = \frac{n}{a}, \qquad \delta k = \frac{1}{a},$$

and finally making a infinite, we obtain

$$f(\varpi) = \int_0^\infty J_0(k\varpi)\,k\,dk\int_0^\infty f(\varpi')J_0(k\varpi')\varpi'd\varpi'.$$

[*January*, 1905.—The remark in the text is not new; for the history of the theorem see Heine, *Kugelfunktionen*, Bd. I., p. 442, and Nielsen, *Cylinderfunktionen* (Leipzig, 1904), p. 360.]

we have
$$\phi = \frac{g}{2\pi}\int_0^\infty \frac{\sin\sigma t}{\sigma} e^{kz} J_0(k\varpi)\, k\, dk. \tag{62}$$

Expanding, and making use of (7), we get

$$\phi = \frac{gt}{2\pi}\int_0^\infty \left\{k - \frac{gt^2}{3!}k^2 + \frac{(gt^2)^2}{5!}k^3 - \ldots\right\} e^{kz} J_0(k\varpi)\, dk. \tag{63}$$

If we put
$$z = -r\cos\theta, \qquad \varpi = r\sin\theta, \tag{64}$$

we have*
$$\int_0^\infty e^{kz} J_0(k\varpi)\, k^n\, dk = n!\,\frac{P_n(\mu)}{r^{n+1}}, \tag{65}$$

where $\mu = \cos\theta$. Hence

$$\phi = \frac{gt}{2\pi}\left\{\frac{P_1(\mu)}{r^2} - \frac{gt^2}{3!}\frac{2!\,P_2(\mu)}{r^3} + \frac{(gt^2)^2}{5!}\frac{3!\,P_3(\mu)}{r^4} - \ldots\right\}. \tag{66}$$

From this the value of ζ is to be obtained by (53). Since

$$P_{2n+1}(0) = 0, \qquad P_{2n}(0) = (-)^n\,\frac{1.3\ldots(2n-1)}{2.4\ldots 2n}, \tag{67}$$

this gives

$$\zeta = \frac{1}{2\pi\varpi^2}\left\{\frac{1^2}{2!}\frac{gt^2}{\varpi} - \frac{1^2.3^2}{6!}\left(\frac{gt^2}{\varpi}\right)^3 + \frac{1^2.3^2.5^2}{10!}\left(\frac{gt^2}{\varpi}\right)^5 - \ldots\right\}. \tag{68}†$$

It appears that any particular phase of the motion is associated with a particular value of gt^2/ϖ, and thence that the various phases travel radially outwards from the origin, each with a constant acceleration.

No exact equivalent for (68), analogous to (23) in the two-dimensional form of the problem, and accordingly suitable for discussion in the case where gt^2/ϖ is large, has as yet been discovered. Cauchy‡ and Poisson§ have, however, given processes of approximation. The method of the latter writer is substantially as follows. Since

$$J_0(k\varpi) = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \cos(k\varpi\cos\beta)\, d\beta, \tag{69}$$

the formula (60) may be written, under the present circumstances,

$$\zeta = \lim_{z=0}\left[-\frac{1}{\pi^2 g}\frac{\partial^2}{\partial t^2}\int_0^\infty \cos\sigma t\, e^{kz}\, dk \int_0^{\frac{1}{2}\pi} \cos(k\varpi\cos\beta)\, d\beta\right]. \tag{70}‖$$

It may be shown, exactly as in § 1 above, that when $gt^2/2\varpi$, and therefore,

---

* Hobson, *Proc. London Math. Soc.*, Vol. xxv., pp. 72, 73. This formula may, however, be dispensed with; see the first foot-note on p. 374, *ante*.

† Cf. "P.," p. 153; "C.," p. 118.

‡ "C.," pp. 119, 237; there is a weak point in the method, which might probably, however, be fortified.

§ "P.," p. 156.

‖ Where z is, of course, supposed to be negative before the limit.

a fortiori, $gt^2/(2\varpi\cos\beta)$ is large,

$$\lim_{s=0}\int_0^\infty \cos\sigma t\cos(k\varpi\cos\beta)e^{ks}\,dk = \frac{\sqrt{(\pi\omega')}}{2\varpi\cos\beta}(\cos\tfrac{1}{2}\omega'+\sin\tfrac{1}{2}\omega'), \quad (71)$$

where $\omega' = gt^2/(2\varpi\cos\beta)$. Hence

$$\zeta = -\frac{1}{(2\pi\varpi)^{\frac{1}{2}}g^{\frac{1}{2}}}\frac{\partial^2}{\partial t^2}t\int_0^{\frac{1}{2}\pi}\{\cos(\tfrac{1}{2}\omega\sec\beta)+\sin(\tfrac{1}{2}\omega\sec\beta)\}\frac{d\beta}{\cos^{\frac{1}{2}}\beta}, \quad (72)$$

where $$\omega = \frac{gt^2}{2\varpi}. \quad (73)$$

The definite integral in (72) is equivalent to

$$\int_0^\infty\{\cos(\tfrac{1}{2}\omega\cosh u)+\sin(\tfrac{1}{2}\omega\cosh u)\}\sqrt{(\cosh u)}\,du, \quad (74)$$

or

$$\int_0^\infty\{\cos(\tfrac{1}{2}\omega+\omega\sinh^2\tfrac{1}{2}u)+\sin(\tfrac{1}{2}\omega+\omega\sinh^2\tfrac{1}{2}u)\}$$
$$\times\left\{2d(\sinh\tfrac{1}{2}u)+\frac{2\tanh\tfrac{1}{2}u\,d(\sinh^2\tfrac{1}{2}u)}{\sqrt{(\cosh u)}+\cosh\tfrac{1}{2}u}\right\}. \quad (75)$$

Since

$$\int_0^\infty\cos(\omega\sinh^2\tfrac{1}{2}u)\,d(\sinh\tfrac{1}{2}u) = \int_0^\infty\sin(\omega\sinh^2\tfrac{1}{2}u)\,d(\sinh\tfrac{1}{2}u) = \tfrac{1}{2}\sqrt{\frac{\pi}{2\omega}}, \quad (76)$$

the first part of the integral (75) reduces to

$$\sqrt{\frac{2\pi}{\omega}}\cos\tfrac{1}{2}\omega. \quad (77)$$

The coefficient of $d(\sinh^2\frac{1}{2}u)$, in the second part, vanishes for $u = 0$ and $u = \infty$; and Poisson proceeds to show by a series of partial integrations that, when ω is large, this second part may be neglected. Hence, substituting the form (77) in (72), we find

$$\zeta = -\frac{1}{2^{\frac{1}{2}}\pi\varpi g}\frac{\partial^2}{\partial t^2}\cos\frac{gt^2}{4\varpi}, \quad (78)$$

or keeping, for consistency, only the most important term,

$$\zeta = \frac{gt^2}{2^{\frac{1}{2}}\pi\varpi^3}\cos\frac{gt^2}{4\varpi}. \quad (79)^*$$

It is not necessary to dwell on the interpretation of this result, which will be readily understood from what has been said in § 3 with respect to the two-dimensional case. The consequences were worked out in some detail by Poisson on the hypothesis of an initial paraboloidal depression.†

---

* "C.," p. 242; "P.," p. 163.

† "P.," p. 165 *et seq.*

5. When the initial data are of impulse, the typical solution is

$$\phi = \cos\sigma t\, e^{kz} J_0(k\varpi), \tag{80}$$

$$\zeta = -\frac{1}{g}\sigma \sin\sigma t\, J_0(k\varpi), \tag{81}$$

which, being generalized, gives, for the initial conditions

$$\rho\phi_0 = F(\varpi), \qquad \zeta = 0, \tag{82}$$

the solution

$$\phi = \frac{1}{\rho}\int_0^\infty \cos\sigma t e^{kz} J_0(k\varpi)\, k\, dk \int_0^\infty F(a) J_0(ka)\, a\, da, \tag{83}$$

$$\zeta = -\frac{1}{g\rho}\int_0^\infty \sigma \sin\sigma t\, J_0(k\varpi)\, k\, dk \int_0^\infty F(a) J_0(ka)\, a\, da. \tag{84}$$

In particular, for a concentrated impulse at the origin, such that

$$\int_0^\infty F(a)\, 2\pi a\, da = 1, \tag{85}$$

we find

$$\phi = \frac{1}{2\pi\rho}\int_0^\infty \cos\sigma t\, e^{kz} J_0(k\varpi)\, k\, dk. \tag{86}$$

Since this may be written

$$\phi = \frac{1}{2\pi\rho}\frac{\partial}{\partial t}\int_0^\infty \frac{\sin\sigma t}{\sigma} e^{kz} J_0(k\varpi)\, k\, dk, \tag{87}$$

we find, performing $1/g\rho\,.\,\partial/\partial t$ on the results of § 4,

$$\phi = \frac{1}{2\pi\rho}\left\{\frac{P_1(\mu)}{r^2} - \frac{gt^2}{2!}\frac{2!\,P_2(\mu)}{r^3} + \frac{(gt^2)^2}{4!}\frac{3!\,P_3(\mu)}{r^4} - \ldots\right\}, \tag{88}$$

$$\zeta = \frac{t}{2\pi\rho\varpi^3}\left\{1 - \frac{1^2.3^2}{5!}\left(\frac{gt^2}{\varpi}\right)^2 + \frac{1^2.3^2.5^2}{9!}\left(\frac{gt^2}{\varpi}\right)^4 - \ldots\right\}. \tag{89*}$$

Again, when $gt^2/2\varpi$ is large, we have, in place of (79),

$$\zeta = -\frac{gt^3}{2^{\frac{7}{2}}\pi\rho\varpi^4}\sin\frac{gt^2}{4\varpi}. \tag{90}$$

Waves due to a Periodic Surface-Disturbance.

6. From the effects of an instantaneous impulse we might, by superposition, pass to the case of a surface-pressure varying with the time according to any assigned law. The only case of special interest, however,

* Cf. "C.," p. 122.

is that of a periodic (simple-harmonic) pressure; and this is most easily treated independently.

To avoid indeterminateness,* we introduce a small frictional force proportional to the velocity. Hence (1) is replaced by

$$p = \rho\left(\frac{\partial\phi}{\partial t} - gy + \mu\phi\right), \tag{91}$$

where μ now denotes the frictional coefficient.† This gives

$$\eta = \frac{1}{g}\frac{\partial\phi_0}{\partial t} + \frac{\mu}{g}\phi_0 - \frac{p_0}{g\rho}, \tag{92}$$

where the suffixes indicate surface-values, as before. The kinematical relation (4) holds as always.

Hence, corresponding to an applied surface-pressure

$$p_0 = e^{i\sigma t}\cos k(x-a), \tag{93}$$

we find

$$g\rho\eta = -\frac{gk}{gk-\sigma^2+i\mu\sigma}e^{i\sigma t}\cos k(x-a), \tag{94}$$

or, writing

$$\kappa = \frac{\sigma^2}{g}, \quad \mu_1 = \frac{\mu\sigma}{g}, \tag{95}$$

$$g\rho\eta = -\frac{k}{k-(\kappa-i\mu_1)}e^{i\sigma t}\cos k(x-a). \tag{96}$$

It will be noticed that $2\pi/\kappa$ would be the length of *free* waves having the imposed period $2\pi/\sigma$.

Generalizing (96) by Fourier's method, we find that a surface-pressure

$$p_0 = f(x)e^{i\sigma t} \tag{97}$$

produces a surface-elevation given by

$$g\rho\eta = -\frac{1}{\pi}e^{i\sigma t}\int_0^\infty \frac{k\,dk}{k-(\kappa-i\mu_1)}\int_{-\infty}^\infty f(a)\cos k(x-a)\,da, \tag{98}$$

or, as it may conveniently be written,

$$g\rho\eta = -\frac{1}{2\pi i}e^{i\sigma t}\frac{\partial}{\partial x}\int_0^\infty \frac{dk}{k-(\kappa-i\mu_1)}\int_{-\infty}^\infty \left\{e^{ik(x-a)} - e^{-ik(x-a)}\right\}f(a)\,da. \tag{99}$$

We shall suppose, for the most part, that $f(a)$ is sensible only for values of a lying between certain finite limits. At points of the surface beyond these limits, to the right of the origin, $x-a$ will be positive, and we then have‡

* Cf. *Hydrodynamics*, § 226.

† The symbol μ will no longer be required in its former sense.

‡ *Hydrodynamics*, § 227.

$$\int_0^\infty \frac{e^{-ik(x-a)}dk}{k-(\kappa-i\mu_1)} = -2\pi i e^{-i(\kappa-i\mu_1)(x-a)} + \int_0^\infty \frac{e^{-m(x-a)}dm}{m-i\kappa-\mu_1}, \tag{100}$$

$$\int_0^\infty \frac{e^{ik(x-a)}dk}{k-(\kappa-i\mu_1)} = \int_0^\infty \frac{e^{-m(x-a)}dm}{m+i\kappa+\mu_1}. \tag{101}$$

Substituting in (99), and putting $\mu_1 = 0$, since the frictional term in (91) has now served its purpose, we find

$$g\rho\eta = -e^{i\sigma t}\frac{\partial}{\partial x}\int_{-\infty}^{\infty}\left\{e^{-i\kappa(x-a)} - \frac{\kappa}{\pi}\int_0^\infty \frac{e^{-m(x-a)}dm}{m^2+\kappa^2}\right\} f(a)\,da. \tag{102}$$

At distances from the seat of the applied pressure which are great compared with $2\pi/\kappa$, the definite integral with respect to m is negligible, and we have

$$g\rho\eta = i\kappa\int_{-\infty}^{\infty} e^{i[\sigma t-\kappa(x-a)]} f(a)\,da = i(A+iB)e^{i(\sigma t-\kappa x)}, \tag{103}$$

provided

$$A = \kappa\int_{-\infty}^{\infty} f(a)\cos\kappa a\,da, \qquad B = \kappa\int_{-\infty}^{\infty} f(a)\sin\kappa a\,da. \tag{104}$$

Hence, taking the real parts of our expressions, we find that the waves produced by an applied surface-pressure

$$p_0 = f(x)\cos\sigma t \tag{105}$$

are represented, at a sufficiently great distance on the positive side of the origin, by

$$g\rho\eta = -A\sin(\sigma t-\kappa x) - B\cos(\sigma t-\kappa x). \tag{106*}$$

This represents a train of progressive waves whose wave-length and wave-velocity are related in the usual way to the imposed period. If the pressure p_0 be symmetrical with respect to the origin, $B = 0$.

When the surface-pressure (105) is concentrated at the origin, so that $f(a)$ vanishes for all but infinitesimal values of a, we write

$$\int_{-\infty}^{\infty} f(a)\,da = P, \tag{107}$$

and obtain from (102)

$$\begin{aligned} g\rho\eta &= i\kappa P e^{i(\sigma t-\kappa x)} - \frac{\kappa P e^{i\sigma t}}{\pi}\int_0^\infty \frac{e^{-mx} m\,dm}{m^2+\kappa^2} \\ &= i\kappa P e^{i(\sigma t-\kappa x)} - \frac{\kappa P}{\pi} e^{i\sigma t}\left\{\frac{1}{\kappa^2 x^2} - \frac{3!}{\kappa^4 x^4} + \frac{5!}{\kappa^6 x^6} - \dots\right\}, \end{aligned} \tag{108}$$

where the series in { } is semi-convergent.

In the case of an integral pressure $Pe^{i\sigma t}$ uniformly distributed over the

* Cambridge Mathematical Tripos, Part II., 1901.

space between $x = \pm a$, we should obtain

$$g\rho\eta = iP\frac{\sin\kappa a}{a}e^{i(\sigma t-\kappa x)} - \frac{\kappa P}{\pi a}e^{i\sigma t}\int_0^\infty \frac{e^{-mx}\sinh ma}{m^2+\kappa^2}\,dm. \quad (109)$$

This is on the supposition that x is positive and greater than a. If x lie *between* $\pm a$, the details of the work require some modification; I find

$$g\rho\eta = -\frac{P}{a}e^{i(\sigma t-\kappa a)}\cos\kappa x + \frac{\kappa P}{\pi a}e^{i\sigma t}\int_0^\infty \frac{e^{-ma}\cosh mx}{m^2+\kappa^2}\,dm. \quad (110$$

At the point $x = a$ the values of η given by (109) and (110) differ by

$$\frac{P}{2g\rho a}e^{i\sigma t},$$

the amplitude of this discontinuity being exactly that which would be caused by a statical difference of pressure $P/2a$.

Again, if

$$p_0 = \frac{P}{\pi}\frac{b}{b^2+x^2}e^{i\sigma t} = \frac{P}{\pi}e^{i\sigma t}\int_0^\infty e^{-kb}\cos kx\,dk, \quad (111)$$

we should find

$$g\rho\eta = i\kappa P e^{-\kappa b}e^{i(\sigma t-\kappa x)} - \frac{P}{\pi}e^{i\sigma t}\int_0^\infty \frac{\kappa\cos mb + m\sin mb}{m^2+\kappa^2}e^{-mx}m\,dm. \quad (112)$$

It may be of interest to calculate the mean rate at which an integral pressure $P\cos\sigma t$ does work in generating waves. In the case of a concentrated pressure, taking the real part of (108), we have, at the origin ($x = 0$),

$$-\frac{\partial\eta}{\partial t} = \frac{\kappa\sigma P}{g\rho}\cos\sigma t + \text{a term in }\sin\sigma t. \quad (113)$$

The required mean rate is therefore

$$\frac{\kappa\sigma P^2}{2g\rho} \quad \text{or} \quad \frac{\sigma^3 P^2}{2g^2\rho}, \quad (114)$$

varying as the *cube* of the frequency. The circumstance that the ineffectual term involving $\sin\sigma t$, in (113), is infinite need not dismay us. The difficulty does not occur when the pressure is diffused. Thus, in the case of a uniform pressure distributed over the space between $x = \pm a$, we easily find from (110) that the mean rate of work is

$$\frac{\sigma^3 P^2}{2g^2\rho}\left(\frac{\sin\kappa a}{\kappa a}\right)^2. \quad (115)$$

This agrees with (114) when a is infinitely small.

7. In the three-dimensional form of the problem, we have, in place of (91),

$$p = \rho\left(\frac{\partial\phi}{\partial t} - gz + \mu\phi\right). \tag{116}$$

This gives

$$\zeta = \frac{1}{g}\frac{\partial\phi_0}{\partial t} + \frac{\mu}{g}\phi_0 - \frac{p_0}{g\rho}, \tag{117}$$

subject to the kinematical condition (54).

In the case of symmetry about the axis of z, assuming

$$p_0 = J_0(k\varpi)\,e^{i\sigma t}, \tag{118}$$

$$\phi = C\,e^{i\sigma t + kz}\,J_0(k\varpi), \tag{119}$$

we find

$$(gk^2 - \sigma^2 + i\mu\sigma)\,C = \frac{i\sigma}{\rho}, \tag{120}$$

and thence

$$g\rho\zeta = -\frac{k}{k-(\kappa - i\mu_1)}\,J_0(k\varpi)e^{i\sigma t}, \tag{121}$$

where the definitions of κ and μ_1 are as in (95).

Hence, corresponding to a symmetrical surface-pressure

$$p_0 = f(\varpi)\,e^{i\sigma t}, \tag{122}$$

we have, by (57),

$$g\rho\zeta = -e^{i\sigma t}\int_0^\infty \frac{J_0(k\varpi)\,k^2\,dk}{k-(\kappa-i\mu_1)}\int_0^\infty f(a)\,J_0(ka)\,a\,da. \tag{123}$$

Thus, for an integral pressure $Pe^{i\sigma t}$ distributed uniformly over a circle of radius a, we have

$$\int_0^\infty f(a)\,J_0(ka)\,a\,da = \frac{P}{\pi a^2}\int_0^a J_0(ka)\,a\,da = \frac{P}{\pi ka}\,J_1(ka), \tag{124}$$

and thence

$$g\rho\zeta = -\frac{P}{\pi a}\,e^{i\sigma t}\int_0^\infty \frac{J_0(k\varpi)J_1(ka)}{k-(\kappa-i\mu_1)}\,k\,dk. \tag{125}$$

Returning to the general case, since

$$k^2 J_0(k\varpi) = -\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial}{\partial\varpi}\,J_0(k\varpi), \tag{126}$$

we have

$$g\rho\zeta = e^{i\sigma t}\,\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\varpi\frac{\partial}{\partial\varpi}\int_0^\infty \frac{J_0(k\varpi)\,dk}{k-(\kappa-i\mu_1)}\int_0^\infty f(a)\,J_0(ka)\,a\,da. \tag{127}$$

For a concentrated pressure $Pe^{i\sigma t}$ at the origin, this becomes

$$g\rho\zeta = \frac{Pe^{i\sigma t}}{2\pi}\,\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\,\varpi\,\frac{\partial}{\partial\varpi}\int_0^\infty \frac{J_0(k\varpi)\,dk}{k-(\kappa-i\mu_1)}. \tag{128}$$

Now, since

$$J_0(k\varpi) = \frac{2}{\pi}\int_0^\infty \sin(k\varpi\cosh u)\,du, \tag{129}$$

we have*

$$\int_0^\infty \frac{J_0(k\varpi)\,dk}{k-(\kappa-i\mu_1)} = -\frac{i}{\pi}\int_0^\infty du\int_0^\infty \frac{e^{ik\varpi\cosh u}-e^{-ik\varpi\cosh u}}{k-(\kappa-i\mu_1)}\,dk$$

$$= -\frac{i}{\pi}\int_0^\infty \left\{2\pi i\, e^{-i(\kappa-i\mu_1)\varpi\cosh u} - 2(\mu_1+i\kappa)\int_0^\infty \frac{e^{-m\varpi\cosh u}\,dm}{m^2-(\mu_1+i\kappa)^2}\right\}du. \tag{130}$$

We may now put $\mu_1 = 0$ without inconvenience; thus

$$\lim_{\mu_1=0}\int_0^\infty \frac{J_0(k\varpi)\,dk}{k-(\kappa-i\mu_1)} = \pi D_0(\kappa\varpi) - \frac{2\kappa}{\pi}\int_0^\infty du\int_0^\infty \frac{e^{-m\varpi\cosh u}\,dm}{m^2+\kappa^2}. \tag{131}$$

The symbol D_0 is here used to denote that particular combination of Bessel's functions of the first and second kinds which is appropriate to the expression of diverging periodic waves,† viz.,

$$D_0(\kappa\varpi) = \frac{2}{\pi}\int_0^\infty e^{-i\kappa\varpi\cosh u}\,du = K_0(\kappa\varpi) - iJ_0(\kappa\varpi). \tag{132}$$

Hence (128) becomes

$$g\rho\zeta = -\tfrac{1}{2}\kappa^2 Pe^{i\sigma t}.D_0(\kappa\varpi) - \frac{\kappa P}{\pi^2}\,e^{i\sigma t}\int_0^\infty du\int_0^\infty \frac{e^{-m\varpi\cosh u}\,m^2\,dm}{m^2+\kappa^2} \tag{133‡}$$

$$= -\tfrac{1}{2}\kappa^2 Pe^{i\sigma t}.D_0(\kappa\varpi) - \frac{\kappa^2 P}{2\pi}\,e^{i\sigma t}\left\{\frac{1^2}{\kappa^3\varpi^3} - \frac{1^2.3^2}{\kappa^5\varpi^5} + \frac{1^2.3^2.5^2}{\kappa^7\varpi^7} - \ldots\right\}, \tag{134}$$

where the series in $\{\ \}$ is of the semi-convergent kind. At distances ϖ which are great compared with $2\pi/\kappa$, the first term only is sensible, and we have

$$g\rho\zeta = -\tfrac{1}{2}\kappa^2 Pe^{i\sigma t}D_0(\kappa\varpi) = -\frac{\kappa^2 P}{\sqrt{(2\pi\kappa\varpi)}}\,e^{i(\sigma t-\kappa\varpi-\frac{1}{4}\pi)}, \tag{135}$$

---

* Cf. equations (100), (101).

† Cf. Rayleigh, *Phil. Mag.*, Vol. XLIII., p. 259, 1897; *Sc. Papers*, Vol. IV., p. 283. In H. Weber's notation,

$$K_0(\kappa\varpi) = \frac{2}{\pi}\int_0^\infty \cos(\kappa\varpi\cosh u)\,du.$$

‡ Use is here made of the identity

$$\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\,\varpi\,\frac{\partial}{\partial\varpi}\,e^{-m\varpi\cosh u} = m^2 e^{-m\varpi\cosh u} - \frac{m}{\varpi}\frac{\partial}{\partial u}\left(e^{-m\varpi\cosh u}\sinh u\right).$$

approximately. This gives a system of annular waves whose amplitude varies inversely as the square root of the distance from the origin.

Taking the real part of our expressions, we find, from (134),

$$-\frac{\partial \zeta}{\partial t} = \frac{\kappa^2 \sigma P}{2g\rho} J_0(\kappa\varpi)\cos\sigma t + \text{terms in } \sin\sigma t. \tag{136}$$

Hence the mean rate at which the concentrated pressure $P\cos\sigma t$ does work in generating waves is

$$\frac{\kappa^2\sigma P^2}{4g\rho} \quad \text{or} \quad \frac{\sigma^5 P^2}{4g^3\rho}, \tag{137}$$

varying as the *fifth* power of the frequency. The contrast with the case of a periodic force acting at a point in an infinite elastic solid,* and other similar problems, where the rate of work varies as the *square* of the frequency, is instructive from the point of view of the dynamics of dispersive media.

TABLE I.

$$M = \omega - \frac{\omega^3}{3.5} + \frac{\omega^5}{3.5.7.9} - \ldots.$$

ω	$\sqrt{\omega}$	M	$1/\omega$	ωM
0	0	0	∞	0
1	1·	+0·93438	1·	+ 0·93438
2	1·4142	1·4996	·5	2·9992
3	1·7321	1·4411	·33333	4·3233
4	2·	+0·7030	·25	+ 2·8120
5	2·2361	−0·5513	·2	− 2·7565
6	2·4495	1·9753	·16667	11·852
7	2·6458	3·1370	·14286	21·959
8	2·8284	3·6427	·125	29·142
9	3·	3·2571	·11111	29·314
10	3·1623	1·9821	·1	19·821
11	3·3166	−0·0736	·090909	− 0·809
12	3·4641	+2·0131	·083333	+ 24·157
13	3·6056	3·7363	·076923	48·571
14	3·7417	4·6114	·071429	64·559
15	3·8730	4·3463	·066667	65·194

* See *Phil. Trans.*, Vol. CCIII. A, p. 32 (1904).

TABLE I.—*continued.*

ω	$\sqrt{\omega}$	M	$1/\omega$	ωM
16	4·	2·9319	·0625	46·911
17	4·1231	+0·6617	·058824	+ 11·248
18	4·2426	−1·9296	·055556	− 34·733
19	4·3589	4·1932	·052632	79·670
20	4·4721	5·5300	·05	110·600
21	4·5826	5·5500	·047619	116·551
22	4·6904	4·1825	·045455	92·016
23	4·7958	−1·7091	·043478	− 39·309
24	4·8990	+1·2934	·041667	+ 31·042
25	5·	4·0884	·04	102·209
26	5·0990	5·9617	·038462	155·008
27	5·1962	6·4047	·037037	172·926
28	5·2915	5·2516	·035714	147·044
29	5·3852	+2·7340	·034483	+ 79·286
30	5·4772	−0·5639	·033333	− 16·916
31	5·5678	3·8410	·032258	119·07
32	5·6569	6·2751	·03125	200·80
33	5·7446	7·2295	·030303	238·57
34	5·8310	6·4190	·029412	218·25
35	5·9161	3·9929	·028571	139·75
36	6·	−0·5096	·027778	− 18·35
37	6·0828	+3·1917	·027027	+118·09
38	6·1644	6·1941	·026316	235·38
39	6·2450	7·7304	·025641	301·49
40	6·3246	7·3796	·025	295·18
41	6·4031	5·1810	·024390	212·42
42	6·4807	+1·6358	·023810	+ 68·70
43	6·5574	−2·4066	·023256	−103·48
44	6·6332	5·9529	·022727	261·93
45	6·7082	8·1101	·022222	364·96
46	6·7823	8·3106	·021739	382·29
47	6·8557	6·4613	·021277	303·68
48	6·9282	−2·9765	·020833	−142·87
49	7·	+1·3138	·020408	+ 64·38
50	7·0711	5·3622	·02	268·11
51	7·1414	8·1598	·019608	416·15
52	7·2111	8·9884	·019231	467·40
53	7·2801	7·6040	·018868	403·01
54	7·3485	+4·3074	·018519	+232·60
55	7·4162	−0·1209	·018182	− 6·65
56	7·4833	4·6050	·017857	257·88
57	7·5498	8·0358	·017544	458·04
58	7·6158	9·5451	·017241	553·62
59	7·6811	8·7257	·016949	514·82
60	7·7460	−5·7403	·016667	−344·42

TABLE II.

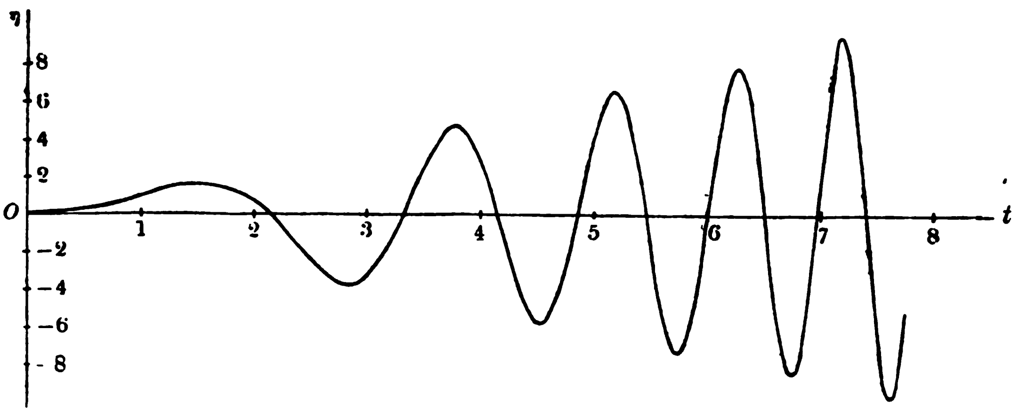
$$\frac{dM}{d\omega} = 1 - \frac{3}{1.3.5}\,\omega^2 + \frac{5}{1.3.5.7.9}\,\omega^4 - \ldots .$$

ω	$dM/d\omega$	$\surd\omega$	$\surd\omega \, . \, dM/d\omega$	$1/\omega$	$\omega^2 . \, dM/d\omega$
0	+1	0	0	∞	0
1	0·80524	1·	+ 0·80524	1·	+ 0·80524
2	+0·28141	1·4142	+ 0·39797	·5	+ 1·1256
3	−0·40761	1·7321	− 0·70600	·33333	− 3·6685
4	1·04137	2·	2·0827	·25	16·662
5	1·40790	2·2361	3·1482	·2	35·198
6	1·36594	2·4495	3·3459	·16667	49·174
7	0·88966	2·6458	2·3538	·14286	43·594
8	−0·08127	2·8284	− 0·2299	·125	− 5·201
9	+0·85247	3·	+ 2·5574	·11111	+ 69·050
10	1·65399	3·1623	5·2304	·1	165·40
11	2·08519	3·3166	6·9158	·090909	252·31
12	1·99538	3·4641	6·9122	·083333	287·33
13	1·36789	3·6056	4·9320	·076923	231·17
14	+0·33205	3·7417	+ 1·2424	·071429	+ 65·08
15	−0·86411	3·8730	− 3·3467	·066667	− 194·42
16	1·91461	4·	7·6584	·0625	490·14
17	2·53456	4·1231	10·4502	·058824	732·49
18	2·53704	4·2426	10·7637	·055556	822·00
19	1·88739	4·3589	8·2270	·052632	681·35
20	−0·71946	4·4721	− 3·2175	·05	− 287·78
21	+0·69172	4·5826	+ 3·1699	·047619	+ 305·05
22	1·99540	4·6904	9·3593	·045455	965·77
23	2·85299	4·7958	13·6825	·043478	1509·23
24	3·02604	4·8990	14·8245	·041667	1743·00
25	2·44168	5·	12·2084	·04	1526·05
26	+1·21774	5·0990	+ 6·2093	·038462	+ 823·19
27	−0·36033	5·1962	− 1·8724	·037037	− 262·68
28	1·90648	5·2915	10·0882	·035714	1494·7
29	3·02897	5·3852	16·3115	·034483	2547·4
30	3·42984	5·4772	18·7860	·033333	3086·9
31	2·98382	5·5678	16·6132	·032258	2867·5
32	1·77546	5·6569	10·0435	·03125	1818·1
33	−0·08431	5·7446	− 0·4843	·030303	− 91·8
34	+1·67994	5·8310	+ 9·7956	·029412	+ 1942·0
35	3·07701	5·9161	18·2038	·028571	3769·3
36	3·74626	6·	22·4776	·027778	4855·2
37	3·49965	6·0828	21·2876	·027027	4791·0
38	2·37381	6·1644	14·6331	·026316	3427·8
39	+0·62660	6·2450	+ 3·9131	·025641	+ 953·1
40	−1·32172	6·3246	− 8·3593	·025	− 2114·7
41	2·99002	6·4031	19·1455	·024390	5026·2
42	3·95525	6·4807	25·6329	·023810	6977·1
43	3·95984	6·5574	25·9664	·023256	7321·8
44	2·98005	6·6332	19·7674	·022727	5769·4
45	−1·23716	6·7082	− 8·2991	·022222	− 2505·2

TABLE II.—*continued.*

ω	$dM/d\omega$	$\sqrt{\omega}$	$\sqrt{\omega}\,.\,dM/d\omega$	$1/\omega$	$\omega^2\,.\,dM/d\omega$
46	+0·85221	6·7823	+ 5·7800	·021739	+ 1803·3
47	2·77588	6·8557	19·0305	·021277	6131·9
48	4·05196	6·9282	28·0728	·020833	9335·7
49	4·34962	7·	30·4473	·020408	10443·4
50	3·57465	7·0711	25·2766	·02	8936·6
51	+1·89779	7·1414	+13·5529	·019608	+ 4936·1
52	−0·28252	7·2111	− 2·0373	·019231	− 763·9
53	2·43501	7·2801	17·7271	·018868	6839·9
54	4·02508	7·3485	29·5778	·018519	11737·0
55	4·64772	7·4162	34·4685	·018182	14059·4
56	4·13093	7·4833	30·9131	·017857	12954·6
57	2·58208	7·5498	19·4943	·017544	8389·2
58	−0·36674	7·6158	− 2·7930	·017241	− 1233·7
59	+1·97787	7·6811	+15·1923	·016949	+ 6885·0
60	+3·87327	7·7460	+30·0023	·016667	13943·8

FIG. 1.



The unit of the horizontal scale is $\sqrt{(2x/g)}$.

The unit of the vertical scale is $Q/\pi x$, if Q be the sectional area of the initially elevated fluid.

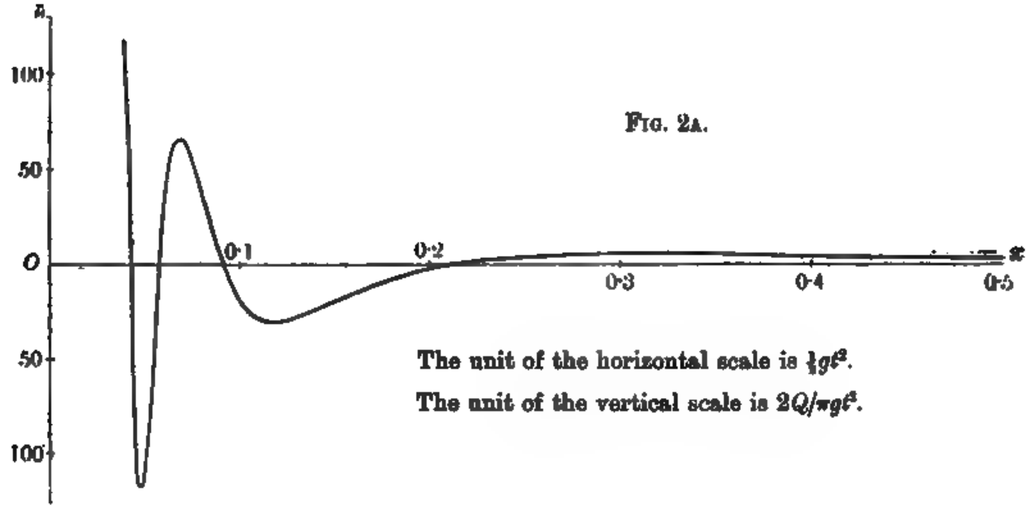


FIG. 2A.

The unit of the horizontal scale is $\frac{1}{2}gt^2$.
The unit of the vertical scale is $2Q/\pi gt^2$.

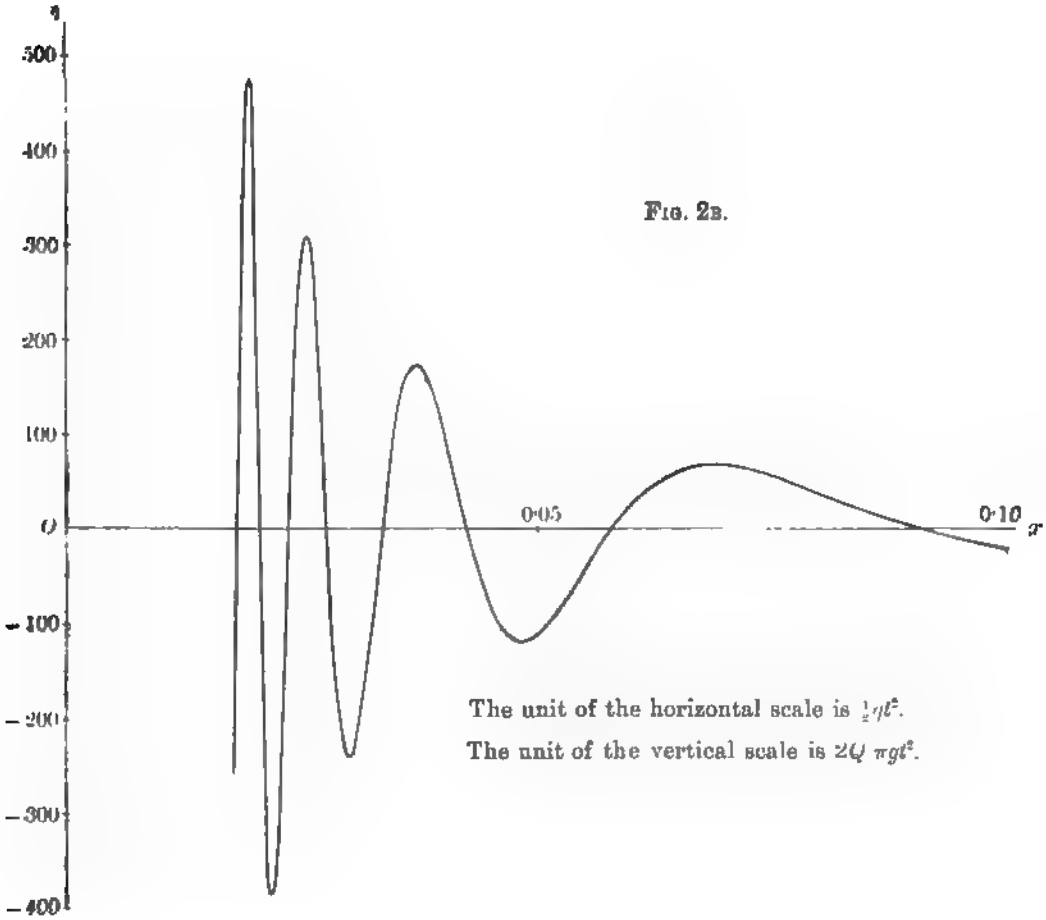
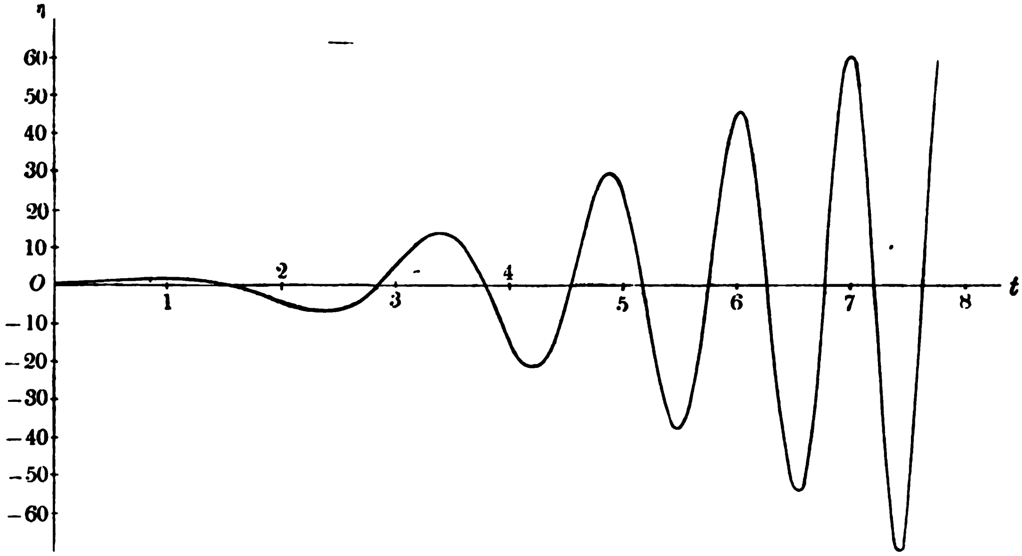


FIG. 2B.

The unit of the horizontal scale is $\frac{1}{2}gt^2$.
The unit of the vertical scale is $2Q/\pi gt^2$.

FIG. 3.



The unit of the horizontal scale is $\sqrt{(2x/g)}$.

The unit of the vertical scale is $\dfrac{P}{\pi\rho x}\sqrt{\dfrac{2}{gx}}$, where P represents the total initial impulse.

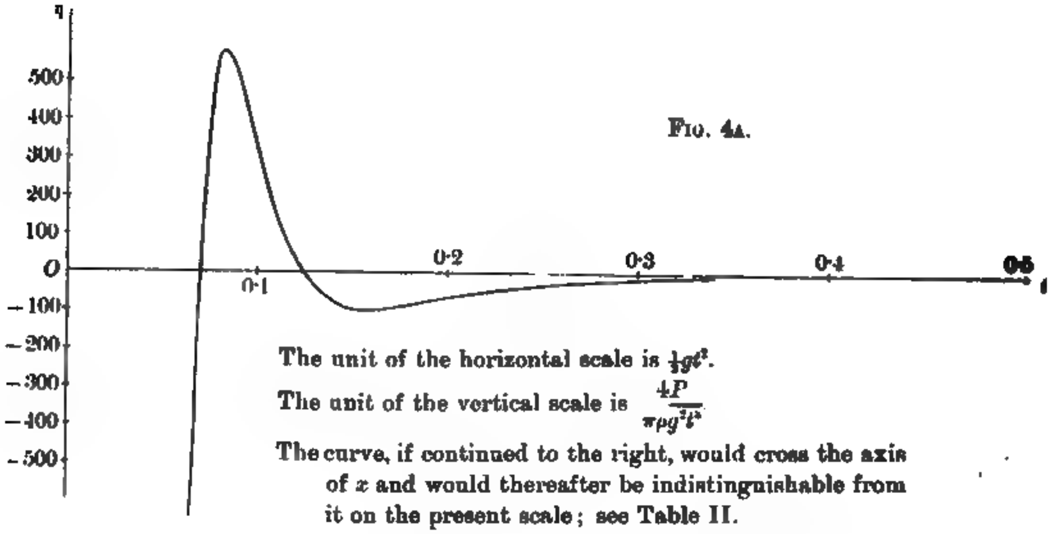


Fig. 4A.

The unit of the horizontal scale is $\frac{1}{2}gt^2$.

The unit of the vertical scale is $\dfrac{4P}{\pi\rho g^2 t^4}$.

The curve, if continued to the right, would cross the axis of x and would thereafter be indistinguishable from it on the present scale; see Table II.

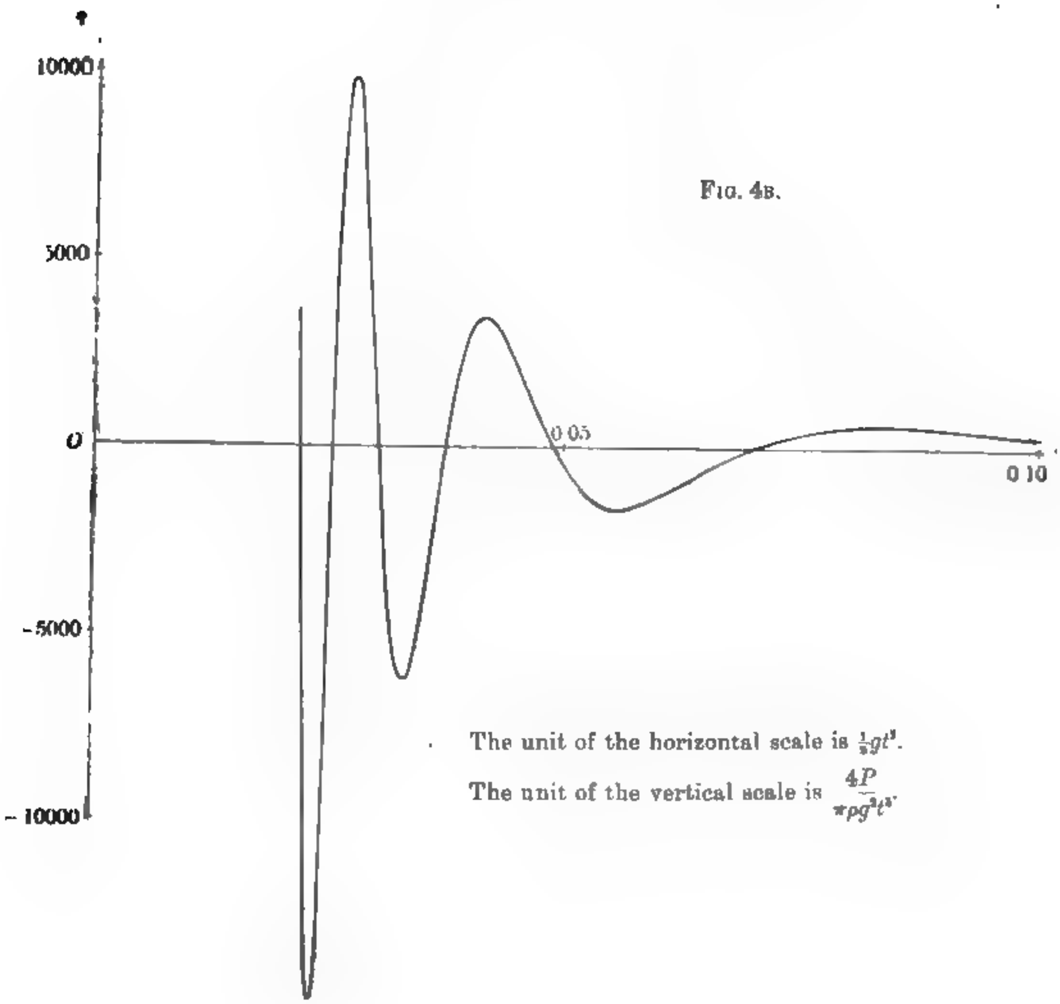


Fig. 4B.

The unit of the horizontal scale is $\frac{1}{2}gt^2$.

The unit of the vertical scale is $\dfrac{4P}{\pi\rho g^2 t^4}$.

Authors of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. d.	*s. d.*	*s. d.*
4 pages	Free	3 6	5 0	6 0
8 „	„	4 6	6 6	8 0
12 „	„	7 0	10 6	13 0
16 „	„	8 0	12 0	15 0
32 „	„	14 0	21 0	26 0

It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. Francis Hodgson, 89 Farringdon Street, E.C., at the following rates:—

Vol. I.	Nos.	1–8	... Jan. 1865	to Nov. 1866	...	10*s.*
„ II.	„	9–20	... Nov. 1866	„ 1869	...	16*s.*
„ III.	„	21–40	... „ 1869	„ 1871	...	20*s.*
„ IV.	„	41–66	... „ 1871	„ 1873	...	31*s.* 6*d.*
„ V.	„	67–75	... „ 1873	„ 1874	...	15*s.*
„ VI.	„	76–88	... „ 1874	„ 1875	...	21*s.*
„ VII.	„	89–103	... „ 1875	„ 1876	...	21*s.*
„ VIII.	„	104–123	... „ 1876	„ 1877	...	21*s.*
„ IX.	„	124–140	... „ 1877	„ 1878	...	21*s.*
„ X.	„	141–155	... „ 1878	„ 1879	...	18*s.*
„ XI.	„	156–165	... „ 1879	„ 1880	...	12*s.* 6*d.*
„ XII.	„	166–179	... „ 1880	„ 1881	...	16*s.*
„ XIII.	„	180–194	... „ 1881	„ 1882	...	18*s.*
„ XIV.	„	195–214	... „ 1882	„ 1883	...	20*s.*
„ XV.	„	215–232	... „ 1883	„ 1884	...	20*s.*
„ XVI.	„	233–252	... „ 1884	„ 1885	...	20*s.*
„ XVII.	„	253–279	... „ 1885	„ 1886	...	25*s.*
„ XVIII.	„	280–304	... „ 1886	„ 1887	...	25*s.*
„ XIX.	„	305–342	... „ 1887	„ 1888	...	30*s.*
„ XX.	„	343–369	... „ 1888	„ 1889	...	25*s.*
„ XXI.	„	370–398	... „ 1889	„ 1890	...	25*s.*
„ XXII.	„	399–429	... „ 1890	„ 1891	...	27*s.* 6*d.*
„ XXIII.	„	430–449	... „ 1891	„ 1892	...	20*s.*
„ XXIV.	„	450–474	... „ 1892	„ 1893	...	22*s.*
„ XXV.	„	475–499	... „ 1893	„ 1894	...	22*s.*
„ XXVI.	„	500–534	... „ 1894	„ 1895	...	30*s.*
„ XXVII.	„	535–574	... „ 1895	„ 1896	...	35*s.*
„ XXVIII.	„	575–611	... „ 1896	„ 1897	...	32*s.* 6*d.*
„ XXIX.	„	612–657	... „ 1897	„ 1898	{ Pt. I. 20*s.* Pt. II. 20*s.*	
„ XXX.	„	658–678	... „ 1898	to Mar. 1899	...	20*s.*
„ XXXI.	„	679–709	... April 1899	„ Dec. 1899	...	20*s.*
„ XXXII.	„	710–737	... Jan. 1900	„ June 1900	...	20*s.*
„ XXXIII.	„	738–763	... Nov. 1900	„ Feb. 1901	...	20*s.*
„ XXXIV.	„	764–789	... Mar. 1901	„ April 1902	...	20*s.*
„ XXXV.	„	790–819	... May 1902	„ Jan. 1903	...	20*s.*

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s.* each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25*s.* each Volume.

The price of cases for binding the Volumes of the Second Series is 1*s.* 3*d.*

Vol. 1 of Series 2 (1904) has been published.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2. VOL. 2.—PART 6.

CONTENTS.

Issued March 2, 1905.

LONDON:

PRINTED BY C. F. HODGSON & SON,

2 NEWTON STREET, HIGH HOLBORN, W.C.,

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. LOVE. / J. H. GRACE.

34 St. Margaret's Road, Oxford.—A. E. H. LOVE.

Peterhouse, Cambridge.—J. H. GRACE.

ON THE ZEROES OF CERTAIN CLASSES OF INTEGRAL TAYLOR SERIES. PART II.—ON THE INTEGRAL FUNCTION $\sum_{n=0}^{\infty} \frac{x^n}{(n+a)^s n!}$ AND OTHER SIMILAR FUNCTIONS

By G. H. HARDY.

[Received August 20th, 1904—Revised* October-November, 1904—Read November 10th, 1904.]

1. This paper is a continuation of one recently published in these *Proceedings*.† The general object of the two papers is the same, but the methods used and the results obtained are of entirely different types, and I have therefore judged it best to keep the two distinct.

Introductory Remarks.

2. The principal end which I have had in view is *to determine asymptotically the zeroes of the function*

$$(1) \qquad F_{a,s}(x) = \sum_0^{\infty} \frac{x^n}{(n+a)^s n!}$$

[where $(n+a)^s = e^{s \log (n+a)}$, the imaginary part of the logarithm being between $-\pi i$ and πi] for all values of a and s, real or complex, except, of course, negative integral or zero values of a. In endeavouring to obtain the complete solution of this problem I have naturally been led to consider other varieties of functions of similar types, some of which include the particular function (1) as a special case. I have not hesitated to include proofs of these results in this paper, when such developments do not diverge far from the natural course of the analysis required for the discussion of the function (1); but, when this course would have led to a considerable increase in the length and complexity of the paper, I have confined myself to a general indication of the results and of the methods by which they can be proved.

* This part has been entirely rewritten, and some of the results considerably extended.
† *Supra*, p. 332.

In what follows I consider only the half of the plane of the complex variable

(2) $$x = \xi + i\eta = re^{i\theta}$$

for which $\eta \geqslant 0$. The corresponding results for the lower half of the plane may be deduced immediately. By drawing a semicircle whose centre is the origin and whose radius is a sufficiently large fixed quantity R_0, and the radii vectores $\theta = \frac{1}{2}\pi \mp \delta$, where δ is also fixed, but arbitrarily small, we divide the distant part of the plane into three regions

(D) $$r \geqslant R_0, \quad 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi - \delta,$$

(D′) $$r \geqslant R_0, \quad \tfrac{1}{2}\pi + \delta \leqslant \theta \leqslant \pi,$$

(E) $$r \geqslant R_0, \quad \tfrac{1}{2}\pi - \delta \leqslant \theta \leqslant \tfrac{1}{2}\pi + \delta,$$

within which the behaviour of the functions which I shall consider is entirely different. It will perhaps be convenient if I state at once the principal results which I obtain concerning the function (1).

I. Throughout D

(3) $$F_{a,s}(x) = \frac{e^x}{x^s}(1+\epsilon_x)$$

where ϵ_x is a function of x which tends uniformly to zero with $1/r$.*

II. Throughout D′

(4) $$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x).$$

In these equations x^s, $(-x)^{-a}$, $\{\log(-x)\}^{s-1}$ are so chosen as to be real when x, a, s are real.

III. Throughout E

(5) $$F_{a,s}(x) = \frac{e^x}{x^s}(1+\epsilon_x) + \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x).$$

In these results a and s may have any values, real or complex, save zero or negative integral values. From III. the nature of the zeroes may be very precisely determined. Thus

IV. If a and s are real, the zeroes of $F_{a,s}(x)$ are given asymptotically† by the equations

(6) $$\begin{cases} \xi = (s-a)\log(2k\pi) + (s-1)\log\log k + \log\dfrac{\Gamma(a)}{\Gamma(s)}, \\ \eta = (2k+1)\pi + \tfrac{1}{2}(s+a)\pi, \end{cases}$$

where k is a large positive integer.

* I use ϵ_x generally in this sense, sometimes omitting the suffix. Of course ϵ_x is not the same in different equations.

† For a precise definition of what is meant by this see Part I., *supra*, p. 333.

If a and s are complex, these formulæ require a slight modification.

So far, zero or negative integral values of a and s have been excluded altogether. If a is zero or a negative integer, $F_{a,s}(x)$ is no longer defined. On the other hand, if s is zero or a negative integer,

V. The function $F_{a,s}(x)$ reduces to the product of e^x by a polynomial, and has but a *finite* number of zeroes. It will be seen in the sequel that greater precision can be given to some of these results, notably to I.

The paper is divided into four sections. In Section I. (§§ 3–13) I consider the region D, in which the functions under consideration have no zeroes; and in Section II. (§§ 14–22) the region D′, of which the same is true. I have in Section I. considered the function $F_{a,s}(x)$ as a particular case of a certain class of functions, and I have endeavoured to make my method as direct and elementary as possible, avoiding the use of contour integrals and other contrivances, which, although very powerful aids to the obtaining of particular results, are apt to obscure the basis on which they rest. In Section III. (§§ 23–31) I consider the region E. The analysis in this Section (as, in a less degree, in Section II.) is more difficult and indirect, the problem being inherently less simple, and I have not attempted to deal with more than a few special functions besides the function $F_{a,s}(x)$. Finally, Section IV. (§§ 32–37) is taken up with a brief discussion of several matters naturally arising out of the previous work.

It will be found in the case of the asymptotic expansions discussed in Sections I. and II. that to each of the regions D and D′ corresponds a special function whose asymptotic expansion is of a particularly simple character, reducing, in fact, to *one term*. For D the function is

$$1+\frac{x}{s+1}+\frac{x^2}{(s+1)(s+2)}+\dots,$$

and for (D′) it is $$1+\frac{x}{(1+a)\,1!}+\frac{x^2}{(2+a)\,2!}+\dots.$$

The results, so far as they concern these functions, are particular cases of results which have already been arrived at from a different point of view —that of the theory of linear differential equations—by Horn, Jacobsthal, and others. Horn has also considered the question as to the nature of the places in which the integrals of the equations assume assigned values, and, in particular, of their zeroes,* but his approximations are much less

* *Crelle*, Bd. cxx., p. 1.

precise than mine. The general types of functions considered here are, of course, not solutions of linear differential equations at all.

I. THE REGION D ($r \geqslant R_0,\ 0 \leqslant \theta \leqslant \frac{1}{2}\pi - \delta$).

The Function $\Sigma \frac{\Gamma(s)}{\Gamma(s+n+1)} x^n$.

3. It will be found that there are certain functions whose behaviour in D (or D′) can be specified in a particularly simple manner. In the case of D the function

$$(7)\qquad f_s(x) = \sum_{n=0}^{\infty} \frac{\Gamma(s)}{\Gamma(s+n+1)} x^n = \frac{1}{s} + \frac{x}{s(s+1)} + \frac{x^2}{s(s+1)(s+2)} + \dots$$

is such a function.

Let us suppose first that $\mathbf{R}(s) > 0$. Then, by the help of the formula

$$(8)\qquad \int_0^1 u^n (1-u)^{s-1} du = \frac{\Gamma(n+1)\Gamma(s)}{\Gamma(n+1+s)},$$

we find that*

$$(9)\qquad f_s(x) = \int_0^1 e^{xu} (1-u)^{s-1} du,$$

where $(1-u)^{s-1} = e^{(s-1)\log(1-u)}$, the logarithm being real. Hence

$$(10)\qquad f_s(x) = e^x \int_0^1 e^{-x\omega} \omega^{s-1} d\omega = \phi_s(x) - \psi_s(x)$$

where

$$(11)\qquad \phi_s(x) = e^x \int_0^\infty e^{-x\omega} \omega^{s-1} d\omega = \Gamma(s) x^{-s} e^x$$

(x^{-s} being defined by $x^{-s} = e^{-s \log x}$ where the logarithm is real with x) and

$$(12)\qquad \psi_s(x) = e^x \int_1^\infty e^{-x\omega} \omega^{s-1} d\omega.$$

Thus

$$(13)\qquad f_s(x) = \Gamma(s) x^{-s} e^x - e^x \int_1^\infty e^{-x\omega} \omega^{s-1} d\omega$$

when $\mathbf{R}(s) > 0$. But it is easy to see that the right-hand side of (12) represents an analytic function of s regular for all values of s, save negative integral (including zero) values. The equation (13) consequently holds for all values of s, with these exceptions. Now, we find easily by

---

* The term by term integration is easily justified.

integration by parts that

$$(14)\quad \psi_s(x) = \frac{1}{x} + \frac{s-1}{x^2} + \ldots + \frac{(s-1)(s-2)\ldots(s-\nu+2)}{x^{\nu-1}}$$
$$+\frac{(s-1)(s-2)\ldots(s-\nu+1)}{x^{\nu-1}} e^x \int_1^\infty e^{-x\omega}\,\omega^{s-\nu}\,d\omega.$$

But

$$\left| e^x \int_1^\infty e^{-x\omega}\,\omega^{s-\nu}\,d\omega \right| < e^{r\cos\theta} \int_1^\infty e^{-r\omega\cos\theta} \,|\,\omega^{s-\nu}|\,d\omega < \int_0^\infty e^{-rt\cos\theta}\,|\,1+t\,|^{s-\nu}\,dt < K_\nu.^*$$

Hence, changing ν into $\nu+1$,

$$(15)\qquad f_s(x) = \Gamma(s)\,x^{-s}e^x - \sum_{\mu=1}^{\nu-1} \frac{(s-1)(s-2)\ldots(s-\mu+1)}{x^\mu} + R$$

where
$$|R| < \frac{K_\nu}{r^\nu}.$$

That is to say, *the asymptotic expansion of* $f_s(x)$ *in* D *is*

$$(16)\qquad \Gamma(s)\,x^{-s}e^x - \frac{1}{x} - \frac{s-1}{x^2} - \frac{(s-1)(s-2)}{x^3} - \ldots,$$

and the asymptotic expansion of $e^{-x}f_s(x)$ *is*

$$(17)\qquad \Gamma(s)\,x^{-s}$$

simply; for $|e^{-x}f_s(x) - \Gamma(s)\,x^{-s}|$ decreases with $1/r$ more rapidly than any power of $1/r$.

I may remark that the extension of (13) to general values of s, which was inferred from the principle of analytic continuity, may be deduced directly from the formulæ

$$(18)\quad \int_0^1 \left\{ u^n - \sum_{\nu=0}^{p} (-)^\nu \binom{n}{\nu} (1-u)^\nu \right\} (1-u)^{s-1}\,du = \frac{\Gamma(n+1)\,\Gamma(s)}{\Gamma(n+1+s)} - \sum_{\nu=0}^{p} \frac{(-)^\nu \binom{n}{\nu}}{s+\nu},$$

$$(19)\qquad \int_0^\infty \left\{ e^{-x\omega} - \sum_{\nu=0}^{p} (-)^\nu \frac{(x\omega)^\nu}{\nu!} \right\} \omega^{s-1}\,d\omega = x^{-s}\,\Gamma(s),$$

which hold for a wider range of values of s than the ordinary Eulerian formulæ;† in each of them p is the greatest integer in $\mathbf{R}(-s)$, which is supposed not to be integral.

* For an explanation of my use of K see Part I., p. 336 (footnote). By using K_ν I imply that the limits of K_ν may depend on ν, but not on x.

† The formula (19) is attributed by Mr. Whittaker to Saalschütz (see *Modern Analysis*, p. 184); but it really dates from Cauchy ("Sur un nouveau genre d'Integrales," *Exercices de Math.*, t. I., p. 57).

The Function $\int_0^1 e^{xu}(1-u)^{s-1}\psi(1-u)\,du$.

4. I shall now consider the more general function

$$\Psi_s(x) = \int_0^1 e^{xu}(1-u)^{s-1}\psi(1-u)\,du \tag{20}$$

where $\mathbf{R}(s) > 0$. I suppose that $\psi(\omega)$ is a function expansible in a Taylor series

$$c_0 + c_1\omega + c_2\omega + \ldots$$

whose radius of convergence is *at least* equal to unity, and that, if

$$\overline{\psi}(\omega) = \Sigma\,|c_\nu|\,\omega^\nu, \tag{21}$$

$\int^1 \overline{\psi}(\omega)\,d\omega$ is convergent.*

I propose first to show that under these circumstances $\Psi_s(x)$ can be expanded in a series

$$\sum_0^\infty c_\nu f_{s+\nu}(x) \tag{22}$$

convergent throughout D.

5. Consider the integral

$$I(\mu, \mu') = \int_0^1 e^{xu}(1-u)^{s-1}\,du \sum_\mu^{\mu'} c_k(1-u)^k. \tag{23}$$

We divide the range of integration into the two parts $(0, \epsilon)$, $(\epsilon, 1)$. We can determine a positive quantity K, independent of μ and μ', and greater than the maximum of

$$|\,e^{xu}(1-u)\,|^{s-1}$$

in $(0, \delta)$, δ being any small fixed quantity greater than ϵ. We can then choose ϵ so small that the modulus of the integral over $(0, \epsilon)$ is less than any given small quantity σ; for it is obviously

$$< K\int_0^\epsilon du \sum_\mu^{\mu'} |\,c_k\,|\,(1-u)^k < K\int_0^\epsilon \overline{\psi}(u)\,du.$$

When ϵ is fixed the series $\Sigma c_k(1-u)^k$ is *uniformly* convergent in $(\epsilon, 1)$. Hence μ can be so chosen that

$$\left|\int_\epsilon^1 e^{xu}(1-u)^{s-1}\,du \sum_\mu^{\mu'} c_k(1-u)^k\right| < \sigma$$

* A more stringent condition than that which merely asserts the *absolute* convergence of $\int^1 \psi(\omega)\,d\omega$.

for all values of $\mu' > \mu$, and hence so that

$$|\,\mathrm{I}(\mu, \mu')\,| < 2\sigma$$

for all values of $\mu' > \mu$. It follows that

$$\Psi_s(x) = \sum_{\nu=0}^{\infty} c_\nu f_{s+\nu}(x), \tag{24}$$

by a deduction so obvious that I need not set it out in detail.*

6. Again, if μ is any positive integer and

$$\Psi_{s,\mu}(x) = \sum_{\nu=0}^{\mu-1} c_\nu f_{s+\nu}(x), \tag{25}$$

$$\Psi_s(x) = \psi_{s,\mu}(x) + \int_0^1 e^{xu}(1-u)^{s-1}\,du \sum_{\mu}^{\infty} c_k(1-u)^k. \tag{26}$$

Let us divide the range of integration into the two parts

$$(0,\ 1-\delta), \qquad (1-\delta,\ 1)$$

where $$\delta = \xi^{-\lambda} \qquad (0 < \lambda < 1).$$

Then

$$\left|\int_0^{1-\delta}\right| < Ke^{(1-\delta)\xi}\int_0^1 \bar{\psi}(u)\,du < Ke^{\xi-\xi^{1-\lambda}}. \tag{27}$$

Moreover, it is plain that throughout $(1-\delta,\ 1)$

$$\left|\sum_{\mu}^{\infty} c_k(1-u)^k\right| < K(1-u)^\mu < \delta^\mu = \xi^{-\lambda\mu},$$

and so

$$\left|\int_{1-\delta}^1\right| < Ke^{\xi}\xi^{-\lambda\mu}. \tag{28}$$

Thus $$\left|\int_0^1\right| < Ke^{\xi}(e^{-\xi^{1-\lambda}}+\xi^{-\lambda\mu}) < Ke^{\xi}\xi^{-\lambda\mu}$$

(for sufficiently large values of r), or

$$< K\left|\frac{e^x}{x^{\lambda\mu}}\right|, \tag{29}$$

since $\xi > Kr$ throughout D.

---

* Generally, if

$$\int_a^{A-\epsilon} \Theta(u)\,\Sigma\,\phi_n(u)\,du = \Sigma\int_a^{A-\epsilon}\Theta\phi_n\,du$$

for any $\epsilon > 0$, we may replace ϵ by 0 if (i.) Θ is continuous up to A, and (ii.) $\int^A \bar{\phi}(u)\,du$ is convergent, where $$\bar{\phi}(u) = \Sigma\,|\phi_n(u)|.$$
This set of sufficient conditions for the integration of an infinite series is often useful in practice, as it covers certain cases which frequently occur and are excluded by the ordinary tests.

Again, we can choose R so that, if $r > R$,

$$\Psi_{s,\mu}(x) = e^x \sum_0^{\mu-1} \frac{c_\nu \Gamma(s+\nu)}{x^{s+\nu}} + \rho$$

where

(30)
$$|\rho| < K_\mu \left| \frac{e^x}{x^{\lambda\mu}} \right|.$$

Hence
$$\Psi_s(x) = e^x \sum_0^{\mu-1} \frac{c_\nu \Gamma(s+\nu)}{x^{s+\nu}} + \rho,$$

where ρ again satisfies an equation of the form (30). Now $\lambda\mu$ tends to infinity with μ, and so, finally, putting $m = [\lambda\mu]$,

(31)
$$\Psi_s(x) = e^x \sum_0^m \frac{c_\nu \Gamma(s+\nu)}{x^{s+\nu}} + \rho$$

where $|\rho| < K_m |x^{-m} e^x|$. It is easy to see that we can, if we like, replace this last inequality by $|\rho| < K_m |x^{-(s+m+1)} e^x|$: and we may sum up the result by saying that *the function*

$$x^s e^{-x} \Psi_s(x)$$

possesses an asymptotic expansion

(32)
$$\sum \frac{c_\nu \Gamma(s+\nu)}{x^\nu}$$

valid throughout D.

Extension to General Values of s.

7. Throughout §§ 5, 6 it was supposed that $\mathbf{R}(s) > 0$. In fact the integral by means of which $\Psi_s(x)$ was defined is evidently divergent when $\mathbf{R}(s) \leqslant 0$. We might generalise our result by the use of contour integrals instead of line integrals, but for my present purpose it is more convenient to proceed as follows :—

We shall, in the sequel, be occupied with functions $\Psi_s(x)$ which (i.) are analytic functions of s for all values of s save (possibly) negative integral or zero values ; (ii.) are expressible, when $\mathbf{R}(s) > 0$, in the forms (20) and (22).

Now suppose $\mathbf{R}(s) \leqslant 0$. We can choose k so that $\mathbf{R}(s+k) > 0$. This being so, it can be shown, by the method of §§ 5, 6, that the series

(33)
$$\Psi_s^{(k)}(x) = \sum_k^\infty c_\nu f_{s+\nu}(x)$$

is convergent, and that $x^s e^{-x}\Psi_s^{(k)}(x)$ possesses the asymptotic expansion

$$\sum_k c_\nu x^{-\nu}\Gamma(s+\nu).$$

Moreover, it appears that the function (33) is an analytic function of s for all values of s for which $\mathbf{R}(s+k)$ is positive. The equation

$$\Psi_s(x) = \sum_0^{k-1} c_\nu f_{s+\nu}(x)+\Psi_s^{(k)}(x)$$

is therefore valid for all such values of s save $0, -1, \ldots, -(k-1)$. From this it follows at once that the conclusion of § 6 holds for the function $\Psi_s(x)$ for all values of s save zero or negative integral values.

Examples.

8. Before proceeding further I shall illustrate this result by some examples.

(i.) Suppose that we write a for s, and that

$$\psi(1-u) = u^{\beta-1} = \{1-(1-u)\}^{\beta-1}$$

where $R(\beta) > 0$. Then

$$c_\nu = \frac{(1-\beta)(2-\beta)\ldots(\nu-\beta)}{1.2\ldots\nu}.$$

It is easy to see (by a glance at a figure) that, however small κ may be,

$$|\nu-\beta| < \nu-\mathbf{R}(\beta)+\kappa,$$

after a certain value of ν. Hence it follows that, after a certain value of ν, $|c_\nu|$ is less than the coefficient of x^ν in the expansion of

$$K(1-x)^{-[1+\kappa-\mathbf{R}(\beta)]}.$$

Now κ can be so chosen that $1+\kappa-\mathbf{R}(\beta) < 1$, in which case

$$\int^1 \frac{dx}{(1-x)^{[1+\kappa-\mathbf{R}(\beta)]}}$$

is convergent. The condition of § 5 relative to $\overline{\psi}(u)$ is therefore satisfied. We find easily that

$$\Psi_a(x) = \frac{\Gamma(a)\Gamma(\beta)}{\Gamma(a+\beta)}\left\{1+\frac{\beta}{1.(a+\beta)}x+\frac{\beta(\beta+1)}{1.2.(a+\beta)(a+\beta+1)}x^2+\ldots\right\}$$

and the asymptotic expansion of $e^{-x}\Psi_a(x)$ is

$$\sum\frac{\Gamma(a+\nu)\Gamma(\nu+1-\beta)}{\Gamma(1-\beta)\Gamma(\nu+1)}x^{-a-\nu},$$

provided that $\mathbf{R}(\beta) > 0$, and that neither a nor $a+\beta$ is a negative integer.

The first restriction is easily removed by the help of an obvious recurrence formula for $\Psi_a(x)$, unless β is a negative integer. The cases in which β or $a+\beta$ is a negative integer are obviously trivial. If a is a negative integer, the function $1+\frac{\beta}{1.a+\beta}x+\dots$ may be easily reduced to the form

$$\Sigma P(\nu)\frac{x^\nu}{\nu!},$$

where $P(\nu)$ is a polynomial. This is one of a class of functions which may be reduced to the product of e^x by a polynomial.

The asymptotic expansion found for $\psi_a(x)$ in D has been otherwise obtained by W. Jacobsthal* from the point of view of the theory of linear differential equations.

9. (ii.) Suppose that

$$\psi(\omega) = (1-\omega)^{a-1}\left\{\frac{1}{\omega}\log\left(\frac{1}{1-\omega}\right)\right\}^{s-1}, \tag{34}$$

a and s having their real parts positive. Then

$$\Psi_s(x) = \int_0^1 e^{xu}u^{a-1}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}du = \sum_{\nu=0}^{\infty}\frac{x^\nu}{\nu!}\int_0^1 u^{a+\nu-1}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}du \tag{35}$$

$$= \Gamma(s)\sum_0^\infty\frac{x^\nu}{(\nu+a)^s\nu!} = \Gamma(s)F_{a,s}(x),$$

if $u^a = e^{a\log u}$ and $\{\log(1/u)\}^{s-1} = e^{(s-1)\log\log(1/u)}$, $\log u$ and $\log\log(1/u)$ being real, while $(\nu+a)^s$ is defined as in § 2. If a and s are real, it is almost obvious that, from a certain ν, c_ν is positive. Otherwise it may be verified by an extension of the argument used in the preceding section, that the condition concerning $\bar{\psi}(\omega)$ is satisfied. Thus we find the asymptotic expansion

$$\Gamma(s)x^s e^{-x}F_{a,s}(x) = \Sigma\frac{c_\nu\Gamma(s+\nu)}{x^\nu} \tag{36}$$

where c_ν is the coefficient of ω^ν in $(1-\omega)^{a-1}\left\{\frac{1}{\omega}\log\left(\frac{1}{1-\omega}\right)\right\}^{s-1}$; so that, in particular,

$$c_0 = 1,\quad c_1 = \tfrac{1}{2}(1+s)-a,\quad \dots. \tag{37}$$

This is valid throughout D if the real parts of a and s are positive. This

* "Asymptotische Darstellung von lösungen linearer Diff.-gleichungen," *Math. Annalen*, Vol. LVI., p. 129.

restriction is not, however, essential. The restriction as to s may be removed by an argument similar to that of § 7, notwithstanding that the coefficients c_ν depend upon s. The restriction as to a may be removed if we choose k so that $\mathbf{R}(a+k)$ is positive, and consider the function

$$(38)\quad F_{a,s}(x)-\sum_{\mu=0}^{k-1}\frac{x^\mu}{(a+\mu)^s\,\mu!}=\int_0^1\left\{e^{xu}-\sum_{\mu=0}^{k-1}\frac{(xu)^\mu}{\mu!}\right\}u^{a-1}\left\{\log\left(\frac{1}{1-u}\right)\right\}^{s-1}du.$$

But I do not propose to go into the details of this here; for when I come to consider the region E, which it is difficult to deal with satisfactorily by the comparatively simple and direct methods of this part of the paper, I shall have to apply to the function $F_{a,s}(x)$ a different and less elementary treatment which leads with greater ease to the desired extension. The only exceptional cases are those in which a or s is zero or a negative integer, the cases which were indicated as exceptional in § 2.

10. When x, a, and s are real and x *positive*, the dominant terms in the asymptotic equations for $f_s(x)$, $F_{a,s}(x)$ may be deduced very easily from a formula given by M. le Roy, who has proved* that, if

$$f(x)=\sum_0^\infty a_n x^n,\qquad a_n=e^{-\phi(n)},$$

where $\phi(n)$ is a positive function such that it and its first derivate $\phi'(n)$ tend steadily to ∞ for $n=\infty$, while $\phi''(n)$ tends steadily to 0, then for large values of x

$$f(x)\sim\sqrt{(2\pi)}\,\frac{e^{\xi\phi'(\xi)-\phi(\xi)}}{\sqrt{\{\phi''(\xi)\}}}$$

where ξ is defined by the equation $\phi'(\xi)=\log x$.

11. The general form of the coefficient a_n in the Taylor's expansion of $\Psi_s(x)$ is easily seen to be

$$a_n=\sum_{\nu=0}^\infty c_\nu\,\frac{\Gamma(s+\nu)}{\Gamma(s+n+\nu+1)}.$$

Thus, for instance, in the first example of § 8,

$$a_n=\frac{1}{\Gamma(1-\beta)}\sum_{\nu=0}^\infty\frac{\Gamma(1-\beta+\nu)\,\Gamma(a+\nu)}{\Gamma(1+\nu)\,\Gamma(a+n+\nu+1)}$$

$$=\frac{\Gamma(a)}{\Gamma(a+n+1)}\,F(1-\beta,\ a,\ a+n+1,\ 1).$$

* *Bull. des Sciences Math.*, t. XXIV., p. 245.

We saw otherwise that

$$a_n = \frac{\Gamma(a)\,\Gamma(\beta+n)}{\Gamma(n+1)\,\Gamma(a+\beta+n)};$$

and the two results agree in virtue of a well known property of the hypergeometric series.* A more general form of a_n which we might take would be

$$a_n = \frac{\Gamma(a)}{\Gamma(a+n+1)} F(1-\beta,\ a,\ a+n+1,\ t),$$

corresponding to $\quad \psi(1-u) = \{1-t(1-u)\}^{\beta-1}.$

12. Instead of starting with the function $f_s(x)$, we might have started with the function

$$f_{s,t}(x) = \Sigma \frac{1.2\ldots n}{s(s+1)\ldots(s+n)\,t(t+1)\ldots(t+n)} x^n$$

defined, when $\mathbf{R}(s)$ and $\mathbf{R}(t)$ are positive, by the double integral

$$\int_0^1\int_0^1 e^{xuv}(1-u)^{s-1}(1-v)^{t-1}\,du\,dv;$$

or from other more general functions which suggest themselves immediately.† But, as I said in § 2, I shall content myself for the present with indicating these generalisations.

13. Before leaving this part of the subject I may point out an interesting application of these results to the theory of multiform functions, defined by Taylor's series, with finite radii of convergence.

Application to the Function $\phi(x) = 1+\dfrac{\beta+1}{a+1}x+\dfrac{(\beta+1)(\beta+2)}{(a+1)(a+2)}x^2+\ldots.$

If

(39) $$\phi(x) = \frac{\Gamma(a+1)}{\Gamma(\beta+1)} \sum_{\nu=0}^{\infty} \frac{\Gamma(\beta+1+\nu)}{\Gamma(a+1+\nu)} x^\nu,$$

it is easy to see that $\phi(x) = \dfrac{a}{\Gamma(\beta+1)}\displaystyle\int_0^\infty e^{-t}t^\beta f_a(tx)\,dx'$

if $|x|<1$ and $R(\beta)>-1$. If x approaches $x=1$ along any path which does not meet the circle of convergence, $tx=u$ approaches infinity along a path lying entirely within the region D in the u-plane. Hence

$$f_a(tx) = \Gamma(a)\frac{e^{tx}}{(tx)^a}+R$$

where $\quad |R|<K.$

* Forsyth, *Differential Equations*, p. 199.

† The dominant term of $f_{s,t}(x)$ is easily proved to be $\Gamma(s)\,\Gamma(t)\,x^{-s-t}e^x$.

Now, if $\beta-a+1$ has its real part positive,

$$\Gamma(a)\int_0^{\infty} e^{-t+tx}t^{\beta}(tx)^{-a}\,dt = \frac{\Gamma(a)\,\Gamma(\beta-a+1)}{x^a(1-x)^{\beta-a+1}}.$$

Moreover,
$$\left|\int_0^{\infty} e^{-t}t^{\beta}R\,dt\right| < K.$$

It follows that

$$\phi(x) = \frac{\Gamma(a+1)\,\Gamma(\beta-a+1)}{\Gamma(\beta+1)\,x^a(1-x)^{\beta-a-1}}+\Theta(x) \tag{40}$$

where $\Theta(x)$ remains numerically below a finite limit as x approaches $x = 1$ along any path lying inside the circle of convergence. This result may be verified by means of the relations between the particular integrals of the hypergeometric differential equation.

The condition $\mathbf{R}(\beta) > 0$ may be removed without difficulty, either by a recurrence formula or by the use of a contour instead of a line integral. If $\mathbf{R}(\beta-a+1) < 0$, the series for $\phi(1)$ is convergent. If $\mathbf{R}(\beta-a+1) = 0$, the result still holds unless $\beta-a+1 = 0$, in which case the part of $\phi(x)$ which becomes infinite is easily found to be

$$\frac{a}{x}\log\left(\frac{1}{1-x}\right).$$

And, obviously, similar results may be obtained for such functions as

$$\sum_{n=0}^{\infty}\frac{x^n}{(n+a)^s}. \tag{41}$$

It is not difficult to determine the limit of $\Theta(x)$ in (40), and of the corresponding term in the similar formula for the function (41); but to enter into this would carry me too far from my subject.

II.—THE REGION D′ ($r \geqslant R_0$, $\frac{1}{2}\pi+\delta \leqslant \theta \leqslant \pi$).

14. The functions which we have been considering belong to a class of which it may roughly be said that they exhibit their most characteristic behaviour in the region D; and, notably, for *real positive* values of x. An obvious illustration is provided by the function $e^x-P(x)$, where P is a polynomial. The dominant term of all such functions is *the same* in D; in D′ it depends on the particular polynomial chosen. It is then not to be expected that the easy analysis of I. will be be equally effective now.

In this section I shall consider the function $\Phi_{a,s}(x)$ defined (when $\mathbf{R}(a)$ and $\mathbf{R}(s)$ are positive) by the equation

(42) $$\Phi_{a,s}(x) = \int_0^1 e^{xu} u^{a-1} \left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} \psi(u)\, du,$$

where $\psi(u)$ is a function of u subject to certain conditions. I shall not, however, treat the function in its most general form, but I shall consider only two cases: (i.) the case in which $\psi(u) \equiv 1$, (ii.) the case in which $s = 1$.

The Function $\int_0^1 e^{xu}u^{a-1}\psi(u)\,du.$

15. I shall consider first the function which is defined, when $\mathbf{R}(a) > 0$, by the equation

(43) $$\Phi_a(x) = \int_0^1 e^{xu}u^{a-1}\psi(u)\,du,$$

where $\psi(u)$ is a function satisfying the same conditions as $\psi(\omega)$ in § 4. The simplest case is that in which $\psi(u) \equiv 1$, *i.e.*,

(44) $$\Phi_a(x) = \sum_0^\infty \frac{x^n}{(a+n)\,n!} = F_{a,1}(x);$$

this function more or less fulfils the *rôle* of "simple element" fulfilled by $f_s(x)$ in I.

It is evident that

(45) $$F_{a,1}(x) = \frac{\Gamma(a)}{(-x)^a} - \int_1^\infty e^{xu}u^{a-1}\,du$$

where $(-x)^a = e^{a\log(-x)}$, the logarithm being real for real negative values of x; and this formula holds for all values of a save negative integral values (including zero). We easily find that

(46) $$\int_1^\infty e^{xu}u^{a-1}\,du = e^x\chi(x)$$

where $\chi(x)$ is a function which possesses the asymptotic expansion

(47) $$\frac{1}{x} + \frac{1-a}{x^2} + \frac{(1-a)(2-a)}{x^3} + \dots;$$

so that

(48) $$F_{a,1}(x) = \frac{\Gamma(a)}{(-x)^a} + \rho$$

where $|\rho|$ is for sufficiently large values of r less than any power of $1/r$; in other words, we may say that *the complete asymptotic expansion of* $F_{a,1}(x)$ *is*

$$(-x)^{-a}\Gamma(a).$$

16. Now consider the more general function $\Phi_a(x)$ of (48). We can prove, as in §§ 5, 6, that, if μ is a sufficiently large positive integer and

(49) $$\Phi_{a,\mu}(x) = \sum_{\nu=0}^{\mu-1} c_\nu F_{a+\nu,1}(x),$$

then

(50) $$\Phi_a(x) = \Phi_{a,\mu}(x) + \int_0^1 e^{xu} u^{a-1}\, du \sum_{\mu}^{\infty} c_\nu u^\nu.$$

We divide the range of integration into the two parts $(0, \delta)$, $(\delta, 1)$ where

$$\delta = (-\xi)^{-\lambda} \qquad (0 < \lambda < 1),$$

and we prove by analysis similar to that of § 6 that

$$\left|\int_0^1\right| \leqslant \left|\int_0^\delta\right| + \left|\int_\delta^1\right| < Ke^{\xi\delta} + K\delta^{\mu+a-1} < K\left\{e^{-(-\xi)^{1-\lambda}} + (-\xi)^{-\lambda(\mu+a-1)}\right\},$$

finally deducing that, throughout D′, $\Phi_a(x)$ possesses the asymptotic expansion

(51) $$\sum \frac{c_\nu \Gamma(a+\nu)}{(-x)^{a+\nu}}.$$

Thus, for example, if $a = \beta$, and

$$\psi(u) = (1-u)^{a-1},$$

where $\mathbf{R}(a) > 0$, we obtain for the function (i.) of § 8 the asymptotic expansion

$$\frac{1}{\Gamma(1-a)} \sum \frac{\Gamma(\beta+\nu)\,\Gamma(1-a+\nu)}{\Gamma(\nu+1)} (-x)^{-\beta-\nu}.$$

In particular, if $\beta = 1$, we obtain for the function $f_a(x)$ the expansion

$$\sum \frac{(1-a)(2-a)\ldots(\nu-a)}{(-x)^{1+\nu}}.$$

This again agrees with a result of Herr Jacobsthal's,* and the restriction on a is easily removed.

The Function $F_{a,s}(x)$.

17. I come now to the question of the behaviour of $F_{a,s}(x)$ in D′; and it is at this point that we begin to feel the need of more powerful analytical machinery.

I start from the equation

$$F_{a,s}(x) = \sum \frac{x^\nu}{\nu!\,(\nu+a)^s} = \frac{1}{\Gamma(s)} \int_0^1 e^{xu} u^{a-1} \left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} du,$$

* *L. c.*

valid when the real parts of a and s are positive. To obtain a formula for $F_{a,s}(x)$, valid for other values of a and s, I consider the integral

$$\int e^{xu}u^{a-1}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}du \tag{52}$$

taken round the contour in the plane of $u=\sigma e^{i\phi}$ formed by (i.) the positive real axis from ρ to $1-\rho$ and from $1+\rho$ to R, ρ being small and R large; (ii.) the radius vector $\phi=\pi-\theta$, from $\sigma=R$ to $\sigma=\rho$; (iii.) arcs of circles whose centres are at the origin and whose radii are ρ and R; (iv.) a small semicircle of radius ρ described around and above the point $u=1$ (see Fig. 1).

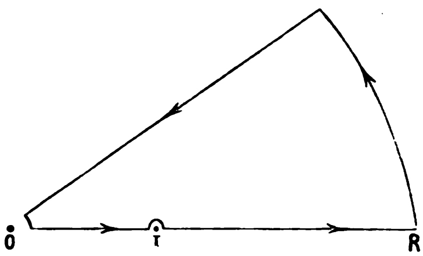


FIG. 1.

It is easy to see that the contributions of all the curvilinear parts of this contour tend to zero when ρ tends to zero and R to infinity.

We start from ρ towards $1-\rho$ with

$$u^{a-1}=e^{(a-1)\log u},\qquad \left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}=e^{(s-1)\log\log(1/u)},$$

$\log u$ and $\log\log(1/u)$ being real. If $u=1-\rho e^{i\psi}$,

$$\log\frac{1}{1-\rho e^{i\psi}}=\rho e^{i\psi}+\ldots,\qquad \log\log\left(\frac{1}{u}\right)=\log\rho+i\psi+\ldots.$$

When u is at $1-\rho$, $\psi=0$, and as u goes round the small semicircle ψ decreases to $-\pi$. When u is at $1+\rho$, $\log\log(1/u)=\log\rho-i\pi+\ldots$, and so the value of $\{\log(1/u)\}^{s-1}$ along the line $(1+\rho, R)$ is defined by

$$\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}=e^{-(s-1)\pi i}(\log u)^{s-1},$$

where
$$(\log u)^{s-1}=e^{(s-1)\log\log u},$$

$\log\log u$ being real. Thus the contribution of (i.) is ultimately

$$\int_0^1 e^{xu}u^{a-1}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1}du+e^{-(s-1)\pi i}\int_1^\infty e^{xu}u^{a-1}(\log u)^{s-1}du, \tag{53}$$

with the above assumptions as to the values of the many-valued functions involved.

Again it is easy to see that the contribution of (ii.) is

$$(54)\qquad -e^{a(\pi-\theta)i}\int_0^\infty e^{-\sigma r}\sigma^{a-1}\left\{\log\left(\frac{1}{\sigma e^{i\phi}}\right)\right\}^{s-1}d\sigma,$$

where the path of integration is real, $\sigma^{a-1}=e^{(a-1)\log\sigma}$, $\log\sigma$ being real, and

$$\left\{\log\left(\frac{1}{\sigma e^{i\phi}}\right)\right\}^{s-1}=e^{(s-1)\log[-\log\sigma+i(\theta-\pi)]},$$

that branch of $\log\{-\log\sigma+i(\theta-\pi)\}$ being taken whose imaginary part is very small with σ. Thus, by Cauchy's theorem, we arrive at the equation

$$\begin{aligned}(55)\quad \Gamma(s)F_{a,s}(x) &= e^{a(\pi-\theta)i}\int_0^\infty e^{-\sigma r}\sigma^{a-1}\{-\log\sigma+i(\theta-\pi)\}^{s-1}d\sigma\\ &\quad -e^{(s-1)\pi i}\int_1^\infty e^{xu}u^{a-1}(\log u)^{s-1}\,du\\ &= e^{a(\pi-\theta)i}\,r^{-a}\int_0^\infty e^{-t}t^{a-1}\{\log r-\log t+i(\theta-\pi)\}^{s-1}dt\\ &\quad -e^{x-(s-1)\pi i}\int_0^\infty e^{xt}(1+t)^{a-1}\{\log(1+t)^{s-1}\}\,dt,\end{aligned}$$

on transforming the two integrals by the substitutions $\sigma r=t$ and $u=1+t$. This formula, which I shall write in the form

$$(56)\qquad \Gamma(s)F_{a,s}(x)=e^{a(\pi-\theta)i}r^{-a}A-e^{x-(s-1)\pi i}B,$$

is valid if the real parts of a and s are positive.

Introduction of Loop Integrals.

20. This formula is easily generalised so as to cover all values of s save negative integral values. For consider the integral

$$\int e^{xt}(1+t)^{a-1}\{\log(1+t)\}^{s-1}dt,$$

taken round the contour C shown in the figure (Fig. 2), including the

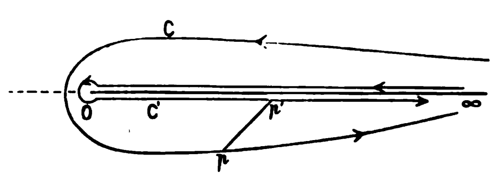


Fig. 2.

positive half of the real axis, but excluding the point $t = -1$. The values of the many-valued functions are to be so chosen that, if t moves along pp', they assume at p' the values already assigned to them in B. Then, if $\mathrm{R}(s) > 0$, the contour C may be transformed into the limit of the contour C' (see the figure) and it is easy to see that

$$(57) \qquad \int_{(C)} = \{1-e^{-2\pi i(s-1)}\}\, B,$$

$\{\log(1+t)\}^{s-1}$ being multiplied by $e^{2\pi i(s-1)}$ by a positive circuit round the origin. Thus

$$(58) \qquad \Gamma(s)F_{a,s}(x) = e^{a(\pi-\theta)i}\, r^{-a} A - \frac{e^{x-(s-1)\pi i}}{1-e^{-2\pi i(s-1)}} \int_{(C)}$$
$$= e^{a(\pi-\theta\, i} r^{-a} A - e^{x-(s-1)\pi i} B', \text{ say.}$$

This formula is valid for all non-integral values of s; while (56) is valid for all values of s whose real part > 0. Thus *one* of (56) or (57) is valid for all values of s save negative integral or zero values of s. In both, however, $\mathrm{R}(a) > 0$.

21. Now it is easy to see that throughout D′

$$(59) \qquad |\, B' \,| < K(-\xi)^{\gamma},$$

where γ is a real constant. For, if we take C' to be formed by two lines practically coinciding with the real axis, and a small circle of radius ρ, then along the circle

$$|\, e^{-xt} \,| < e^{-\xi\rho}, \qquad |\, (1+t)^{a-1} \{\log(1+t)\}^{s-1} \,| < K\rho^{s'-1},$$

where $s' = \mathrm{R}(s)$; so that the modulus of the contribution of the circle is less than $K\rho^{s'}e^{-\xi\rho}$. If we take $\rho = -1/\xi$, this is less than $K(-\xi)^{-s'}$. Again, the contribution of the rectilinear parts is in absolute value

$$< K\int_{\rho}^{\delta} t^{s'-1}\,dt + Ke^{\delta\xi},$$

where δ is any small quantity $> \rho$. It is easy to see that, if we take $\delta = \log(-\xi)/(-\xi)$, the first of these terms is less than $K(-\xi)^{1-s'}$, and the second less than $K/(-\xi)$. Hence the second term of (58) is in absolute value less than $Ke^{\xi}(-\xi)^{\gamma}$, or, what is the same thing, less than $Ke^{\xi}r^{\gamma}$.

22. Again

$$A = (\log r)^{s-1} \int_0^{\infty} e^{-t}\, t^{a-1} \left\{1 - \frac{\log t - i(\theta-\pi)}{\log r}\right\}^{s-1} dt,$$

and it is easy to see that the limit of this integral for $r = \infty$ is

$$\int_0^{\infty} e^{-t}\, t^{a-1}\, dt = \Gamma(a).$$

I omit the formal proof of this, which is a little tedious, and in no way particularly interesting. Hence we arrive at the following conclusion:—for all values of θ such that $\frac{1}{2}\pi+\delta \leqslant \theta < \pi$

$$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}(\log r)^{s-1}(1+\epsilon_x) \tag{60}$$

where ϵ_x is a function of x which tends to 0 for $r=\infty$, and that *uniformly* for all values of θ in question, being, in fact, numerically less than $K/\log r$. Here

$$(-x)^{-a} = r^{-a}e^{a(\pi-\theta)i},$$

which is real for $\theta=\pi$. Again $(\log r)^{s-1} = \{\log(-x)\}^{s-1}(1+\epsilon)$. Finally, from the uniformity of the convergence of $\lim \epsilon_x$, we infer that the equation (60) is valid also for $\theta = \pi$ (as may be proved independently). We have thus proved theorem II. of § 2 with the sole restriction that $\mathrm{R}(a) > 0$.

This last restriction also may be removed unless a is zero or a negative integer. For, if x_0 is a fixed point in D′ and $|x|$ is large, and the path of integration is rectilinear,

$$\int_{x_0}^{x} F_{a,s}(x)\,dx = \sum_0^\infty \frac{x^{\nu+1}}{(\nu+1)!\,(\nu+a)^s}+C = F_{a-1,s}(x)-1+C,$$

where C is independent of x. Now

$$\int_{x_0}^{x}(-x)^{-a}\{\log(-x)\}^{s-1}dx$$
$$= \left[-\frac{(-x)^{1-a}}{1-a}\{\log(-x)\}^{s-1}\right]_{x_0}^{x} + \frac{s-1}{1-a}\int_{x_0}^{x}(-x)^{-a}\{\log(-x)\}^{s-2}dx.$$

The first term may, if $0 < \mathrm{R}(a) \leqslant 1$, be put in the form

$$-\frac{(-x)^{1-a}}{1-a}\{\log(-x)\}^{s-1}(1+\epsilon_x),$$

while the second is in absolute value less than

$$K\int_{r_0}^{r} r^{-a}(\log r)^{s-2}dr < Kr^{1-a}(\log r)^{s-2}.$$

Finally, it is easy to deduce from the inequality $|\epsilon_x| < K/\log r$ that

$$\left|\int_{x_0}^{x}(-x)^{-a}\{\log(-x)\}^{s-1}\epsilon_x\,dx\right| < Kr^{1-a}(\log r)^{s-2}.$$

Thus, finally, $F_{a-1,s}(x) = \dfrac{\Gamma(a-1)}{\Gamma(s)}(-x)^{1-a}(\log r)^{s-1}(1+\epsilon_x).$

If we write a for $a-1$, the range of (60) is extended to all values of a other than $a=0$, for which $\mathrm{R}(a) > -1$. Repeating this process of extension, we arrive finally at the complete proof of II.

III. THE REGION E: THE ZEROES OF $F_{a,s}(x)$.

23. It follows from the results of I. and II. that there are infinitely many zeroes of $F_{a,s}(x)$ within the region E. In order to determine them more precisely it is necessary to determine an asymptotic formula for $F_{a,s}(x)$ valid within this region. We must distinguish three cases—the cases in which $\xi \lesseqgtr 0$.

The case $\xi < 0$.

24. The analysis which led to (58) assumed only that $\xi < 0$, and the formula is therefore valid for all such points of E. The same is true of the reduction of the first term on the right-hand side of (58) to the form

$$\Gamma(a)(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_r).$$

But we must now consider the second term more precisely. We therefore turn our attention to the integral

$$I = \int_{(C)} e^{xt}(1+t)^{a-1}\{\log(1+t)\}^{s-1}\,dt. \tag{61}$$

The real part of x being negative, it is easy to see that we may replace the contour of integration by a similar contour C_1 enclosing the origin and the straight line for which
$$t = \tau e^{i\phi}, \qquad \phi = \pi - \theta.$$

This contour we replace by a contour C_1' similar to the contour C' of § 21, taking the radius of the small circle to be $1/r$. Now

$$I = \int_{(C_1)} e^{xt}t^{s-1}\,dt + R = \{1-e^{-2\pi i(s-1)}\}\frac{\Gamma(s)}{(-x)^s}+R,$$

where $(-x)^s = r^s e^{(\theta-\pi)si}$ and

$$R = \int_{(C_1)} e^{xt}[(1+t)^{a-1}\{\log(1+t)\}^{s-1}-t^{s-1}]\,dt. \tag{62}$$

We can prove, as in § 21, that the absolute value of the contribution of the circular part of the contour is less than $Kr^{-s'-1}$. The absolute value of the remaining part of R is less than

$$K\int_{1/r}^{\infty} e^{-r\tau}\,|\,(1+\tau e^{i\phi})^{a-1}\log(1+\tau e^{i\phi})^{s-1}-\tau^{s-1}e^{(s-1)i\phi}\,|\,d\tau = K\int_{1/r}^{\delta}+\int_{\delta}^{\infty}$$

where δ is less than unity. The first term is less than

$$K\int_{1/r}^{\delta}\tau^{s'}d\tau < K(\delta^{s'+1}-r^{-s'-1}),$$

and the second than $\qquad Ke^{-r\delta}.$

If we take $\delta = \kappa \log r/r$, where $\kappa > s'+1$, both terms are small in comparison with $r^{-s'}$, and so

(63) $$\lim_{r=\infty} r^{s'} R = 0,$$

and that uniformly for all values of x whose real part is negative. Hence

(64) $$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x)+\frac{e^x}{x^s}(1+\epsilon_x')$$

where $(-x)^{-a} = e^{-a\log(-x)}$, and $x^s = e^{-s\log(x)}$, the logarithms being real on the negative and positive halves of the real axis respectively, and ϵ_x, ϵ_x' are quantities which tend uniformly to zero with $1/r$ for all values of x whose real part is negative. Owing to the uniformity of the convergence of the limit in (63) the formula is also valid when $\mathbf{R}(x) = 0$. The extension to values of a whose real part is less than 0 is much the same as before. We have only to make the almost obvious additional remark that when r is large

$$\int_{x_0}^{x} \frac{e^x}{x^s}\,dx = \frac{e^x}{x^s}(1+\epsilon_x)+C$$

where C is independent of x.

The case $\xi > 0$.

25. It is of importance for our present purpose to prove that the formula (64) is valid also for those points of E whose real part is positive. The proof of this is so similar to the preceding analysis that I shall merely indicate the principal steps in it.

We start from the formula

(65) $$\Gamma(s)\,F_{a,s}(x) = e^x\int_0^1 e^{-xu}(1-u)^{a-1}\left\{\log\left(\frac{1}{1-u}\right)\right\}^{s-1}du,$$

valid, like (35), so long as the real parts of a and s are positive; and we consider the integral

$$\int e^{-xu}(1-u)^{a-1}\left\{\log\left(\frac{1}{1-u}\right)\right\}^{s-1}du$$

taken round a contour which only differs from that of § 17 in that the radius vector (ii.) is defined by $\phi = -\theta$ and that the semicircle (iv.) is turned downwards. By arguments similar to those of § 17, we arrive at the formula

(66) $$\Gamma(s)F_{a,s}(x) = e^{x-i\theta}\int_0^\infty e^{-\sigma r}(1-\sigma e^{-i\theta})^{a-1}\left\{\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right)\right\}^{\sigma-1}d\sigma$$
$$-e^{x+(a-1)\pi i}\int_1^\infty e^{-xu}(u-1)^{a-1}\{-\log(u-1)-\pi i\}^{s-1}du$$

where in the first integral $(1-\sigma e^{-i\theta})^{a-1} = e^{(a-1)\log(1-\sigma e^{-i\theta})}$, the logarithm vanishing for $\sigma = 0$, and

$$\left\{\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right)\right\}^{s-1} = \exp\left[(s-1)\log\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right)\right],$$

wherein

$$\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right) = \sigma e^{i\theta}+\ldots, \quad \log\log\left(\frac{1}{1-\sigma e^{-i\theta}}\right) = \log\sigma - i\theta+\ldots$$

when σ is small; while in the second integral $(u-1)^{a-1} = e^{(a-1)\log(u-1)}$, the logarithm being real, and

$$\{-\log(u-1)-\pi i\}^{s-1} = \exp[(s-1)\log\{-\log(u-1)-\pi i\}],$$

$\log(u-1)$ being real and $\log\{-\log(u-1)-\pi i\}$ having its imaginary part small when $(u-1)$ is small.

We transform each of these integrals as in § 19, obtaining

(67) $\Gamma(s) F_{a,s}(x)$

$$= e^{x-si\theta} r^{-s} \int_0^\infty e^{-t} t^{s-1}\left(1-\frac{t}{r}e^{-i\theta}\right)^{a-1}\left\{\frac{re^{i\theta}}{t}\log\left(\frac{1}{1-\frac{t}{re^{i\theta}}}\right)\right\}^{s-1} dt$$

$$-e^{(a-1)\pi i}\int_0^\infty e^{-xt} t^{a-1}\{-\log t-\pi i\}^{s-1} dt$$

where in the first integral the last bracket, when expanded in powers of t, starts with the term $1+\ldots$.

The first of these integrals must be replaced by a loop integral when $\mathbf{R}(s) \leqslant 0$, as in § 20. Finally, by arguments similar to those of §§ 22–24, we arrive at the asymptotic formula (64).

The Zeroes of $F_{a,s}(x)$.

26. We have then the asymptotic formula *

(64) $$F_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}(1+\epsilon_x)+\frac{e^x}{x^s}(1+\epsilon'_x),$$

valid for all points of E, and all values of a and s other than negative integral values. If $x = \xi+i\eta$ is a zero of $F_{a,s}(x)$,

$$e^{\xi+i\eta-si\theta} r^{-s} = -\frac{\Gamma(a)}{\Gamma(s)}(\log r)^{s-1} r^{-a} e^{-(\theta-\pi)ia}(1+\epsilon).$$

* In this section I suppose, for simplicity, that a and s are real. The necessary modifications when they are not are easily made.

Now $r = \eta(1+\epsilon)$, $\log r = \log \eta (1+\epsilon)$, and $\theta = \frac{1}{2}\pi + \epsilon$. Hence

$$e^{\xi + i\eta - \frac{1}{2}s\pi i - \epsilon i} = -\frac{\Gamma(a)}{\Gamma(s)} (\log \eta)^{s-1} \eta^{s-a} e^{\frac{1}{2}\pi i a} (1+\epsilon). \tag{68}$$

Equating the moduli of the two sides, we find

$$e^{\xi} = \frac{\Gamma(a)}{\Gamma(s)} (\log \eta)^{s-1} \eta^{s-a} (1+\epsilon), \tag{69}$$

$$\xi = (s-a) \log \eta + (s-1) \log \log \eta + \log \frac{\Gamma(a)}{\Gamma(s)} + \epsilon. \tag{70}$$

Dividing (68) by (69),

$$e^{(\eta - \frac{1}{2}s\pi - \epsilon)i} = -e^{\frac{1}{2}a\pi i}, \tag{71}$$

or

$$\eta = \tfrac{1}{2}(a+s)\pi + (2k+1)\pi + \epsilon, \tag{72}$$

k being a positive integer. From (70) and (71) we deduce the asymptotic formulæ

$$\begin{cases} \xi = (s-a) \log (2k\pi) + (s-1) \log \log k + \log \dfrac{\Gamma(a)}{\Gamma(s)} + \epsilon, \\ \eta = (2k+1)\pi + \frac{1}{2}(a+s)\pi + \epsilon. \end{cases} \tag{73}$$

Thus, the zeroes of $F_{a,s}(x)$ are associated with some or all of the points obtained by giving k any large positive integral value in the above formulæ. The real part of the zeroes is therefore ultimately positive if $s > a$, negative if $s < a$. If $s = a$, its sign depends on that of $s-1$. If $s = a = 1$,

$$F_{a,s}(x) = \sum_0^\infty \frac{x^n}{(n+1)!} = \frac{e^x - 1}{x},$$

and the zeroes are all purely imaginary.

27. It still remains to be proved that *one, and only one*, zero of $F_{a,s}(x)$ corresponds to each of the points (73). The proof of this is not difficult, though a little tedious. I shall only indicate the argument briefly; it is as follows:—

In the first place, it is easy to show that the function

$$\Theta_{a,s}(x) = \frac{\Gamma(a)}{\Gamma(s)} (-x)^{-a} \{\log(-x)\}^{s-1} + \frac{e^x}{x^s} \tag{74}$$

vanishes, when k is large, once, and only once, in the immediate neighbourhood of each of the points (73). To prove this, we have only (following a line of argument which I have employed on several occasions in the

papers already referred to) to draw the portions of the curves

$$\left|\frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}\right| = \left|\frac{e^x}{x^s}\right|,$$

$$\operatorname{am}\frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1} = \operatorname{am}\left(-\frac{e^x}{x^s}\right),$$

which lie in the part of the plane in question, and to satisfy ourselves that there is in fact just one intersection near each of the points (73).

Now let δ be a fixed, but fairly small, positive quantity (such as $\frac{1}{10}$). Let us surround each of the points (73) by a closed contour, say a square with its sides parallel to the coordinate axes and all at unit distance from the point. First we prove that for all points on this square

$$(75)\qquad |\Theta_{a,s}(x)| > \delta\left|\frac{\Gamma(a)}{\Gamma(s)}(-x)^{-a}\{\log(-x)\}^{s-1}\right|.$$

Then we have only to show that for points on the square the ratio of the moduli of the two terms of $\Theta_{a,s}(x)$ lies between certain fixed limits in order to satisfy ourselves that along the contour of the square

$$(76)\qquad F_{a,s}(x) = \Theta_{a,s}(x)(1+\epsilon),$$

where ϵ is small. It follows that $F_{a,s}(x)$ has within the square the same number of zeroes as $\Theta_{a,s}(x)$, that is to say, *one*.

The Zeroes of the Two Simple Functions $f_s(x)$ and $F_{a,1}(x)$.

28. When $s = 1$ we obtain, for the zeroes of the function $F_{a,1}(x)$ which served as "simple element" in D′, the asymptotic formula

$$(77)\quad \xi = (1-a)\log(2k\pi)+\log\Gamma(a),\qquad \eta = (2k+1)\pi+\tfrac{1}{2}(a+1)\pi.$$

29. The corresponding investigation for the function $f_s(x)$, which served as our simple element in D, is simpler, and I shall not set it out in detail, as the formula

$$(78)\qquad f_s(x) = \Gamma(s)x^{-s}e^x(1+\epsilon)-x^{-1}(1+\epsilon)$$

is already known.* From this we deduce the asymptotic formula for the zeroes, viz.,

$$(79)\quad \xi = (s-1)\log(2k\pi)-\log\Gamma(s),\qquad \eta = 2k\pi+\tfrac{1}{2}(s-1)\pi.$$

It may be shown, as above, that one and only one zero is associated with each of these points.

* See *e.g.*, Jacobsthal, *loc. cit.*

30. I do not propose to attempt a similar discussion for the more general functions considered in I. and II. It is obvious that in order to apply the preceding methods assumptions would have to be made not only as regards the behaviour of the arbitrary functions ψ along the line (0, 1), but also as regards their analytic nature for complex values of u. To take a simple example, consider the function defined by the integral

$$G_a(x) = \int_0^1 e^{xu} u^{a-1} \psi(u) du \quad (a > 0)$$

and its continuation in the a-plane, $\psi(u)$ being real and expansible in a Taylor's series which converges for $u = 1$. Then the dominant terms of the asymptotic expressions for $G_a(x)$ in D and D′ respectively are $x^{-1}e^x\psi(1)$ and $\Gamma(a)(-x)^{-a}\psi(0)$ respectively. It is natural to suppose after what has preceded that in E

$$G_a(x) = \frac{e^x}{x}\psi(1)(1+\epsilon)+\Gamma(a)(-x)^{-a}\psi(0)(1+\epsilon),$$

in which case the zeroes are, when a is real, given by the points

$$(1-a)\log(2k\pi)+\log\Gamma(a)+\log\frac{\psi(0)}{\psi(1)}+i\{(2k+1)\pi+\tfrac{1}{2}(a+1)\pi\}.$$

But I do not intend now to attempt to investigate the conditions regarding $\psi(u)$ which are sufficient to establish the truth of this.

The case in which $s = 1$ and a is a Positive Integer.

31. If $s = 1$ and a is a positive integer, we can obtain an easy and interesting verification of our results. In fact, in this case,

$$(80)\quad F_{a,s}(x) = \int_0^1 e^{xu}u^{a-1}du = \frac{e^x}{x}\sum_0^{a-1}\frac{(-)^\nu(a-1)\dots(a-\nu)}{x^\nu}+\frac{(-)^a(a-1)!}{x^a},$$

as is easily found by repeated integration by parts. In the first place, this verifies the formula (64). Again, the equation $F_{a,s}(x) = 0$ takes the form

$$e^x = (-)^{a-1}(a-1)!/x^{a-1}-\dots,$$

and it follows from results which I have proved elsewhere* that the

---

* *Quarterly Journal*, Vol. xxxv., p. 261.

asymptotic solution of this equation is given by

$$\xi = (1-a)\log(2k\pi)+\log\Gamma(a), \qquad \eta = (2k+1)\pi+\tfrac{1}{2}(a+1)\pi,$$

which is in agreement with the general result. The case in which $a = 1$ has been already disposed of (§ 26, end).

IV.

32. I shall conclude this paper by a short discussion of one or two points of a miscellaneous character.

The Function $F_{a,\,-n}(x)$.

In all the preceding analysis it has been assumed that neither a nor s is a negative integer. If a is one, $F_{a,s}(x)$ is no longer defined. But the case in which s is a negative integer $-n$ is of considerable interest. In fact, in this case $F_{a,s}(x)$ reduces to the product of e^x by a polynomial $P_n(x)$ of degree n. For

$$F_{a,\,-n}(x) = \sum_0^\infty \frac{(\nu+a)^n x^\nu}{\nu!}$$

is the coefficient of t^n in the expansion of

$$n!\sum_0^\infty \frac{e^{(\nu+a)t}x^\nu}{\nu!} = n!\,e^{at+xe^t};$$

so that

$$F_{a,\,-n}(x) = \left[\left(\frac{d}{dt}\right)^n e^{at+xe^t}\right]_{t=0},$$

which is easily seen to be of the form*

(81) $$e^x P_n(x).$$

From the method of formation of the polynomials P_n it is easy to deduce the recurrence formula

(82) $$P_{n+1}(x) = (x+a)P_n(x)+x\frac{dP_n(x)}{dx};$$

so that

(83) $$P_0(x) = 1, \quad P_1(x) = x+a, \quad P_2(x) = x^2+(2a+1)x+a^2, \ \ldots.$$

If a is real and positive, the roots of $P_n(x)$ are all real and negative, and separated by those of $P_{n-1}(x)$. This is easily proved by induction.

---

* A result substantially equivalent to this was proposed as a problem in the Mathematica Tripos for 1903.

Another interesting property of these polynomials is that

$$\int_{-\infty}^{0} e^x P_n(x)\,dx = (a-1)^n. \tag{84}$$

The Equation $F_{a,s}(x) = c$.

33. The question is naturally suggested whether the functions $F_{a,s}(x)$ possess the property that for *one* value of the constant c the distribution of the roots of $F_{a,s}(x) = c$ is abnormal. It is easy to see that in certain cases they do, though the peculiarity is far less marked than in the case in which $s = 0$ (or, more generally, s is a negative integer). Suppose, for simplicity, that $s = 1$, and that a and c are real. Then we have to satisfy the equation

$$\frac{e^x}{x}(1+\epsilon)+\Gamma(a)(-x)^{-a}(1+\epsilon) = c.$$

It is easy to infer from this that ξ must be positive and large (though small in comparison with η), whatever be the value of a. If $a < 0$, we approximate to the roots by taking

$$e^x = \Gamma(a)(-x)^{1-a}(1+\epsilon),$$

and the value of c is indifferent. But, if $a > 0$, we must take

$$e^x = cx\,(1+\epsilon),$$

i.e., $\quad \xi = \log(2k\pi)+\log c+\epsilon, \qquad \eta = (2k+\tfrac{1}{2})\,\pi,$

unless $c = 0$, in which case the approximation (77) still holds. Thus the case of $c = 0$ is abnormal, provided $a > 0$ [and, more generally, provided $R(a) > 0$].

The Function $F_s(x) = \sum\limits_{1}^{\infty} \dfrac{x^n}{n^s n!}$.

34. In the case in which a is zero or a negative integer the definition of $F_{a,s}(x)$ by means of a series fails. But, if, for instance, $a = 0$, it is natural to define $F_s(x) = F_{0,s}(x)$ as

$$\lim_{a=0}\{F_{a,s}(x)-a^{-s}\} = \sum_{1}^{\infty}\frac{x^n}{n^s n!}. \tag{85}$$

If $\mathbf{R}(s) > 0$, $\quad \Gamma(s)\,F_s(x) = \displaystyle\int_0^1 \frac{e^{xu}-1}{u}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} du$

and $\quad \Gamma(s)\,F_s'(x) = \displaystyle\int_0^1 e^{xu}\left\{\log\left(\frac{1}{u}\right)\right\}^{s-1} du = \Gamma(s)\,F_{1,s}(x).$

The asymptotic expressions in D and D′ for $F_{1,s}(x)$ are

$$x^{-s}e^{x} \quad \text{and} \quad -\frac{1}{\Gamma(s)}\frac{\{\log(-x)\}^{s-1}}{x},$$

and it may be shown, in the first place, that the dominant terms in the expressions for $F_s(x)$ are dominant terms in the integrals of these expressions, namely

$$x^{-s}e^{x} \quad \text{and} \quad -\frac{\{\log(-x)\}^{s}}{\Gamma(s+1)};$$

and, in the second place, that the equation $F_s(x) = 0$ is equivalent to

$$x^{-s}e^{x} = \frac{\{\log(-x)\}^{s}}{\Gamma(s+1)}(1+\epsilon),$$

from which we deduce as an asymptotic formula for the zeroes

(86) $$\xi = s\log(2k\pi)+s\log\log k-\log\Gamma(s+1), \qquad \eta = (2k+\tfrac{1}{2}s)\pi.$$

In the particularly interesting case in which $s = 1$, so that

(87) $$F_1(x) = \sum_{1}^{\infty}\frac{x^n}{n.n!} = \mathrm{li}(e^{x})-\log(-x)-\gamma,$$

where γ is Euler's constant, the asymptotic expressions are*

(88) $$e^{x}/x, \quad -\log(-x),$$

and the formula for the zeroes is

(89) $$\log(2k\pi)+\log\log k+i(2k+\tfrac{1}{2})\pi.$$

Functions analogous to the Sine Function.

35. All the functions which have been considered so far are in many ways analogous to the ordinary exponential function. Their increase is substantially that of e^{r}, and the distribution of their zeroes is substantially similar to that of the zeroes of $e^{r}-c$ $(c \neq 0)$. Even in the case of those functions whose zeroes have not been approximated to by the methods of Section III., the asymptotic expressions obtained in I. and II. show that the zeroes ultimately lie inside any small angle issuing from O and including the imaginary axis.

* See Barnes, "On Integral Functions," *Phil. Trans.* (A), Vol. 199, p. 411, and Horn, *Crelle*, Bd. cxx., p. 1, where complete asymptotic expansions of this function are obtained.

By means of combinations of these functions we can form a variety of functions similarly related to the simple function $\sin x$.

Consider, for instance, the function

$$(90) \qquad \psi_a(x) = \int_0^1 \sin(xu) u^{a-1} du \quad [\mathrm{R}(a) > 0]$$

$$= \frac{1}{2i}\{F_{a,1}(ix) - F_{a,1}(-ix)\} = \sum_{n=0}^{\infty} \frac{(-)^n x^{2n+1}}{(2n+1+a)(2n+1)!}.$$

We easily find that in the domain D_1 for which $0 < \delta \leqslant \theta \leqslant \pi - \delta < \pi$

$$(91) \qquad \psi_a(x) = -\frac{e^{-xi}}{2x}(1+\epsilon),$$

while within the corresponding domain below the real axis

$$(92) \qquad \psi_a(x) = -\frac{e^{xi}}{2x}(1+\epsilon).$$

Thus $\psi_a(x)$ possesses the property of $\sin x$ that its modulus tends to infinity along any line issuing from the origin and going to infinity save along the real axis.

On the other hand, if ξ and ξ/η are large and ξ positive, $\psi_a(x)$ may be expressed in the form

$$(93) \qquad -\frac{e^{xi}+e^{-xi}}{2x}(1+\epsilon) + \frac{\Gamma(a)}{2i}\{(-xi)^{-a} - (xi)^{-a}\}(1+\epsilon),$$

where $(-xi)^{-a}$ has an argument nearly equal to $\frac{1}{2}\pi i a$ and $(xi)^{-a}$ one nearly equal to $-\frac{1}{2}\pi i a$. From this formula an asymptotic formula for the zeroes may be deduced. If, *e.g.*, a is real, positive, and less than unity, ξ, η, and ξ/η are all large and positive and

$$\frac{e^{-i\xi+\eta}}{\xi} = \frac{\Gamma(a)\sin\frac{1}{2}\pi a}{\xi^a}(1+\epsilon),$$

and so

$$(94) \qquad \xi = 2k\pi + \epsilon, \qquad \eta = (1-a)\log(2k\pi) + \log\{\Gamma(a)\sin\tfrac{1}{2}\pi a\} + \epsilon.$$

In the special case in which $a = 1$,

$$\psi_1(x) = \int_0^1 \sin xu\, du = \frac{1-\cos x}{x},$$

so that all the zeroes are real; in fact, $\xi = 2k\pi$, $\eta = 0$, which agrees with the general result. The close analogy between $\psi_a(x)$ and $\sin x$ is now apparent.

Functions analogous to the Function $\sum\limits_0^\infty \dfrac{x^n}{\Gamma(an+1)}$.

36. Prof. Mittag-Leffler has defined a function

$$E_a(x) = \sum_0^\infty \frac{x^n}{\Gamma(an+1)},$$

and has summarily indicated some of its properties, which are in many ways analogous to those of the exponential

$$e^{x^{1/a}}.$$

It is natural to suppose that the function

$$F_{a,\,a,\,s}(x) = \sum_0^\infty \frac{x^n}{(n+a)^s\,\Gamma(an+1)}$$

would be, to some extent at any rate, amenable to analysis similar to that of this paper. But, as Prof. Mittag-Leffler's extended memoir on the subject has not yet appeared, I shall not discuss this question further at present.

Conclusion.

37. The behaviour in D of the series

$$\sum \frac{c_n x^n}{n!}$$

where
$$c_n = \frac{1}{(n+a)^s}\left(b_0 + \frac{b_1}{n} + \frac{b_2}{n^2} + \dots\right),$$

the series $b_0 + b_1/n + \dots$ being convergent for $n \geqslant 1$, may be determined in certain cases by means of the results of Section I. But the corresponding investigations for D′ and E seem to present serious difficulties, the nature of which I have to some extent already indicated. And, if, instead of postulating the entire analytic nature of the coefficients c_n, we confine ourselves to the information furnished by inequalities, however precise, we find at once that very little progress can be made. Suppose, for instance, that we consider the function

$$F(x) = \Sigma c_n \frac{x^n}{n!}$$

where
$$c_n = \frac{1}{n+a} + \rho_n, \qquad |\rho_n| < \frac{K}{(n+a)^2}.$$

Then
$$F(x) = F_{a,\,1}(x) + \phi(x)$$

where $$| \phi(x) | < K \Sigma \frac{r^n}{(n+a)^2 n!} < K \frac{e^r}{r^2}.$$

Thus at a zero of $F(x)$

$$| F_{a,1}(x) | < K \frac{e}{r^2},$$

which, if, *e.g.*, $\xi > 0$, gives

$$\frac{e^\xi}{r} < K \frac{e^r}{r^2}, \qquad \xi < K + r - \log r,$$

an inequality which conveys very little information indeed. And this is only as it should be. Consider, for instance, the case in which

$$c_n = \frac{1}{n+a} + \frac{(-i)^n}{(n+a)^2}, \qquad F(x) = F_{a,1}(x) + F_{a,2}(-ix).$$

The modulus of this function becomes exponentially infinite when x approaches infinity along any radius vector situated in the angle $(\frac{1}{2}\pi - \delta, \frac{1}{2}\pi + \delta)$; and its zeroes are distributed over the plane in a manner entirely different from that of the zeroes of $F_{a,1}(x)$.

ON GROUPS OF ORDER $p^\alpha q^\beta$

(SECOND PAPER)

By W. BURNSIDE.

[Received November 17th, 1904—Read December 8th, 1904.]

HAVING shown that all groups of order $p^\alpha q^\beta$ are soluble, the enquiry naturally suggests itself as to whether any general law can be laid down with respect to the orders of the self-conjugate sub-groups of such groups. I have here shown that, subject to certain specified exceptions when the order is even, a group of order $p^\alpha q^\beta$ $(p^\alpha > q^\beta)$ must have a characteristic sub-group of order p^a, where a satisfies the inequality $p^a > p^\alpha q^{-\beta}$. The exceptions are interesting, as emphasizing the marked distinction that may exist between groups of even and groups of odd order.

LEMMA.—If p, q are primes, and α, β positive integers, such that $p^\alpha > q^\beta$, then p^α cannot be a factor of

$$(q-1)(q^2-1)\dots(q^\beta-1)$$

unless (i.) p is 2 and q is of the form $1+2^{2^m}$, or (ii.) q is 2 and p is of the form 2^n-1.

Let m be the index to which q belongs (mod p), and let

$$q^m = 1+yp^x \quad (y \not\equiv 0, \text{ mod } p).$$

Then p will divide q^t-1 only when t is a multiple of m; and, if

$$t = kmp^s \quad (k \not\equiv 0, \text{ mod } p),$$

p^{x+s} is the highest power of p which divides q^t-1.

Let $$\gamma m \leqslant \beta < (\gamma+1)\, m,$$

so that the only terms in the product $\prod_1^\beta (q^t-1)$ which p divides are

$$(q^m-1),\ (q^{2m}-1),\ \dots,\ (q^{\gamma m}-1).$$

Further, let $$\gamma = a_0+a_1p+a_2p^2+\dots+a_rp^r,$$

where the a's are zeroes or positive integers less than p. Then, in the product $\prod_{t=1}^{t=\gamma}(q^{tm}-1)$ there are just

$$a_s+pa_{s+1}+\ldots+p^{r-s}a_r-(a_{s+1}+pa_{s+2}+\ldots+p^{r-s-1}a_r)$$

terms which are divisible by p^{x+s} and not by p^{x+s+1}. Hence, if p^d is the highest power of p which divides the product, then

$$d=\sum_{s=0}^{s=r}[a_s+pa_{s+1}+\ldots+p^{r-s}a_r-(a_{s+1}+pa_{s+2}+\ldots+p^{r-s-1}a_r)](x+s)$$

$$=\gamma x+\frac{\gamma-a_0-a_1-\ldots-a_r}{p-1}.$$

Now $$p^d<q^\beta$$
if $$p^{md/\beta}<1+yp^x$$
or $$p^{[1/(p-1)](m\gamma/\beta)}<(yp^{x[(\beta-m\gamma)/\beta]}+p^{-m\gamma x/\beta})\,p^{m\Sigma a_i/[\beta(p-1)]}.$$

Since $\beta \geqslant m\gamma$, the right-hand side is greater than y, and the left-hand side is not greater than $p^{1/(p-1)}$. But, if p and q are both odd primes, y cannot be less than 2, and $p^{1/(p-1)}<2$. Hence, when p and q are both odd, the highest power of p which divides $\prod_{t}^{\beta}(q^t-1)$ is less than q^β, and the same holds if either p or q is 2, and y is greater than unity.

It remains to consider the two cases in which either p or q is 2, and at the same time y is 1.

If p is 2, m is unity, and therefore, when y is unity, q is a prime of the form $1+2^{2^m}$. That this gives an exception to the lemma may be verified at once by considering a particular case. Thus

$$p=2,\quad q=5,\quad x=2,\quad y=1,\quad \beta=3$$
give $$d=7,$$
and, in fact, $2^7>5^3$.

If q is 2, and y is unity, then x must be unity,* and p is a prime

* That 2^n-1 cannot be a power, higher than the first, of an integer may be proved as follows:—

If $$2^n-1=p^x,$$
then $$2(2^{n-1}-1)=p^x-1=(p-1)(p^{x-1}+p^{x-2}+\ldots+1).$$
Hence x must be odd, since the second factor on the right is not divisible by 2. But, if x is odd,
$$2^n=p^x+1=(p+1)(p^{x-1}-p^{x-2}+\ldots+1).$$
Hence $p+1=2^\nu$ $(\nu \leqslant n)$, and $2^n-1=(2^\nu-1)^x$.

If x is greater than unity, this gives
$$2^n=2^{\nu x}-x.2^{\nu(x-1)}+\ldots+x.2^\nu,$$
and therefore $x\equiv 0 \pmod 2$, in contradiction to the fact that x is odd.

of the form 2^m-1. In this case, if β be taken equal to $m\gamma+1$,

$$d=\gamma\frac{p}{p-1}-\frac{\Sigma a}{p-1},$$

and $$p^d > 2^\beta$$

if $$p^{\gamma[p/(p-1)]} > 2^{1+m\gamma}p^{\Sigma a/(p-1)},$$

or if $$p^{p/(p-1)} > (1+p)\,2^{1/\gamma}p^{\Sigma a/[\gamma(p-1)]}.$$

Now $$p^{p/(p-1)} > 1+p,$$

and therefore in this case, by taking γ (and therefore β) large enough, it can always be insured that $$p^d > 2^\beta.$$

This case, again, then gives an exception in which the lemma is not necessarily true.

COROLLARY.—If p and q are primes, the highest power of p which divides

$$\prod_{i=1}^{i=t}(q-1)(q^2-1)\dots(q^{\gamma_i}-1)$$

is less than $q^{\sum_1^t \gamma_i}$, with the same exceptions.

Let G be a group of order p^aq^β, where p and q are primes and $p^a > q^\beta$, while p and q do not come under either of the above two exceptional cases. Since G is soluble, it must have a self-conjugate sub-group whose order is a power of either p or q. Suppose that H, of order p^a, is a self-conjugate sub-group of G, and that G contains no self-conjugate sub-group whose order is a greater power of p than p^a. Then either $p^a > p^a/q^\beta$ or the factor group G/H, of order $p^{a-a}q^\beta$, where $p^{a-a} > q^\beta$, has no self-conjugate sub-group whose order is a power of p.

We are led therefore to consider the case in which a group G of order p^aq^β, where $p^a > q^\beta$, has a self-conjugate sub-group whose order is a power of q. If this is the case, let K, of order q^b, be the greatest self-conjugate sub-group of G whose order does not contain p as a factor. Then G has a sub-group G' of order p^aq^b, containing K self-conjugately; and every operation of a sub-group of order p^a in G' gives an isomorphism of K. Let $K, K_1, \dots, K_n, 1$, of orders $q^b, q^{b_1}, \dots, q^{b_n}, 1$, be a characteristic series of K. Every isomorphism of K, whose order is relatively prime to q, which transforms every operation of each of the factor-groups

$$K/K_1,\ K_1/K_2,\ \dots,\ K_n,$$

into itself, is the identical isomorphism.*

* *Theory of Groups*, p. 249.

Now the order of the group of isomorphisms of K_i/K_{i+1} is

$$(q^{b_i-b_{i+1}}-1)(q^{b_i-b_{i+1}}-q)\dots(q^{b_i-b_{i+1}}-q^{b_i-b_{i+1}-1}).$$

Hence, the greatest power of p which can be the order of a group of isomorphisms of K cannot exceed the greatest power of p which divides

$$\prod_{i=0}^{i=n}(q-1)(q^2-1)\dots(q^{b_i-b_{i+1}}-1);$$

and, by the corollary to the lemma, the greatest power of p, say p^d, which divides this product is less than q^b. Hence, the isomorphisms of K, given by the operations of a sub-group of G' of order p^{α}, must be alike in sets of $p^{\alpha-d'}$, where $d' \leqslant d$; and every operation of some sub-group of order $p^{\alpha-d'}$ must give the identical isomorphism of K, *i.e.*, must be permutable with every operation of K. Now

$$p^{d'} \leqslant p^d < q^b < q^{\beta},$$

and therefore there must be a sub-group of G, of order p^a, where $p^a > p^{\alpha}/q^{\beta}$, every one of whose operations is permutable with every operation of K. Let a represent the greatest number for which this is true. The totality of the operations of G which are permutable with every operation of K constitute a self-conjugate sub-group I of G. The order of this sub-group is $p^a q^{b'+c}$, where $q^{b'}$ is the order of the sub-group L of K formed of its self-conjugate operations and $c \geqslant 0$. The greatest sub-group common to I and K is L. If I has a self-conjugate sub-group whose order is a power of q greater than $q^{b'}$, let it be L' of order $q^{b'+c_1}$, where c_1 is as great as possible. Then L' is a self-conjugate sub-group of G, and $\{K, L'\}$ is a self-conjugate sub-group of G whose order is a higher power of q than q^b. No such sub-group exists, since it was supposed above that q^b is the highest power of q which is the order of a self-conjugate sub-group. Hence I/L has no self-conjugate sub-group whose order is a power of q. It must therefore have a self-conjugate sub-group of maximum order p^{a_1}. This is necessarily a characteristic sub-group, and I therefore contains a characteristic sub-group of order $p^{a_1}q^{b'}$. This is the direct product of L and a group of order p^{a_1}; and the latter therefore is a self-conjugate sub-group of G.

Finally, then, G must in any case have a self-conjugate sub-group whose order is a power of p; and, since, when $p^{a_1} < p^{\alpha}/q^{\beta}$, the same reasoning may be applied to the factor group of order $p^{\alpha-a_1}q^{\beta}$, G must have a self-conjugate sub-group of order p^a, where $p^a > p^{\alpha}/q^{\beta}$.

If p^{a_1} is the greatest power of p that is the order of a self-conjugate sub-group of G, then G has a characteristic sub-group G_1 of

order p^{α_1}. Similarly, G/G_1, of order $p^{\alpha-\alpha_1}q^{\beta}$ has a characteristic sub-group of order q^{β_1}, where q^{β_1} is the greatest power of q which is the order of a self-conjugate sub-group of G/G_1. Thus G has a characteristic sub-group H_1, of order $p^{\alpha_1}q^{\beta_1}$. The system of characteristic sub-groups of orders p^{α_1}, $p^{\alpha_1}q^{\beta_1}$, $p^{\alpha_1+\alpha_2}q^{\beta_1}$, $p^{\alpha_1+\alpha_2}q^{\beta_1+\beta_2}$, ... and the indices $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ corresponding to them may be extended till G itself is arrived at.

When p and q are odd, or when, one of them being 2, the other is not of the form 2^n-1 or $2^{2^m}+1$, these indices are subject to a system of inequalities, materially limiting the extent of the system of characteristic sub-groups. It has already been proved that

$$p^{\alpha_1} > \frac{p^{\alpha}}{q^{\beta}} \quad \text{and} \quad q^{\beta_1} > \frac{q^{\beta}}{p^{\alpha-\alpha_1}}.$$

Now in G/G_1 there is a characteristic sub-group of order q^{β_1}, and no self-conjugate sub-group whose order is a power of p. Every operation of G/G_1 gives an isomorphism of the characteristic sub-group of order q^{β_1}, and, unless $p^{\alpha-\alpha_1} < q^{\beta_1}$, there will be a sub-group, whose order is a power of p, every one of whose operations is permutable with every operation of the characteristic sub-group. But this, as shown in the preceding paragraph, would involve that G/G_1 had a self-conjugate sub-group whose order was a power of p, which is not the case. Hence

$$q^{\beta_1} > p^{\alpha-\alpha_1},$$

and, taking this with $$q^{\beta_1} > \frac{q^{\beta}}{p^{\alpha-\alpha_1}},$$

it follows that $$\beta_1 > \tfrac{1}{2}\beta.$$

Similarly it may be shown that

$$\alpha_2 > \tfrac{1}{2}(\alpha-\alpha_1),$$

and generally that

$$\beta_{i+1} > \tfrac{1}{2}(\beta-\beta_1-\ldots-\beta_i), \qquad \alpha_{i+1} > \tfrac{1}{2}(\alpha-\alpha_1-\ldots-\alpha_i).$$

Again, since G/G_1 has a characteristic sub-group of order $p^{\alpha_2}q^{\beta_1}$ and no self-conjugate sub-group whose order is a power of p,

$$p^{\alpha_2} < q^{\beta_1}.$$

Similarly $$q^{\beta_2} < p^{\alpha_2},$$

and so on. Thus $\beta_1 > \beta_2 > \ldots$ and $\alpha_2 > \alpha_3 > \ldots$;

but it cannot be inferred that $\alpha_1 > \alpha_2$, since G may have a self-conjugate sub-group whose order is a power of q.

It is not without interest to show how these results lend themselves to

the discussion of possible types of groups when the order is given. As a very simple illustration, I take the case of a group of order $3^8 . 5^2$. Here

$$3^5 < \frac{3^8}{5^2} < 3^6;$$

so that there must be characteristic sub-groups of order 3^i and $3^i 5^2$, i being not less than 6. The factor group of order $5^2 3^{8-i}$ must be such that in it no operation whose order is a power of 3 gives the identical isomorphism of the group of order 5^2. Hence i must be either 7 or 8. If i is 7, the group of order 5^2 is non-cyclic, and the factor group of order $5^2 . 3$ has no self-conjugate sub-group of order 5. Hence the characteristic sub-group, of order $3^7 . 5^2$, can have no sub-group, characteristic within itself, of order 5. But, if there were 3^4 sub-groups of order 5^2, there would be such a sub-group of order 5. Hence the group must either contain 5^2 sub-groups of order 3^8, with a self-conjugate sub-group of order $3^7 . 5^2$, which is a direct product; or it must contain a self-conjugate sub-group of order 3^8. To push the discussion further would be foreign to the subject of this paper.

Finally it should be remarked that for the two exceptional cases noted in the statement, the theorems proved above are not necessarily true. Thus, although 2^{11} is greater than 5^4, a group of order $2^{11} . 5^4$ may have no self-conjugate sub-group whose order is a power of 2. In fact, an Abelian group of order 5^4, whose operations are all of order 5, admits a group of isomorphisms of order 2^{11}. Similarly a group of order $2^{21} . 7^8$, though 7^8 is greater than 2^{21}, may have no self-conjugate sub-group whose order is a power of 7.

THE LINEAR DIFFERENCE EQUATION OF THE FIRST ORDER

By E. W. BARNES.

[Received September 16th, 1904—Read November 10th, 1904.]

CONTENTS.

1. In the present paper it is proposed to investigate the nature of the functions defined by linear difference equations of the first order. The three main questions to be considered are (i.) the existence of a solution, (ii.) its analytical expression, (iii.) its place among transcendental functions. These questions are, of course, bound up with one another: the first is obviously contained in the second; and the third has already formed the subject of a separate paper,* in which the general results here obtained were assumed.

Linear difference equations arose historically in arithmetical investigations connected with the theory of interpolation and with the necessarily discontinuous nature of physical experiments. And, in consequence, until recently such equations have been considered only in cases where the variable is a real number. References to many investigations of this nature are given by Boole.† The first investigation where the variable was supposed to assume all complex values appears to have been given by Guichard.‡ By means of Hermite's *coupures* he proves that there exists a holomorphic solution of

$$f(x+1)-f(x)=\mu(x)$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 2, pp. 280–292.

† Boole, *Finite Differences*, Third Edition, pp. 232, 263.

‡ Guichard, *Annales de l'École Normale Supérieure* (1887), Sér. 3, T. IV., pp. 361-380.

when $\mu(x)$ is holomorphic. He further gives a criterion for the nature of the holomorphic function $\nu(x)$ that

$$f(x+1)-\nu(x)f(x) = 0$$

may admit a holomorphic solution.

Appell* was the next to give an expression for the holomorphic solution of

$$f(x+1)-f(x) = \mu(x)$$

when $\mu(x)$ is holomorphic.

Then Mellin,† starting from a result in the theory of gamma functions due to Prym,‡ considered the nature of the solution of

$$f(x+1)-r(x)f(x) = s(x)$$

where $r(x)$ and $s(x)$ are rational functions of x.

More recently there has appeared a paper by Hurwitz.§ Some of his results anticipate those of the present investigation; they are indicated in the text. Hurwitz solved independently, but by substantially the same method, the problem considered by Appell, and showed how to solve the equation

$$\phi(x)f(x+1)-\chi(x)f(x) = \psi(x)$$

where $\phi(x)$, $\chi(x)$, and $\psi(x)$ are meromorphic functions.

2. The linear difference equation of the first order may be written

$$\phi(z)f(z+\omega)-\chi(z)f(z) = \psi(z),$$

where we assume $\phi(z)$, $\chi(z)$, and $\psi(z)$ to be analytic functions of z. It is at once evident that we may reduce this equation to two others of simpler type,

$$\frac{f_1(z+\omega)}{f_1(z)} = \frac{\phi(z)}{\chi(z)} \tag{A}$$

and

$$f_2(z+\omega)-f_2(z) = \frac{\psi(z)}{\chi(z)}f_1(z), \tag{B}$$

and that then

$$f(z) = \frac{f_2(z)}{f_1(z)}.$$

For, substituting $f_1(z)f(z)$ for $f_2(z)$ in the second equation, we have

$$f_1(z+\omega)f(z+\omega)-f_1(z)f(z) = \frac{\psi(z)}{\chi(z)}f_1(z);$$

---

* Appell, *Liouville* (1891), Sér. 4, T. VII., pp. 157–219, especially chapter i.

† Mellin, *Acta Mathematica*, T. XV., pp. 317–384.

‡ Prym, *Crelle*, Bd. LXXXII., pp. 165–172.

§ Hurwitz, *Acta Mathematica* (1897), T. XX., pp. 285–312; T. XXI., p. 243.

so that, by the first equation,

$$\phi(z)f(z+\omega)-\chi(z)f(z) = \psi(z),$$

which is the equation from which we started.

We may then regard (A) and (B) as the two fundamental equations. Although, by taking logarithms, we may reduce these to a common form, yet it is convenient to consider each separately. By so doing, we not only avoid the deviation from uniformity introduced by the logarithm, but we obtain two expressions representing solutions of either equation which correspond in some degree to the expression by the theorems of Weierstrass and Mittag-Leffler of a uniform transcendental function.

3. We propose to limit ourselves to the case when the coefficients in the difference equation are uniform functions with a single essential singularity at infinity. It is obvious that, with such a restriction, we may take $\phi(z)$, $\chi(z)$, and $\psi(z)$ to be integral functions (holomorphic functions—*fonctions entières*).

In three papers * recently published I have analysed integral functions and introduced certain definitions which it is convenient to repeat here.

A *simple* integral function is a function which may be expressed as a single Weierstrassian product, whose n-th zero a_n depends solely upon n and certain definite constants, and which is such that the law of dependence of a_n upon n is the same for all but a finite number of zeros. The function is called a *non-repeated* function if the n-th primary factor of Weierstrass's product does not correspond to a zero of order depending upon n. If there is such dependence, it is called a *repeated* simple integral function.

Functions of *multiple* linear sequence are functions whose general zero is of the type

$$f(a+m_1\omega_1+\ldots+m_r\omega_r),$$

the m's being the integers which define the particular zero.

The *order* of a simple non-repeated integral function is a real positive quantity ρ such that $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\rho+\epsilon}}$ converges and $\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\rho-\epsilon}}$ diverges, however small the real positive quantity ϵ may be. When ρ is dependent upon n the function is of infinite order.

Analogous definitions can be given for the order of repeated functions

* (1) "A Memoir on Integral Functions," *Phil. Trans. Roy. Soc.* (A), Vol. 199, pp. 411–500; (2) "The Classification of Integral Functions," *Camb. Phil. Trans.*, Vol. XIX., pp. 322–355; (3) "The Asymptotic Expansion of Integral Functions of Multiple Linear Sequence," *ibid.*, Vol. XIX., pp. 426–439.

and functions of multiple linear sequence. For them I may conveniently refer to the papers cited.

Suppose now that we have the equation (A),

$$\frac{f_1(z+\omega)}{f_1(z)} = \frac{\phi(z)}{\chi(z)}$$

where $\phi(z)$ and $\chi(z)$ are simple non-repeated integral fuuctions of finite or infinite order. The equation may be written

$$\frac{f_1(z+\omega)}{f_1(z)} = e^{G(z)} \frac{\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{a_n}\right)e^{f_n(z)}\right]}{\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{b_n}\right)e^{g_n(z)}\right]}$$

where $G(z)$ is an integral function.

Now it is evident that a solution of

$$\frac{f_1(z+\omega)}{f_1(z)} = e^{G(z)}$$

is $e^{G_1(z)}$ where $G_1(z)$ is a solution of $f(z+\omega)-f(z) = G(z)$, which is of the form of equation (B).

We limit ourselves then to the case when the quotient $\frac{\phi(z)}{\chi(z)}$, expressed as a quotient of two Weierstrassian products, involves no extraneous exponential factor.

4. In the first place, we consider the equation

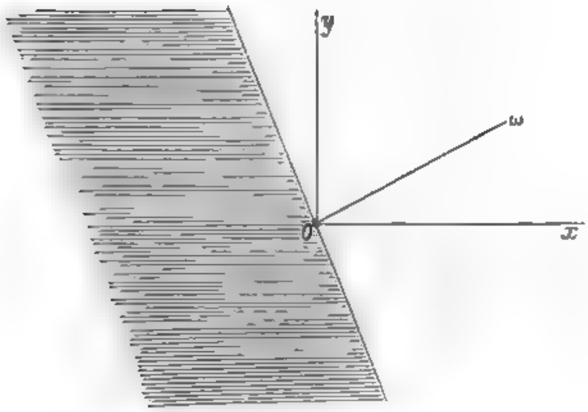
$$\frac{f(z+\omega)}{f(z)} = \phi(z)$$

where $\phi(z)$ is a holomorphic function of finite order with no extraneous exponential factor whose zeros are all negative with respect to ω. By this we mean that, if we draw a line from the origin to the point ω, the zeros all lie in the half plane which is on the other side from ω of the perpendicular through the origin to $O\omega$. This part of the plane is shaded in the figure (p. 442).

When expressed as a Weierstrassian product

$$\phi(z) = \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{a_n}\right)e^{\sum_{s=1}^{\rho-1}\frac{(-)^s}{s}\left(\frac{z}{a_n}\right)^s}\right]$$

where ρ is the integer next greater than the order of the function, and



$\frac{a_1}{\omega}, \ldots, \frac{a_n}{\omega}, \ldots$ form a series of quantities whose real parts are positive and whose moduli are arranged in non-decreasing order of magnitude.

Take the simple modified gamma function*

$$\Gamma_1^{-1}(z) = \omega^{-z/\omega} e^{\gamma z/\omega} z \prod_{m=1}^{\infty} \left[\left(1+\frac{z}{m\omega}\right) e^{-z/m\omega}\right],$$

which satisfies the difference equation

$$\frac{f(z+\omega)}{f(z)} = z.$$

As usual, we write $\qquad \psi_1^{(r)}(z) = \frac{d^r}{dz^r} \log \Gamma_1(z) \quad (r = 1, 2, \ldots).$

Construct the product

$$P(z) = \prod_{n=1} \left[\frac{\Gamma_1(z+a_n)}{\Gamma_1(a_n)} e^{-\sum_{s=1}^{\sigma-1} \frac{z^s}{s!} \psi_1^{(s)}(a_n)}\right]$$

where σ is an integer to be presently determined.

The first $k-1$ terms of this product for $P(z)$ are evidently convergent so long as $P(z)$ is finite. If we suppose that $|z| < |a_k|$, we may expand $\log \frac{\Gamma_1(z+a_n)}{\Gamma_1(a_n)}$ by Taylor's theorem and write the remaining terms in the product for $P(z)$ in the form

$$\exp\left[\sum_{n=k}^{\infty} \sum_{s=\sigma}^{\infty} \psi_1^{(s)}(a_n) \frac{z^s}{s!}\right].$$

---

* This function is chosen with form given on account of its analogy with multiple gamma functions. Its theory was worked out by the author in the *Messenger of Mathematics*, Vol. XXIX., pp. 64–128.

Now $$\psi_1^{(1)}(z) = -\frac{\gamma}{\omega} + \frac{\log \omega}{\omega} - \frac{1}{z} - \sum_{m=1}^{\infty} \left\{ \frac{1}{z+m\omega} - \frac{1}{m\omega} \right\},$$

and therefore, if $s > 1$,

$$\psi_1^{(s)}(z) = (-)^s (s-1)! \sum_{m=0}^{\infty} \frac{1}{(z+m\omega)^s}.$$

The remaining terms in the product for $P(z)$ may therefore be written (assuming that $\sigma > 1$)

$$\exp \left[\sum_{n=k}^{\infty} \sum_{s=\sigma}^{\infty} \sum_{m=0}^{\infty} \frac{(-z)^s}{s(a_n+m\omega)^s} \right] = \exp Z, \text{ say.}$$

Consider first the double series

$$\sum_{s=\sigma}^{\infty} \sum_{m=0}^{\infty} \frac{(-z)^s}{s(a_n+m\omega)^s}.$$

Each series $\sum_{m=0}^{\infty} \frac{(-z)^s}{s(a_n+m\omega)^s}$, being equal to $\psi_1^{(s)}(a_n)\frac{z^s}{s!}$, is absolutely convergent for all values of s.

Again, since $-a_n$ is negative with respect to ω, we may use the asymptotic expansion for $\psi_1^{(s)}(a_n)$, and we see that

$$\sum_{s=\sigma}^{\infty} \sum_{m=0}^{\infty} \frac{(-z)^s}{s(a_n+m\omega)^s} \text{ converges with } \sum_{s=\sigma}^{\infty} \frac{z^s}{s!} \frac{(-)^s(s-2)!}{a_n^{s-1}},$$

and is thus absolutely convergent if $|z/a_n| < 1$.

For when $|z|$ is large and not near the line of poles of $\psi_1^{(1)}(z)$ we have asymptotically*

$$\psi_1^{(1)}(z) = \frac{1}{\omega} \log z + \left(\frac{z}{\omega} - \frac{1}{2} \right) \frac{1}{z} + \sum_{s=1}^{\infty} \frac{(-)^r}{z^{r+1}} B_{r+1}(\omega)$$

and $$\psi_1^{(s)}(z) = \frac{(-)^s(s-2)!}{\omega z^{s-1}} + \text{smaller terms.}$$

Hence, by a result due to Cauchy,† the double series is equal to

$$\sum_{m=0}^{\infty} \sum_{s=\sigma}^{\infty} \frac{(-z)^s}{s(a_n+m\omega)^s}.$$

We may therefore write

$$Z = \sum_{n=k}^{\infty} \sum_{m=0}^{\infty} \sum_{s=\sigma}^{\infty} \frac{(-z)^s}{s(a_n+m\omega)^s};$$

* *Loc. cit.*, Part IV.

† *Analyse Algébrique*, Note VII.; *Résumés Analytiques*, § 8.

and therefore

$$|Z| < \frac{1}{\sigma} \sum_{n=k}^{\infty} \sum_{m=0}^{\infty} \sum_{s=\sigma}^{\infty} \frac{|z|^s}{|a_n+m\omega|^s}$$

$$< \frac{1}{\sigma} \sum_{n=k}^{\infty} \sum_{m=0}^{\infty} \frac{|z|^\sigma}{|a_n+m\omega|^\sigma} \Big/ \left\{1 - \left|\frac{z}{a_n+m\omega}\right|\right\},$$

provided $|z/(a_n+m\omega)| < 1$, that is to say, remembering the distribution of the points a_n, provided $|z| < a_k$.

If now μ be the minimum value of $1-|z/(a_n+m\omega)|$, we have

$$|Z| < \frac{|z|^\sigma}{\mu^\sigma} \sum_{n=k}^{\infty} \sum_{m=0}^{\infty} \frac{1}{|a_n+m\omega|^\sigma}.$$

It is necessary now to investigate series of this type.

5. We proceed to prove that, if

$$\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n,\ \ldots, \qquad \beta_1,\ \beta_2,\ \ldots,\ \beta_n,\ \ldots$$

form two series of quantities whose graphic representations lie within a quadrant of the Argand diagram, and whose moduli form in each case a non-decreasing series, and if $\sum_{n=1}^{\infty} \frac{1}{|\alpha_n|^\rho}$ and $\sum_{n=1}^{\infty} \frac{1}{|\beta_n|^\sigma}$ are absolutely convergent, then is

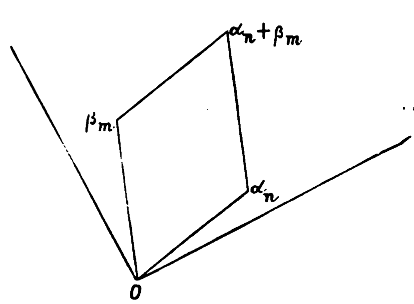
$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(\alpha_n+\beta_m)^s}$$

an absolutely convergent series, provided s is equal to or greater than the greater of the two quantities 2ρ and 2σ.

Since the quantities α_n and β_m lie within a quadrant of the Argand diagram, we readily see from a figure that

$$|\alpha_n+\beta_m|^2 > a_n^2+b_m^2$$

where $\qquad a_n = |\alpha_n| \quad \text{and} \quad b_m = |\beta_m|.$



In the limiting case when the quantities lie respectively on the arms of the quadrant this inequality becomes an equality.

The modulus of the series to be investigated is therefore

$$\leqslant \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(a_n^2+b_m^2)^{\frac{1}{2}s}}.$$

Now $a_n^2+b_m^2 \geqslant 2a_n b_m$; hence the modulus is

$$< \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(2a_n b_m)^{\frac{1}{2}s}} < \frac{1}{2^{\frac{1}{2}s}} \sum_{n=1}^{\infty} \frac{1}{a_n^{\frac{1}{2}s}} \sum_{m=1}^{\infty} \frac{1}{b_m^{\frac{1}{2}s}}.$$

Thus the series is absolutely convergent, in any of the four ways in which we may sum it, and converges to the same definite limit provided s is greater than or equal to the greater of the two quantities 2ρ and 2σ. The proposition is therefore established.

6. Return now to the series $\sum_{n=k}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(a_n+m\omega)^{\sigma}}$, which arose in § 4. The only restriction which we have so far imposed upon σ is that it is greater than 1. The quantities $a_1, \ldots, a_n, \ldots$ were taken to be positive with respect to the ω's, which is equivalent to saying that the a's lie in the Argand diagram within an angle of 90° on either side of the positive direction of the axis of ω.

We assumed that $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\rho}}$ is convergent, and we know that $\sum_{m=1}^{\infty} \frac{1}{|m\omega|^{1+\epsilon}}$ is convergent, if $\epsilon > 0$.

Hence the series $\sum_{n=k}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(a_n+m\omega)^{\sigma}}$ is absolutely convergent, provided $\sigma \geqslant 2\rho$ and $\sigma > 2$.

With these limitations on σ we see that $P(z)$ is absolutely convergent at all points of the plane except the poles of the functions $\Gamma_1(z+a_n)$. $P(z)$ is therefore a one-valued meromorphic function of z with these poles and no zeros.

7. Consider now the quotient $P(z+\omega)/P(z)$.

Since $\Gamma_1(z+\omega) = z\Gamma_1(z)$, we evidently have

$$\frac{P(z+\omega)}{P(z)} = \prod_{n=1}^{\infty} \left[(z+a_n)\, e^{-\sum_{s=1}^{\sigma-1} \frac{\psi_1^{(s)}(a_n)}{s!}[(z+\omega)^s - z^s]}\right]$$

$$= \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{a_n}\right) e^{\sum_{s=1}^{\rho-1} \frac{(-)^s}{s}\left(\frac{z}{a_n}\right)^s}\right]$$

$$\times \prod_{n=1}^{\infty} \left[\exp\left\{\sum_{s=1}^{\rho-1} \frac{(-)^{s-1}}{s}\left(\frac{z}{a_n}\right)^s - \sum_{s=1}^{\sigma-1} \frac{\psi_1^{(s)}(a_n)}{s!}[(z+\omega)^s - z^s]\right\}\right].$$

Now $P(z+\omega)$, $P(z)$ and the first product on the right-hand side are each absolutely convergent. The second product on the right-hand side is

therefore absolutely convergent and may be written

$$\exp\left\{-\sum_{m=0}^{\sigma-2} c_m z^m\right\}$$

where the c's are definite finite functions of the a's.*

We have now

$$\frac{P(z+\omega)}{P(z)} = \phi(z)\exp\left\{-\sum_{m=0}^{\sigma-2} c_m z^m\right\}.$$

Hence the function $P(z)\exp\left[\sum_{m=0}^{\sigma-2} c_m S_m(z)\right]$, where $S_m(z)$ is the Bernoullian function of z,† satisfies the difference equation with which we started in § 4,

$$f(z+\omega)-\phi(z)f(z) = 0.$$

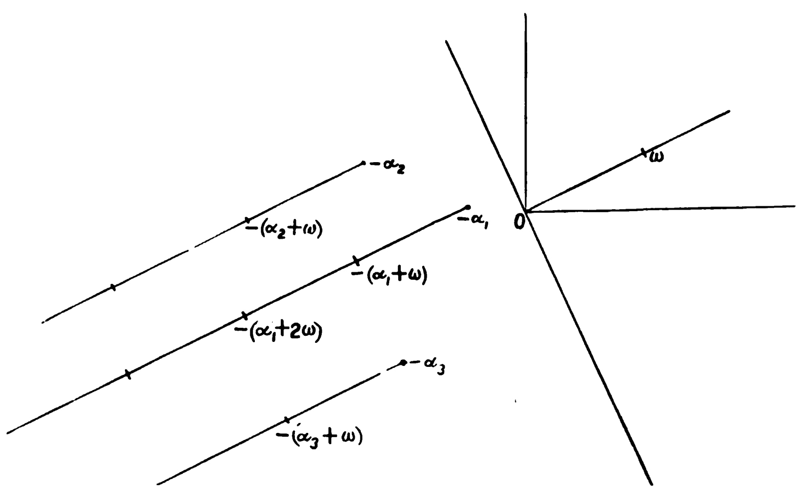
The general solution of this difference equation is the particular solution multiplied by a simply periodic function of z of period ω.

The principal solution is a meromorphic function with no zeros, all of whose poles are at the points

$$z = -a_n - m\omega \quad (m = 0, 1, \ldots, \infty),$$

the zeros of $\phi(z)$ being at the points $z = -a_n$.

We propose to say that the poles of the particular solution are at the points *negatively congruent* to the zeros of $\phi(z)$. These points are the doubly infinite series formed by sequences of points stretching out negatively at intervals ω from the points $z = -a_n$ as in the diagram.



* In certain special cases I have evaluated the c's by means of asymptotic expansions. See "The Theory of the Double Gamma Function," *Phil. Trans. Roy. Soc.* (A), Vol. 146, pp. 265–387, §§ 40 and 41; and, for the extension to multiple gamma functions, *Camb. Phil. Trans.*, Vol. XIX., pp. 374–425, Part 3.

† Vide *Messenger of Mathematics*, Vol. XXIX., pp. 64–128, Part 2.

The reciprocal of the particular solution is a non-repeated integral function of double sequence.

In the course of the proof we have imposed on $\phi(z)$ the limitation that all its zeros should be negative with respect to ω. But it is evident that, in general, a finite number may be at finite points on the positive side of or actually on the perpendicular through the origin to $O\omega$. For, corresponding to such points, we have in general a finite primary gamma factor multiplying the solution obtained. The exceptional case, which may be readily treated, occurs when $-(a_n+m\omega)$ is, for some values of m and n, identically zero.

8. We will consider next the solution of the difference equation

$$f(z+\omega)-\phi(z)f(z)=0,$$

where $\phi(z)$ is a simple non-repeated integral function of finite order without extraneous exponential factor, and all but a finite number of its finite zeros lie on the positive side of the perpendicular through the origin to $O\omega$.

Neglecting the exceptional zeros which may be treated separately, we may put

$$\phi(z)=\prod_{n=1}^{\infty}\left[\left(1-\frac{z}{\beta_n}\right)e^{\sum_{s=1}^{\rho-1}\frac{1}{s}\left(\frac{z}{\beta_n}\right)^s}\right],$$

where ρ is the integer next greater than the order of the function and $\beta_1/\omega, \ldots, \beta_n/\omega, \ldots$ form a series of quantities which have their real parts positive, and are arranged so that their moduli are in non-decreasing order of magnitude. We construct the product

$$P(z)=\prod_{n=1}^{\infty}\left[\frac{\Gamma_1(\omega+\beta_n)}{\Gamma_1(\omega+\beta_n-z)}\exp\left\{\sum_{s=1}^{\sigma-1}\frac{(-z)^s}{s!}\psi_1^{(s)}(\omega+\beta_n)\right\}\right].$$

The terms of $P(z)$ after the first $(k-1)$ are equal to

$$\exp\left[\sum_{n=k}^{\infty}\sum_{s=\sigma}^{\infty}\left\{-\frac{(-z)^s}{s!}\psi_1^{(s)}(\omega+\beta_n)\right\}\right]$$

provided $$|z|<\omega+\beta_k.$$

Therefore, if $\sigma>1$, we see, as in § 4, that they may be written

$$\exp\left[-\sum_{n=k}^{\infty}\sum_{s=\sigma}^{\infty}\sum_{m=0}^{\infty}\frac{z^s}{s(\omega+\beta_n+m\omega)^s}\right]=\exp\left[-\sum_{n=k}^{\infty}\sum_{s=\sigma}^{\infty}\sum_{m=1}^{\infty}\frac{z^s}{s(\beta_n+m\omega)^s}\right].$$

Exactly as in § 6, we can now demonstrate that the product $P(z)$ is absolutely convergent, provided $\sigma \geqslant 2\rho$ and $\sigma > 2$. Now

$$\frac{P(z+\omega)}{P(z)} = \prod_{n=1}^{\infty}\left[\frac{\Gamma_1(\omega+\beta_n-z)}{\Gamma_1(\beta_n-z)} \exp \sum_{s=1}^{\sigma-1} \psi_1^{(s)}(\omega+\beta_n)\frac{(-)^s}{s!}\{(z+\omega)^s-z^s\}\right]$$

$$= \phi(z)\prod_{n=1}^{\infty}\exp\left[\log\beta_n - \sum_{s=1}^{\rho-1}\frac{1}{s}\left(\frac{z}{\beta_n}\right)^s\right.$$

$$\left. + \sum_{s=1}^{\sigma-1}\frac{(-)^s}{s!}\psi_1^{(s)}(\omega+\beta_n)\{(z+\omega)^s-z^s\}\right]$$

$$= \phi(z)\exp\left\{-\sum_{s=0}^{\sigma-2} d_s z^s\right\},$$

where the quantities d are finite functions of the β's. Thus a solution of

$$f(z+\omega)-\phi(z)f(z) = 0$$

is

$$P(z)\exp\left\{\sum_{s=0}^{\sigma-2} d_s S_s(z)\right\},$$

and the general solution consists of this function multiplied by an arbitrary simply periodic function of z of period ω. The particular solution thus obtained is an integral function of z whose zeros are at the points positively congruent to the zeros of $\phi(z)$, these zeros excluded.

When $\omega = 1$ our restriction is that all but a finite number of zeros of $\phi(z)$ should lie to the right of the imaginary axis, in order that a particular solution of $f(z+1) = \phi(z)f(z)$, where $\phi(z)$ is holomorphic, should itself be holomorphic. It may be compared with Guichard's result.*

9. It is now evident that there exists a meromorphic solution of

$$f(z+\omega) - \mu(z)f(z) = 0$$

where $\mu(z)$ is a meromorphic function of z which is a quotient of two simple non-repeated integral functions of finite order which when expressed as Weierstrassian products involve no extraneous exponential factor.

Under the specified conditions we may write

$$\mu(z) = \prod_{n=1}^{\infty}\left\{\frac{\left[\left(1+\frac{z}{a_n}\right)e^{\sum_{s=1}^{\rho-1}\frac{(-z)^s}{s a_n^s}}\right]\left[\left(1-\frac{z}{\beta_n}\right)e^{\sum_{s=1}^{\sigma-1}\frac{1}{s}\left(\frac{z}{\beta_n}\right)^s}\right]}{\left[\left(1+\frac{z}{\gamma_n}\right)e^{\sum_{s=1}^{\tau-1}\frac{(-z)^s}{s\gamma_n^s}}\right]\left[\left(1-\frac{z}{\delta_n}\right)e^{\sum_{s=1}^{\nu-1}\frac{1}{s}\left(\frac{z}{\delta_n}\right)^s}\right]}\right\},$$

---

* *Loc. cit.*, § 1, p. 376. See also Hurwitz, *loc. cit.*, p. 312.

where ρ, σ, τ, υ are integers next greater than the orders of the respective products, and a_1/ω, ..., a_n/ω, ..., and the three corresponding sequences form series whose real parts are positive and whose moduli are arranged in non-decreasing order of magnitude.

And now, from the results of §§ 7 and 8, a solution of

$$f(z+\omega)-\mu(z)f(z) = 0$$

is
$$\prod_{n=1}^{\infty}\left\{\frac{\left[\dfrac{\Gamma_1(z+a_n)}{\Gamma_1(a_n)}e^{-\sum\limits_{s=1}^{\rho'-1}\psi_1^{(s)}(a_n)\frac{z^s}{s!}}\right]\left[\dfrac{\Gamma_1(\omega+\beta_n)}{\Gamma_1(\omega+\beta_n-z)}e^{\sum\limits_{s=1}^{\sigma'-1}\frac{(-z)^s}{s!}\psi_1^{(s)}(\omega+\beta_n)}\right]}{\left[\dfrac{\Gamma_1(z+\gamma_n)}{\Gamma_1(\gamma_n)}e^{-\sum\limits_{s=1}^{\tau'-1}\psi_1^{(s)}(\gamma_n)\frac{z^s}{s!}}\right]\left[\dfrac{\Gamma_1(\omega+\delta_n)}{\Gamma_1(\omega+\delta_n-z)}e^{\sum\limits_{s=1}^{\upsilon'-1}\frac{(-z)^s}{s!}\psi_1^{(s)}(\omega+\delta_n)}\right]}\right\}$$

$$\times\exp\left\{\sum_{s=0}^{\delta-2}d_m S_m(z)\right\},$$

where ρ', σ', τ', υ' are integers such that $\rho' \geqslant 2\rho$ and $\rho' > 2$ and corresponding inequalities, and δ is the greatest of ρ', σ', τ', υ'. The general solution is this meromorphic solution multiplied by a simply periodic function of z of period unity.

The particular solution is a meromorphic function with (1) sequences of zeros proceeding positively from but excluding the points β_n, (2) sequences of zeros proceeding negatively from and including the points $-\gamma_n$, (3) sequences of poles proceeding negatively from and including the points $-a_n$, (4) sequences of poles proceeding positively from but excluding the points δ_n.

10. We have now to consider the solution of the equation

$$f(z+\omega)-\mu(z)f(z) = 0$$

when $\mu(z)$ is a meromorphic function, as in § 9, except that the Weierstrassian products are of infinite order. The investigation can be briefly indicated: it is an almost obvious extension of the process previously employed. Take first the equation

$$f(z+\omega)-\phi(z)f(z) = 0$$

where $\phi(z)$ is a similar function to that considered in § 4, but of infinite order. We may put

$$\phi(z) = \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{a_n}\right)e^{\sum\limits_{s=1}^{\rho_n-1}\frac{(-z)^s}{sa_n^s}}\right]$$

where ρ_n is an integer infinite with n such that the series $\sum_{n=1}^{\infty} \frac{1}{a_n^{\rho_n}}$ is absolutely convergent. (As is well known, it always suffices to take $\rho_n = n$.)

Construct the function

$$P(z) = \prod_{n=1}^{\infty} \left[\frac{\Gamma(z+a_n)}{\Gamma(a_n)} e^{-\sum_{s=1}^{\sigma_n - 1} \frac{z^s}{s!} \psi_1^{(s)}(a_n)} \right].$$

Neglecting a finite number of factors, and taking $|z| < |a_k|$, we see that $P(z)$ converges with

$$\exp \left[\sum_{n=k}^{\infty} \sum_{s=\sigma_n}^{\infty} \sum_{m=0}^{\infty} \frac{(-z)^s}{s(a_n + m\omega)^s} \right].$$

By exactly the same process as before, we prove that this expression is convergent if $\frac{1}{2}\sigma_n > \rho_n$ and $\sigma_n > 2$. The second of these criteria is included in the first, and we see that we must have $\sigma_n > 2\rho_n$.

Proceed again as before, and we see that

$$\frac{P(z+\omega)}{P(z)} = \phi(z) e^{G(z)}$$

where $G(z)$ is an integral function which, when written in the form

$$G(z) = \sum_{n=0}^{\infty} a_n z^n,$$

has its coefficients a definite functions of the quantities a. Before, then, we can completely solve

$$f(z+\omega) - \phi(z) f(z) = 0$$

we must solve

$$f(z+\omega) - e^{G(z)} f(z) = 0,$$

an equation whose solution is $e^{G_1(z)}$ where $G_1(z)$ is a solution of

$$f(z+\omega) - f(z) = G(z).$$

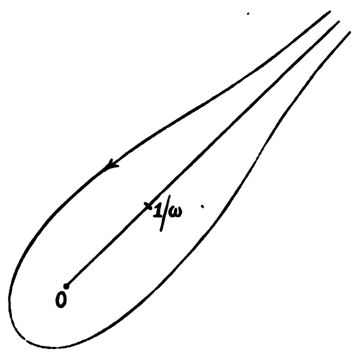
This equation belongs to the type (B) of § 2. We now proceed to consider it.

11. I have previously investigated* the extended Riemann ζ function, which, when $\mathfrak{R}(z/\omega)$ is positive, can be represented by

$$\zeta(s, z, \omega) = \frac{\iota\Gamma(1-s)}{2\pi} \int \frac{e^{-yz}}{1-e^{-\omega y}} (-y)^{s-1} dy.$$

---

* *Loc. cit.*, § 4, Part III.



The integral is taken along a contour extending along the axis of $1/\omega$ from $+\infty/\omega$ round the origin and back again to $+\infty/\omega$, as in the figure, the contour containing no infinities of the subject of integration except the origin. That value of

$$(-y)^{s-1} = \exp\{(s-1)\log(-y)\}$$

is taken which has its principal value with respect to the axis to $1/\omega$.

The function is fundamental in the theory of simple gamma and Bernoullian functions, and, as Mellin* has shown, can be represented by a series of functions valid for all values of z.

When s is a positive integer greater than 1

$$\zeta(s, z, \omega) = \frac{(-)^s}{(s-1)!}\frac{d^s}{dz^s}\log\Gamma_1(z),$$

and when s is a negative integer

$$\zeta(s, z, \omega) = -\{S_{-s}(z)+B_{1-s}(\omega)\}.$$

Also, *qua* function of s, $\zeta(s, z, \omega)$ is finite for all values of s except $s = 1$.

Further, $$\zeta(s, z+\omega, \omega)-\zeta(s, z, \omega) = -z^{-s},$$

the principal value relative to the axis of $-\omega$ being taken when s is not an integer. Thus for all values, real or complex, of s, except $s = 1$, $-\zeta(-s, z, \omega)$ is a solution of the difference equation

$$f(z+\omega)-f(z) = z^s. \tag{1}$$

Not only so, but, if $\zeta_k(s, z, \omega)$ denotes the function defined as above when $\mathfrak{R}(z/\omega)$ is positive, except that the contour of the integral includes $2k$ of the poles of the subject of integration besides the origin, $-\zeta_k(-s, z, \omega)$ is equally a solution of the difference equation (1).

Suppose now that we have the equation

$$f(z+\omega)-f(z) = G(z)$$

where $G(z)$ is the integral function $\sum_{n=0}^{\infty} a_n z^n$. A formal solution will be

$$-\sum_{n=0}^{\infty} a_n \zeta_{k_n}(-n, z, \omega),$$

* Mellin, *Acta Soc. Sci. Fennicæ*, T. xxiv., No. 10, Part I. Mellin's function is obtained by putting $\omega = 1$. I have generalized it and extended the theory to the case of r parameters in my memoir on the multiple gamma function.

and this solution will be valid provided the series of functions is absolutely convergent at all finite points of the plane. Suppose, in the first place, that $G(z)$ is of finite non-zero order ρ. Then* $|a_n|$ behaves when n is very large like $(1/n!)^{1/\rho}$ to a first approximation. And therefore

$$-\sum_{n=0}^{\infty} a_n \zeta(-n, z, \omega)$$

behaves like

$$-\frac{\iota}{2\pi}\int \frac{e^{-yz}}{1-e^{-\omega y}} \sum_{n=0}^{\infty} n!\, a_n (-y)^{-n-1}\, dy = \frac{\iota}{2\pi}\int \frac{e^{-yz}\psi(1/y)}{1-e^{-\omega y}}\frac{dy}{y}$$

where the contour of integration includes the origin, but no other zero of $1-e^{-\omega z}$, and where

$$\psi\left(\frac{1}{y}\right) = \sum_{n=0}^{\infty} \frac{(-)^n a_n n!}{y^n}.$$

When $\rho < 1$, $|(-)^n a_n n!|$ behaves to a first approximation like $(n!)^{-(1-\rho)/\rho}$, and therefore $\psi(z)$ is an integral function of order $\rho/(1-\rho)$. Thus, when $\rho < 1$, a solution of

$$f(z+\omega)-f(z) = G(z)$$

is

$$\frac{\iota}{2\pi}\int \frac{e^{-yz}\psi(1/y)}{1-e^{-\omega y}}\frac{dy}{y}.$$

This integral defines a solution for all values of z and ω. For the integral taken along the prescribed contour may be at once reduced to an integral taken round a circle, centre the origin, and radius less than $|2\pi\iota/\omega|$.

12. We have thus found a solution of

$$f(z+\omega)-f(z) = G(z)$$

where the order of $G(z)$ is less than unity.

If the order of $G(z)$ is not less than unity, one of two things may happen: $\psi(z)$ may still be a function convergent within a circle of finite radius λ, or the series for $\psi(z)$ may diverge absolutely. In the first case our course is evident: we take the previous integral round a circle of radius $> 1/\lambda$, and thus obtain a solution of the difference equation. In the second case the series for $\psi(y)$ is truly asymptotic, *i.e.*, of zero radius of convergence, and the solution breaks down. But in this case the formal solution which we have found is not altogether nugatory. For $G(z)$ and $\psi(z)$ are what Borel† has called *associated* functions, and, as may be

* See the author's "Classification of Integral Functions" (*loc. cit.* § 3), § 35, p. 349, where references to the work of Hadamard, Borel, and others are given.

† Borel, *Annales de l'École Normale Supérieure* (1899), pp. 1–136, especially pp. 89 *et seq.*

deduced from the theory of asymptotic series,* the divergent series for $\psi(y)$ may be used to determine a function which possesses the properties which are required to build up by the method indicated a solution of the difference equation.

By means, however, of a series of modified functions $\zeta_m(-m, z, \omega)$ we may formally give a convergent expansion for this solution.

Let $\zeta_m(-m, z, \omega)$ denote the integral

$$\frac{\iota m!}{2\pi}\int\frac{e^{-zy}}{1-e^{-\omega y}}\frac{dy}{(-y)^{m+1}}$$

taken round a circle of radius $(2m+1)\pi\iota/\omega$, so that the subject of integration has $2m$ poles inside the contour in addition to the origin. At the pole $2k\pi\iota/\omega$ the residue is

$$\frac{-m!\,e^{-2k\pi\iota z/\omega}}{\omega\left(-\dfrac{2k\pi\iota}{\omega}\right)^{m+1}}.$$

The function $\zeta_m(-m, z, \omega)$ therefore differs from $\zeta(-m, z, \omega)$ by a function of z which is simply periodic of period ω. We have

$$\zeta_m(-m, z+\omega, \omega)-\zeta_m(-m, z, \omega) = -z^m,$$

and hence $-\sum\limits_{n=0}^{\infty} a_n\zeta_n(-n, z, \omega)$ is a formal solution of the difference equation

$$f(z+\omega)-f(z) = G(z).$$

Now, when n is large,†

$$|\zeta_n(-n, z, \omega)| < \frac{n!\,e^{\left|z\frac{(2n+1)\pi\iota}{\omega}\right|}}{K\left|(2n+1)\dfrac{\pi\iota}{\omega}\right|^n},$$

where K is the minimum value of $|1-e^{-\omega y}|$ on the circle of radius $(2n+1)\pi\iota/\omega$, and is therefore > 0.

Thus $$|\zeta_n(-n, z, \omega)| < \frac{e^{\left|z\frac{(2n+1)\pi\iota}{\omega}\right|}}{K\left|\dfrac{2\pi\iota}{\omega}e\right|^n} \text{ approximately.}$$

Now, since $G(z)$ is an integral function, $|\sqrt[n]{a_n}|$ tends to zero with $1/n$. Hence $|\sqrt[n]{\{a_n\zeta_n(-n, z, \omega)\}}|$ tends to zero with $1/n$, and therefore the formal solution $\sum\limits_{n=0}^{\infty} a_n\zeta_n(-n, z, \omega)$ is a series of functions absolutely convergent for all finite values of $|z|$. It therefore represents an integral

* See the author's *Memoir on Integral Functions*, § 29, &c.

† Forsyth, *Theory of Functions* (1900), § 15.

function of z. Therefore there exists a solution of

$$f(z+\omega)-f(z) = G(z),$$

where $G(z)$ is any integral function of z, which is an integral function: the general solution is obtained by adding to this particular solution an arbitrary simply periodic function of z of period ω.

The substitution of the function $\zeta_n(-n, z, \omega)$ for $\zeta(-n, z, \omega)$ is substantially the same as the process employed by Mittag-Leffler in his well known theorem. It is from this point of view that the matter has been considered by Hurwitz, who has anticipated the results just obtained.

13. We see now that we have the means of completing our solution of the equation

$$f(z+\omega)-\mu(z)f(z) = 0$$

for all cases in which $\mu(z)$ is a meromorphic function of infinite order with simple sequences of non-repeated zeros and poles.

A particular solution is a meromorphic function of z which may be expressed in the form

$$e^{G_1(z)} \prod_{n=1}^{\infty} \left[\frac{\mathfrak{F}_{1,n}(z)}{\mathfrak{F}_{2,n}(z)} \frac{\mathfrak{F}_{3,n}(z)}{\mathfrak{F}_{4,n}(z)}\right],$$

where $\mathfrak{F}(z)$ denotes a primary gamma factor, and where $G_1(z)$ is an integral function which reduces to a polynomial when the orders of the integral functions $\phi(z)$ and $\chi(z)$ whose quotient $\dfrac{\phi(z)}{\chi(z)}$ forms $\mu(z)$ are finite.

The products
$$\prod_{n=1}^{\infty} [\mathfrak{F}_{r,n}(z)] \qquad (r = 1, 2, 3, 4)$$

are constructed from the integral functions $\phi(z)$ and $\chi(z)$. In such construction there is an element of arbitrariness as regards the finite zeros of $\phi(z)$ and $\chi(z)$, correspondining to the fact that, since

$$\Gamma_1(z)\,\Gamma_1(\omega-z) = \frac{\pi}{\omega \sin \pi z/\omega},$$

we may take either $\Gamma_1(z)$ or $\dfrac{1}{\Gamma_1(\omega-z)}$ as the basis of the primary factor. But ultimately the infinite terms of the products must be such that, corresponding to zeros of $\phi(z)$ and $\chi(z)$ negative with regard to ω, we must have sequences of poles and zeros respectively of the solution negatively congruent to and including these zeros, while, corresponding to positive zeros of $\phi(z)$ and $\chi(z)$, we must have sequences of zeros and

poles respectively which are positively congruent to but exclude the corresponding points.

From a diagram the rule for congruent zeros is evident: the sequences from zeros or poles of $\mu(z)$ whose moduli are very large must not cross the finite part of the plane. And this geometrical point of view explains the restriction introduced by the analysis; if this phenomenon did occur, we should get, in general, critical points of the solution at all points in the finite part of the plane, so that this part of the plane would be a lacunary space for the function. In special cases the distribution of the zeros and poles of $\mu(z)$ may be such that they lie on a finite number of lines parallel to the axis of ω, and then we should get lines of singularity in the finite part of the plane; except in the case when these sets of zeros and poles are all congruent with regard to ω, when isolated essential singularities would arise.

The previous investigation may be readily extended to cases when $\mu(z)$ has repeated zeros or poles or is of multiple sequence. After the previous investigation a discussion of these cases would be tedious: a single example will be given later.

14. It is possible to express the principal solution of

$$f(z+\omega)-\mu(z)f(z) = 0$$

in two other forms. We proceed first to write it as a doubly infinite product of strict Weierstrassian form. As before, we express $\mu(z)$ as a quotient of two pairs of integral functions, each with its ultimate sequence of zeros all positive or all negative with regard to ω, so that a typical factor is the function

$$\phi_1(z) = \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{a_n}\right)e^{\sum_{s=1}^{\rho_n-1}\frac{(-z)^s}{s a_n^s}}\right],$$

defined as in § 10.

The solution of

$$f(z+\omega)-\phi_1(z)f(z) = 0$$

has been shown (§ 10) to be

$$P_1(z) = e^{G_1(z)}\prod_{n=1}^{\infty}\left[\frac{\Gamma(z+a_n)}{\Gamma(a_n)}e^{-\sum_{s=1}^{\sigma_n-1}\frac{z^s}{s!}\psi_1^{(s)}(a_n)}\right],$$

where $\sigma_n \geqslant 2\rho_n$ when ρ_n depends upon n, and where $\sigma \geqslant 2\rho$ and $\sigma > 2$ when ρ, the integer next greater than the order of $\phi_1(z)$, is finite.

Consider now the product

$$Q_1(z) = \prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\left[\left(1+\frac{z}{a_n+m\omega}\right)e^{\sum_{s=1}^{\tau_n-1}\frac{1}{s}\left(\frac{-z}{a_n+m\omega}\right)^s}\right].$$

It converges with

$$\exp\left[\sum_{n}^{\infty}\sum_{m}^{\infty}\sum_{s=\tau_n}^{\infty} -\frac{1}{s}\left(\frac{-z}{a_n+m\omega}\right)^s\right],$$

those terms (finite in number) being omitted from the summation with regard to m and n for which $|a_n+m\omega| \leqslant |z|$.

By an extension of the process employed in §§ 4 and 5, we see that this expression is convergent if $\tau_n \geqslant 2\rho_n$ (a detailed proof for a more complicated case is given in § 16). If, then, we put $\tau_n = \sigma_n$, we see that $P_1(z)\,Q_1(z)$ is a function with no zeros or poles, and therefore of the form $e^{G(z)}$ where $G(z)$ is an integral function. Therefore the principal solution of

$$f(z+\omega)-\phi_1(z)f(z) = 0$$

may be written

$$e^{G_1(z)}\Big/\prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\left[\left(1+\frac{z}{a_n+m\omega}\right)e^{\sum_{s=1}^{\sigma_n-1}\frac{1}{s}\left(\frac{-z}{a_n+m\omega}\right)^s}\right].$$

A solution of

$$f(z+\omega)-\phi_2(z)f(z) = 0$$

where

$$\phi_2(z) = \prod_{n=1}^{\infty}\left[\left(1-\frac{z}{\beta_n}\right)e^{\sum_{s=1}^{\rho_n-1}\frac{1}{s}\left(\frac{z}{\beta_n}\right)^s}\right]$$

and the real parts of $\beta_1/\omega, \ldots, \beta_n/\omega, \ldots$ are positive is, similarly,

$$e^{G_2(z)}\prod_{n=1}^{\infty}\prod_{m=1}^{\infty}\left[\left(1-\frac{z}{m\omega+\beta_n}\right)e^{\sum_{s=1}^{\sigma_n-1}\frac{1}{s}\left(\frac{z}{\beta_n+m\omega}\right)^s}\right]$$

where $\sigma_n \geqslant 2\rho_n$ if ρ_n depends upon n, and $\sigma_n \geqslant 2\rho$ and $\sigma > 2$ if $\phi_2(z)$ is of finite order. Solutions of

$$f(z+\omega)-f(z)/\chi_1(z) = 0, \qquad f(z+\omega)-f(z)/\chi_2(z) = 0$$

can be written down in similar manner, and the principal solution of

$$f(z+\omega)-\mu(z)f(z) = 0$$

will be the product of all four of such solutions.

We see that the products $Q_1(z), \ldots$ which arise are, in the notation to which reference was made in § 3, functions of double sequence, linear with respect to one of the sequences. If $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{|(a_n+m\omega)|^{R+\epsilon}}$ is convergent and $\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{1}{|(a_n+m\omega)|^{R-\epsilon}}$ is divergent, we say that the order of the function is R, this being the natural extension of the definition of order for functions of double linear sequence. With this definition we see (§ 5) that the order of $Q_1(z)$ is at most 2 or $2\rho'$, where ρ' is the order

of the function $\phi_1(z)$ from which $Q_1(z)$ is derived, whichever be the greater of these two quantities.

15. We proceed to write the solution just obtained in yet another form. Take, as before, the equation

$$f(z+\omega)-\phi_1(z)f(z) = 0$$

and consider the product

$$R_1(z) = \prod_{m=0}^{\infty}\left[\frac{\phi_1(m\omega)}{\phi_1(z+m\omega)}\, e^{\sum\limits_{s=1}^{\tau_m-1}\frac{z^s}{s!}\chi_1^{(s)}(m\omega)}\right]$$

where
$$\chi_1^{(r)}(z) = \frac{d^r}{dz^r}\log\phi_1(z).$$

Since a_n is positive with respect to ω, we have $|a_n+m\omega| > |m\omega|$, and therefore, for values of z inside a circle centre the origin and radius $|k\omega|$, $\log\phi_1(z+m\omega)$, where $m > k$, can be expanded in an absolutely convergent series of powers of z. Hence each term of $R_1(z)$ after the first k terms can for such values of z be written in the form

$$\exp\left[-\sum_{s=\tau_m}^{\infty}\frac{z^s}{s!}\chi_1^{(s)}(m\omega)\right].$$

We may choose τ_m so that the series inside the bracket is as small as we please, let us say $< \epsilon_m$. We choose the τ's so that $\sum\limits_{m=k}^{\infty}\epsilon_m$ is absolutely convergent, and thus ensure the convergency of $R_1(z)$.

But $R_1(z)$ has the same poles as $P_1(z)$, and neither function has any finite zeros. Hence the solution of

$$f(z+\omega)-\phi_1(z)f(z) = 0$$

may be written

$$e^{G_1(z)}\prod_{m=0}^{\infty}\left[\frac{\phi_1(m\omega)}{\phi_1(z+m\omega)}\, e^{\sum\limits_{s=1}^{\tau_m-1}\frac{z^s}{s!}\chi_1^{(s)}(m\omega)}\right],$$

where $G_1(z)$ is an integral function of z.

We may now prove that it is sufficient to take τ_m equal to the quantity σ of § 6 in the case when the order of $\phi_1(z)$ is a finite quantity.

For, for values of z such that $|z|<|k\omega|$, the product $R_1(z)$ is convergent with

$$\exp\left[-\sum_{m=k}^{\infty}\sum_{s=\tau_m}^{\infty}\frac{z^s}{s!}\chi_1^{(s)}(m\omega)\right].$$

Now

$$\chi_1^{(s)}(z) = \sum_{n=1}^{\infty}\left[\frac{(-)^{s-1}(s-1)!}{(z+a_n)^s}+(-)^s\sum_{r=0}^{p_n-s-1}\frac{(r+1)(r+2)\dots(r+s-1)(-z)^r}{a_n^{r+s}}\right];$$

so that $R_1(z)$ is convergent with

$$\sum_{m=k}^{\infty}\sum_{s=\tau_m}^{\infty}\sum_{n=1}^{\infty}\frac{(-)^s z^s}{s}\left[\frac{1}{(m\omega+a_n)^s}-\sum_{r=0}^{\rho_n-s-1}\binom{r+s-1}{s-1}\frac{(-m\omega)^r}{a_n^{r+s}}\right].$$

When ρ_n is finite and equal to ρ, the integer next greater than the finite order of $\phi_1(z)$, the summation in the bracket vanishes when $s \geqslant \rho$. The series last written becomes

$$\sum_{m=k}^{\infty}\sum_{s=\tau_m}^{\infty}\sum_{n=1}^{\infty}\frac{(-)^s z^s}{s\,(a_n+m\omega)^s},$$

and will be convergent when $|z| < |\sqrt{(k\omega)}|$ if we take $\tau_m = \sigma$ (a quantity independent of m) and such that $\sigma \geqslant 2\rho$ and $\sigma > 2$. For in this case the series of moduli is

$$< \sum_{m=k}^{\infty}\sum_{s=\sigma}^{\infty}\sum_{n=1}^{\infty}\frac{|z|^s}{s\,|m\omega|^{\frac{1}{2}s}}\frac{1}{|a_n|^{\frac{1}{2}s}}$$

$$< \frac{1}{\sigma}\sum_{m=k}^{\infty}\frac{|z|^\sigma}{(m\omega)^{\frac{1}{2}\sigma}\left\{1-\left|\frac{z}{\sqrt{(m\omega)}}\right|\right\}}\sum_{s=\sigma}^{\infty}\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\frac{1}{2}s}},$$

provided $|z| < |\sqrt{(k\omega)}|$. And therefore within the circle defined by the inequality last written the product is absolutely convergent provided $\sigma \geqslant 2\rho$ and $\sigma > 2$. Since we may make $|\sqrt{(k\omega)}|$ as large as we please, the product must be, when $\tau_m = \sigma$, absolutely convergent for all finite values of $|z|$.

16. As has been stated, the investigation of the solution of

$$f(z+\omega)-\mu(z)f(z) = 0$$

can be extended to cases when $\mu(z)$ is a function with repeated zeros or poles or of multiple sequence. Suppose, as an example, that

$$\mu(z) = \prod_{n=1}^{\infty}\left[\left(1+\frac{z}{a_n}\right)^{\mu_n}\exp\left\{\mu_n\sum_{s=1}^{\sigma_n-1}\frac{1}{s}\left(-\frac{z}{a_n}\right)^s\right\}\right];$$

so that $\mu(z)$ is a function with repeated zeros. The function will be of finite order ρ if we can choose for σ_n a number σ independent of n and such that $\sigma > \rho \geqslant \sigma - 1$, where $\Sigma\frac{\mu_n}{|a_n|^{\rho+\epsilon}}$ is convergent and $\Sigma\frac{\mu_n}{|a_n|^{\rho-\epsilon}}$ is divergent, however small the positive quantity ϵ may be. In other cases we must take σ_n such that $\sum_{n=k}^{\infty}\mu_n\left|\frac{z}{a_n}\right|^{\sigma_n}$ is convergent when $|z|$ is finite. Since the sequence $a_0, a_1, \ldots, a_n, \ldots$ tends to infinity, this is equivalent to saying that $\sum_{n=k}^{\infty}\frac{\mu_n}{|a_n|^{\sigma_n}}$ must be convergent.

If now the quantities a_n be positive with respect to ω, we may, to obtain a solution corresponding to the second of the three preceding

types, form the product

$$\prod_{n=1}^{\infty}\prod_{m=0}^{\infty}\left[\left(1+\frac{z}{a_n+m\omega}\right)^{\mu_n}\exp\left\{\mu_n\sum_{s=1}^{\tau_n-1}\frac{1}{s}\left(\frac{-z}{a_n+m\omega}\right)^s\right\}\right].$$

This product will be convergent for values of z such that $|z|$ is sufficiently small if $\exp\left[-\sum_n^{\infty}\sum_m^{\infty}\sum_{s=\tau_n}^{\infty}\frac{\mu_n}{s}\left(\frac{-z}{a_n+m\omega}\right)^s\right]$ is convergent, those terms (finite in number) being omitted from the summations with regard to m and n for which $|a_n+m\omega| \leqslant |z|$. The series is convergent with

$$\sum_n^{\infty}\sum_m^{\infty}\frac{\mu_n\left|\dfrac{z}{a_n+m\omega}\right|^{\tau_n}}{1-\left|\dfrac{z}{a_n+m\omega}\right|};$$

and therefore with
$$\sum_n^{\infty}\sum_m^{\infty}\mu_n\left|\frac{z}{a_n+m\omega}\right|^{\tau_n}.$$

Suppose now that $|l\omega| > |z| \geqq |(l-1)\omega|$, $|a_k| > |z| \geqq |a_{k-1}|$. Then the double series is less than

$$\sum_{n=0}^{k-1}\sum_{m=0}^{\infty}+\sum_{m=0}^{l-1}\sum_{n=0}^{\infty}+\sum_{m=l}^{\infty}\sum_{n=k}^{\infty}\mu_n\left|\frac{z}{a_n+m\omega}\right|^{\tau_n}.$$

The first set of single series is finite for all finite values of k if $\tau_n > 1$ for all values of n. The second set of series is finite for all finite values of l if $\tau_n \geqq \sigma_n$. The third double series is less than (§ 5)

$$\sum_{m=l}^{\infty}\sum_{n=k}^{\infty}\mu_n\frac{|z|^{\tau_n}}{|2a_n|^{\frac{1}{2}\tau_n}|m\omega|^{\frac{1}{2}\tau_n}} < \sum_{m=l}^{\infty}\left|\frac{z}{m\omega}\right|^{\frac{1}{2}\tau_k}\sum_{n=k}^{\infty}\mu_n\left|\frac{z}{2a_n}\right|^{\frac{1}{2}\tau_n},$$

where τ_k is the value of τ_n when $n = k$, this value being less than any succeeding value for sufficiently large values of k. Thus the third double series is convergent if $\tau_k > 2$, $\tau_n \geqq 2\sigma_n$. Finally, then, the original product is absolutely convergent for all finite values of $|z|$ if $\tau_n \geqq 2\sigma_n$, τ_n being > 1 for all values of n. When σ_n is independent of n* and equal to σ, say, we must have $\tau \geqq 2\sigma$ and $\tau > 2$.

The reciprocal of the product multiplied by a function of the type $e^{G_1(z)}$ is a solution of the difference equation.

The method of extension to cases of more complicated integral or meromorphic functions is now obvious.

We have the general theorem that a solution of the linear difference equation

$$f(z+\omega)-\mu(z)f(z) = 0$$

---

* An example is given in the author's "Memoir on Integral Functions," § 81.

where $\mu(z)$ is a meromorphic function of z can always be obtained as a meromorphic function with sequences of poles and zeros *of the same order* as the corresponding poles or zeros of $\mu(z)$. This solution can be written in three different forms. In each the sequences of zeros proceed positively from but exclude the positive zeros of $\mu(z)$ and negatively from and include the negative poles of $\mu(z)$, while the sequences of poles proceed positively from but exclude the positive poles of $\mu(z)$ and negatively from and include the negative zeros of $\mu(z)$.

17. We have now to consider equations of the type B (§ 2), that is an equation

$$f(z+\omega)-f(z) = \mu(z)$$

where $\mu(z)$ is a meromorphic function of z.

For this purpose we express $\mu(z)$ by Mittag-Leffler's theorem, just as to solve

$$f(z+\omega)-\mu(z)f(z) = 0$$

we expressed $\mu(z)$ as a quotient of Weierstrassian products.

Briefly recapitulated Mittag-Leffler's process is as follows. When the poles of $\mu(z)$ are arranged in order of non-decreasing moduli, let the r-th pole a_r be of order k_r and let the expansion of $\mu(z)$ in its vicinity be

$$\mu(z) = \sum_{s=1}^{k_r} \frac{a_s}{(z-a_r)^s}+b_0+\sum_{s=1}^{\infty} b_s(z-a_r)^s$$

$$= \mathfrak{P}_r(z-a_r)+b_0+\sum_{s=1}^{\infty} b_s(z-a_r)^s, \text{ say.}$$

So long as $|z| < |a_r|$ we have the expansion

$$\mathfrak{P}_r(z-a_r) = \sum_{m=0}^{\infty} {}_r c_m z^m.$$

Take now
$$F_r(z) = \mathfrak{P}_r(z-a_r)-\sum_{m=0}^{m_r-1} {}_r c_m z^m;$$

so that, when $|z| < |a_r|$,

$$F_r(z) = \sum_{m=m_r}^{\infty} {}_r c_m z^m.$$

Choose now a value of m_r sufficiently large and we shall have, when $|z| < |a_r|$, $|F_r(z)| < \epsilon_r$, where the series $\sum_{r=1}^{\infty} \epsilon_r$ is absolutely convergent.

And now $\mu(z) = \sum_{r=0}^{\infty} F_r(z)+G(z)$ where $G(z)$ is an integral function of z.

To solve the equation $f(z+\omega)-f(z) = \mu(z)$,

divide the poles of $\mu(z)$ into two sets $a_0, a_1, \ldots, a_n, \ldots$ and $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ respectively negative and positive with respect to ω.

We know that $\frac{(-)^{s-1}}{(s-1)!}\psi_1^{(s)}(z)$ is a solution of

$$f(z+\omega)-f(z) = \frac{1}{z^s},$$

and therefore we take a function $\phi_r(z)$ corresponding to $\mathfrak{P}_r(z-a_r)$, and such that

$$\phi_r(z) = \sum_{s=1}^{k_r} \frac{(-)^{s-1}}{(s-1)!} a_s \psi_1^{(s)}(z-a_r).$$

Then, since $|m\omega - a_r| > |a_r|$, we shall have, when $|z| < |a_r|$, the expansion

$$\phi_r(z) = \sum_{m=0}^{\infty} {}_rC_m z^m.$$

18. We now need the following lemma:—

If $\phi(z)$ be a solution of the difference equation

$$f(z+\omega)-f(z) = \mathfrak{P}(z-a),$$

where $\mathfrak{P}(z-a)$ can, within a circle of radius $|a|$ and centre the origin, be expanded in a convergent series $\sum_{m=0}^{\infty} c_m z^m$, so that when m is large to a first approximation $|c_m| = 1/|a|^m$, then, if $|a(2\pi\iota/\omega)| > 1/e$, we shall have, within a circle of radius ρ given by

$$e^{\rho|2\pi\iota/\omega|} = e\,|a|\left|\frac{2\pi\iota}{\omega}\right|,$$

an expansion for $\phi(z)$ of the form

$$\varpi(z, a, \omega) - \sum_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega),$$

the series being absolutely convergent so long as $|z| < \rho$ and $\varpi(z, a, \omega)$ being a simply periodic function of z of period ω.

Consider the series $\sum_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega)$. By the results of § 12 it is convergent, provided

$$e^{|z(2\pi\iota/\omega)|} < \left|\frac{2\pi\iota e a}{\omega}\right|,$$

i.e., provided $|z| < \rho$, as above defined.

We note that ρ always exists as a positive quantity provided $|a(2\pi\iota/\omega)| > 1/e$. Again ρ is less than $|a|$. For, putting $|2\pi\iota/\omega| = \theta$, ρ is

defined by the equality $$e^{\rho\theta-1} = |a|\theta.$$

Now, if x be real and positive, or real, negative, and < 1,

$$e^x > 1+x;$$

therefore $$\rho < |a|.$$

Further, we note that ρ increases to infinity with a.

Assume now that $|a| > e^{-1}|\omega/2\pi\iota|$. Then, within a circle of radius ρ, $\sum\limits_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega)$ is convergent. Therefore within this circle

$$\sum_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega) - \sum_{m=0}^{\infty} c_m z^m = \sum_{m=0}^{\infty} c_m \zeta_m(-m, z+\omega, \omega)$$

is convergent. Hence, provided $|z| < \rho$, $\sum\limits_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega)$ is a solution of the difference equation

$$f(z+\omega)-f(z) = \sum_{m=0}^{\infty} c_m z^m,$$

or $$f(z+\omega)-f(z) = \mathfrak{P}(z-a).$$

And the general solution will be

$$\varpi(z, a, \omega) - \sum_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega),$$

where $\varpi(z, a, \omega)$ is a simply periodic function of z of period ω.

We note that, if $\phi(z)$ has no singularities within the circle $|z| < \rho$, neither has $\varpi(z, a, \omega)$ singularities within the same circle. The lemma is thus established.

COROLLARY.—We can deduce an interesting result in the theory of the expansion of an arbitrary function in a series of functions.

Let $f(z)$ be an arbitrary integral function. Then $f(z+\omega)-f(z)$ is an integral function and therefore capable of expansion in the form

$$\sum_{m=0}^{\infty} c_m z^m.$$

The general solution of the equation

$$f(z+\omega)-f(z) = \sum_{m=0}^{\infty} c_m z^m$$

is $$\varpi(z, \omega) - \sum_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega),$$

the series being valid for all finite values of $|z|$.

Hence any integral function of z can be expanded in the form

$$\varpi(z, \omega) - \sum_{m=0}^{\infty} c_m \zeta_m(-m, z, \omega),$$

where $\varpi(z, \omega)$ is a finite simply periodic function of z with no finite poles, and the expansion will hold for all finite values of $|z|$.

19. Take now the function

$$\phi_r(z) = \sum_{s=1}^{k_r} \frac{(-)^{s-1}}{(s-1)!} a_s \psi_1^{(s)}(z-a_r).$$

If $|a_r(2\pi\iota/\omega)| > 1/e$, which will be the case for sufficiently large values of r, since $\mu(z)$ is meromorphic and consequently the sequence of its poles tends to infinity, we may expand $\phi_r(z)$ in the form

$$\varpi_r(z, a_r, \omega) - \sum_{m=0}^{\infty} {}_r c_m \zeta_m(-m, z, \omega),$$

the expansion being valid, provided $|z| < \rho_r$, where ρ_r tends to infinity with $|a_r|$ and therefore with r.

Let us take

$$\Phi_r(z) = \phi_r(z) - \varpi_r(z, a_r, \omega) + \sum_{m=0}^{\mu_r-1} {}_r c_m \zeta_m(-m, z, \omega);$$

then, so long as $|z| < \rho_r$, we have

$$\Phi_r(z) = -\sum_{m=\mu_r}^{\infty} {}_r c_m \zeta_m(-m, z, \omega),$$

and the modulus of this series may for sufficiently large values of μ_r be made as small as we please.

We choose μ_r such that, when $|z| < \rho_r$, $|\Phi_r(z)| < \epsilon_r$, where $\sum_{r=1}^{\infty} \epsilon_r$ is absolutely convergent.

Then, for values of $|z| < \rho_r$, the series $\sum_{s=r}^{\infty} \Phi_s(z)$ is absolutely convergent, and hence $\sum_{s=1}^{\infty} \Phi_s(z)$ is a series convergent at all points of the plane except the points negatively congruent to and including the poles a_r. And we have

$$\Phi_r(z+\omega) - \Phi_r(z) = \sum_{s=1}^{k_r} \frac{a_s}{(z-a_r)^s} - \sum_{m=0}^{\mu_r-1} {}_r c_m z^m = \sum_{m=\mu_r}^{\infty} {}_r c_m z^m,$$

by § 17, when $|z| < a_r$.

20. For the poles β_r of $\mu(z)$ we proceed in an analogous manner.

By the fundamental difference equation for the gamma functions $\psi_1^{(s)}(\omega-z)/(s-1)!$ is a solution of $f(z+\omega)-f(z) = 1/z^s$, when s is an integer.

If near its r-th pole β_r, positive with regard to ω, $\mu(z)$ admit the expansion

$$\mu(z) = \sum_{s=1}^{l_r} \frac{b_s}{(z-\beta_r)^s} + e_0 + e_1(z-\beta_r) + \ldots,$$

we construct the function

$$\psi_r(z) = \sum_{s=1}^{l_r} \frac{b_s}{(s-1)!} \psi_1^{(s)}(\omega+\beta_r-z).$$

Then, when $|z| < \sigma_r$, where σ_r tends to infinity with β_r, we have the absolutely convergent expansion

$$\psi_r(z) = \varpi(z, \beta_r, \omega) - \sum_{m=0}^{\infty} {}_r d_m \zeta_m(-m, z, \omega).$$

We take $$\Psi_r(z) = \psi_r(z) - \varpi(z, \beta_r, \omega) + \sum_{m=0}^{\nu_r-1} {}_r d_m \zeta_m(-m, z, \omega),$$

and, by choosing ν_r sufficiently large, we can ensure that $\sum_{s=1}^{\infty} \Psi_s(z)$ is a series convergent at all points of the plane, except the points positively congruent to but not including the poles β_r.

And now $\sum_{s=1}^{\infty} [\Phi_s(z) + \Psi_s(z)]$ will be a meromorphic function satisfying the difference equation

$$f(z+\omega) - f(z) = \mu_1(z),$$

where $\mu_1(z) - \mu(z)$ is an integral function which may be expanded in the series $\sum_{m=0}^{\infty} e_m z^m$, valid for all finite values of $|z|$.

Finally, then, a solution of the equation $f(z+\omega)-f(z) = \mu(z)$ is given by

$$\sum_{s=1}^{\infty} [\Phi_s(z) + \Psi_s(z) - e_s \zeta_s(-s, z, \omega)],$$

and the general solution is this function plus an arbitrary simply periodic function of z of period ω.

The form of the principal solution just obtained can be modified as regards sequences proceeding from the poles of $\mu(z)$, which are in the finite part of the plane.

21. It is possible to give two other forms to the solution just obtained. Retaining the notation of § 20, consider $\phi_r(z)$. We have

$$\phi_r(z) = \sum_{s=1}^{k_r} \frac{(-)^{s-1}}{(s-1)!} a_s \psi_1^{(s)}(z-a_r)$$
$$= -\sum_{s=2}^{k_r} a_s \sum_{m=0}^{\infty} \frac{1}{(z-a_r+m\omega)^s}$$
$$-a_1 \left\{ \frac{\gamma - \log \omega}{\omega} + \frac{1}{z-a_r} + \sum_{m=1}^{\infty} \left(\frac{1}{z-a_r+m\omega} - \frac{1}{m} \right) \right\}.$$

Therefore $\sum_{s=1}^{\infty} \Phi_s(z)$ has its sole poles at the points negatively congruent to the a's (these points included), and each pole congruent to a_r is an infinity of exactly the same nature as the infinity a_r of $-\mu(z)$.

Hence, if $\mu(z) = \mu_a(z)+\mu_\beta(z)$, the two meromorphic functions possessing respectively the sequences of poles negative and positive with respect to ω, we see that

$$-\{\mu_a(z)+g_0(z)\} - \{\mu_a(z+\omega)+g_1(z)\} - \ldots - \{\mu_a(z+m\omega)+g_m(z)\} - \ldots,$$

the functions $g(z)$ being integral functions of z, will have poles exactly like those of $\sum_{s=1}^{\infty} \Phi_s(z)$.

We can chose the functions $g(z)$ so that the series just written shall be convergent. For the poles of $\mu_a(z+m\omega)$ $(m > k)$ will have moduli greater than $|k\omega|$, and hence, when $|z| < |k\omega|$, we shall have the expansion

$$\mu_a(z+m\omega) = \sum_{s=0}^{\infty} \frac{z^s}{s!} \mu_a^{(s)}(m\omega).$$

If then we take

$$M_a(z+m\omega) = -\mu_a(z+m\omega) + \sum_{s=0}^{s_m-1} \frac{z^s}{s!} \mu_a^{(s)}(m\omega),$$

we shall have, when $|z| < |k\omega|$,

$$M_a(z+m\omega) = -\sum_{s=s_m}^{\infty} \frac{z^s}{s!} \mu_a^{(s)}(m\omega);$$

so that, by choosing s_m sufficiently large, we may make $|M_a(z+m\omega)| < \epsilon_m$ where $\sum_{m=0}^{\infty} \epsilon_m$ is absolutely convergent.

Then $\sum_{m=0}^{\infty} M_a(z+m\omega)$ is absolutely convergent except at the poles of $\sum_{s=1}^{\infty} \Phi_s(z)$, and only differs from the latter series of functions by an integral function.

Repeating the same argument for the β's and constructing the series of analogous functions

$$M_\beta\{z-(m+1)\omega\} = \mu_\beta\{z-(m+1)\omega\} - \sum_{s=0}^{t_m-1} \frac{z^s}{s!}\mu_\beta^{(s)}\{-(m+1)\omega\},$$

we see that

$$\sum_{m=0}^{\infty}[M_\alpha(z+m\omega)+M_\beta\{z-(m+1)\omega\}]+G(z)$$

is a solution of $\qquad f(z+\omega)-f(z) = \mu(z).$

It is a solution analogous to this which Hurwitz * has given.

22. We may now express the solution in a third form. We take, as before,

$$\mu(z) = \sum_{r=1}^{\infty}\left[\sum_{s=1}^{k_r}\frac{a_s}{(z-a_r)^s} - \sum_{m=0}^{m_r-1} {}_rc_m z^m\right]$$
$$+\sum_{r=1}^{\infty}\left[\sum_{s=1}^{l_r}\frac{b_s}{(z-\beta_r)^s} - \sum_{m=0}^{n_r-1} {}_rd_m z^m\right]+G(z),$$

the a poles being negative and the β poles positive with regard to ω. And we construct the function

$$Q(z) = -\sum_{m=0}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{k_r} a_s\left\{\frac{1}{(z-a_r+m\omega)^s}+(-)^{s-1}\sum_{t=0}^{{}_s\sigma_r-1}\binom{s+t-1}{t}\frac{z^t}{(a_r-m\omega)^{s+t}}\right\}$$
$$+\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{l_r} b_s\left\{\frac{1}{(z-\beta_r-m\omega)^s}+(-)^{s-1}\sum_{t=0}^{{}_s\tau_r-1}\binom{s+t-1}{t}\right.$$
$$\left.\times\frac{z^t}{(\beta_r+m\omega)^{s+t}}\right\}.$$

Consider the first series. If those values of m and r be omitted for which $|z| \leqslant |a_r-m\omega|$, the series converges with the group of series

$$\sum_m^{\infty}\sum_r^{\infty}\sum_{t={}_s\sigma_r}^{\infty}\binom{s+t-1}{t}\frac{z^t}{(a_r-m\omega)^{s+t}} \qquad (s = 1, 2 \ldots, k_r).$$

Now, when r is very large, $\binom{s+r-1}{r}$ behaves to a first approximation like $\frac{1}{\Gamma(s)}r^{s-1}$ when $s > 1$. Hence, by the same method of proof as that employed in § 16, the series are absolutely convergent provided ${}_s\sigma_r > 2\rho_r$ for all values of s and r, where ρ_r is the integer next greater than the

---

* Hurwitz, *Acta Mathematica*, T. xx., pp. 308-311.

order-number of the sequence $a_1, a_2, \ldots, a_r, \ldots$. When the order of the sequence is finite, so that $\rho_r = \rho$, we may take σ_r to be a finite quantity σ where

$$\sigma > 2\rho \quad \text{and} \quad \sigma > 1.$$

Similar remarks apply to the second series.

The function $Q(z)$ is a meromorphic function, and, by suitable choice of the integral function $G(z)$, we may make $Q(z)+G(z)$ a solution of the difference equation

$$f(z+\omega)-f(z) = \mu(z).$$

In each case the particular solution which we have obtained of this equation is a meromorphic function whose poles are two series of points, those negatively congruent to the poles of $\mu(z)$ negative with regard to ω (these poles being included), and those positively congruent to the poles of $\mu(z)$ positive with regard to ω (these points being excluded).

23. We have now completed the solution of the two fundamental subsidiary equations (A) and (B) of § 2. Let us then consider the nature of the solution of

$$\phi(z)f(z+\omega)-\chi(z)f(z) = \psi(z).$$

We have

$$f(z) = \frac{f_2(z)}{f_1(z)}$$

where

$$f_1(z+\omega)-\frac{\phi(z)}{\chi(z)}f_1(z) = 0$$

and

$$f_2(z+\omega)-f_2(z) = \frac{\psi(z)}{\chi(z)}f_1(z).$$

Let the n-th positive and negative (with respect to ω) zeros of $\phi(z)$, $\chi(z)$, and $\psi(z)$ respectively be a_n and a'_n, β_n and β'_n, γ_n and γ'_n. We have seen that we may take $f_1(z)$ to be a meromorphic function with poles at the sequences

$$a'_n-m\omega, \quad \beta_n+(m+1)\omega \quad (m = 0, 1, \ldots, \infty)$$

and zeros at the sequences

$$a_n+(m+1)\omega, \quad \beta'_n-m\omega,$$

these poles and zeros being of the same order as the corresponding zeros of $\phi(z)$ and $\chi(z)$.

Hence $\dfrac{\psi(z)}{\chi(z)}f_1(z)$ has poles at the sequences

$$a'_n-m\omega, \quad \beta_n+m\omega \quad (m = 0, 1, \ldots, \infty),$$

these poles being of the same order as the corresponding zeros of $\phi(z)$ and $\chi(z)$.

Consider now the solution of

$$f_2(z+\omega)-f_2(z) = \frac{\psi(z)}{\chi(z)} f_1(z).$$

A particular solution will be a meromorphic function with its sole poles at the sequences

$$a'_n - m\omega, \quad \beta_n+(m+1)\,\omega \quad (m = 0, 1, \ldots, \infty).$$

These poles will be of the order of the corresponding zeros of $\phi(z)$ and $\chi(z)$, but not of the same type, *i.e.*, the coefficients in the expansion near a pole will in general be different.

Finally, the particular solution $f(z) = \dfrac{f_2(z)}{f_1(z)}$ of the difference equation $\phi(z) f(z+\omega) - \chi(z) f(z) = \psi(z)$ will be a one-valued meromorphic function whose sole poles are at the sequences

(1) $$\beta'_n - m\omega \quad (m = 0, 1, \ldots, \infty),$$

each of the order of the corresponding zero β'_n of $\chi(z)$;

(2) $$a_n+(m+1)\,\omega \quad (m = 0, 1, \ldots, \infty),$$

each of the order of the corresponding zero a_n of $\phi(z)$.

The general solution of the difference equation

$$\phi(z) f(z+\omega) - \chi(z) f(z) = \psi(z)$$

is

$$f(z) + \varpi(z, \omega) / f_1(z)$$

where $\varpi(z, \omega)$ is a simply periodic function of z of period ω.

It may be noticed that, unless $\varpi(z, \omega)$ has finite poles, this solution has its only poles at the sequences

$$\beta'_n - m\omega, \quad a_n + (m+1)\,\omega.$$

It must be distinctly understood that we have given the nature of the general solution on the assumption that $\phi(z)$, $\chi(z)$, and $\psi(z)$ are arbitrary integral functions of z. And the solution given may admit of reductions which will simplify its character when there are relations between the zeros of these functions.

24. As a single example of the general theory we may mention Prym's* solution of

$$f(z+\omega) - z f(z) = c,$$

which is

$$f(z) = c e^{1/\omega} \sum_{n=0}^{\infty} \frac{(-)^{n-1}}{n!\,\omega^n (z+n\omega)}.$$

Here a_n, a'_n, β_n are non-existent, and $\beta'_1 = 0$.

* Prym, *Crelle*, Bd. LXXXII., pp. 165–172.

Other examples are furnished by the G-function,* the multiple gamma functions, and the functions constructed by Mellin.†

As an example of the very different form which may be sometimes given to the general solution we may mention that, if $\phi(z)$, $\chi(z)$, and $\psi(z)$ are all simply periodic functions of z of period ω, the solution of

$$\phi(z) f(z+\omega) - \chi(z) f(z) = \psi(z)$$

may be written
$$\frac{\psi(z)}{\phi(z)-\chi(z)} + \varpi(z, \omega) \left[\frac{\chi(z)}{\phi(z)}\right]^{z/\omega}.$$

---

* See a paper by the author, *Quarterly Journal of Mathematics*, Vol. XXXI., pp. 264–314.

† Mellin, *Acta Mathematica*, T. XV., pp. 317–384.

[illegible] OF POWER SERIES

By E. M. Wright.

[Received [illegible].—Read November [illegible].]

1. The theory of [illegible] in the symbols of [illegible] has been dealt with in several recent papers. In the present it is proposed to [illegible] when the form or forms of [illegible] are [illegible] power series.* This is of course the natural [illegible].

The [illegible] of the terms [illegible] may be [illegible] simply as [illegible] when [illegible] is [illegible]. In addition, the variable [illegible] is replaced by unity, as we are concerned with ordinary power series.

Now [illegible]

If we expand the expression on the right, we have

[illegible]

and

[illegible]

and therefore

[illegible]

If in this we make [illegible] very large, we have practically

[illegible]

Calling [illegible] respectively, and omitting the suffix [illegible] from [illegible], we have finally

[illegible]

[illegible]

* [illegible]

where y is replaced by x after differentiation. This shows that the perpetuant $(ab)^\lambda$ of the forms $f(x)$ and $\phi(x)$ is, for all practical purposes,

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^\lambda f(x)\,\phi(y) \qquad (y=x).$$

Exactly similarly, the perpetuant $(bc)^\lambda(ca)^\mu(ab)^\nu$ of three forms f, ϕ, ψ may be taken to be

$$\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)^\lambda \left(\frac{\partial}{\partial z}-\frac{\partial}{\partial x}\right)^\mu \left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^\nu f(x)\,\phi(y)\,\psi(z) \qquad (y=z=x),$$

and the generalisation is obvious.

Incidentally this shows that in the appropriate symbols of MacMahon, wherein $f=e^{ax}$, and a is a symbol, the covariants are of the type $(a-\beta)^\lambda$, $(\beta-\gamma)^\lambda(\gamma-a)^\mu(a-\beta)^\nu$, and so on; and, further, that the rules for writing down complete systems are exactly the same as in the ordinary Aronhold symbols.

2. The method for calculating covariants of a power series shows at once that all the covariants have the same circle of convergence as the original form. In the particular cases considered below the forms converge over the whole of the finite part of the plane.

Consider first the exponential series. We have only one form $f\equiv e^{ax}$, and the Hessian is

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^2 e^{ax}e^{ay}=0 \qquad (y=x),$$

and, in like manner, every other covariant vanishes. This result is to be expected, for every covariant of a perfect n-th power vanishes, and the exponential function is the limiting form of a perfect n-th power when n is infinite.

Conversely, the only form for which $(ab)^2$ vanishes is the exponential function.

In fact, we have

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^2 f_x f_y=0 \qquad (y=x),$$

i.e.,
$$f\frac{d^2f}{dx^2}-\left(\frac{df}{dx}\right)^2=0,$$

and therefore
$$f=Ce^{ax}$$

where C and a are arbitrary constants.

We next take the case of several exponential functions. The covariant $(bc)^{\lambda}(ca)^{\mu}(ab)^{\nu}$ of the symbolical forms e^{ax}, $e^{\beta x}$, $e^{\gamma x}$ is

$$(\beta-\gamma)^{\lambda}(\gamma-a)^{\mu}(a-\beta)^{\nu}e^{(a+\beta+\gamma)x},$$

and for exponential forms a, β, γ are actual numbers. Hence this covariant is a multiple of the product of the forms. This is obviously true in general.

Circular functions are next considered. If $f(x)=\phi(x)+\psi(x)$, it is obvious that

$$(ff)^{2\lambda}=(\phi\phi)^{2\lambda}+2(\phi\psi)^{2\lambda}+(\psi\psi)^{2\lambda}.$$

Hence, if

$$f(x)=Ae^{mx}+Be^{nx},$$

$$(ab)^2=(ff)^2=2AB(m-n)^2e^{(m+n)x},$$

and, in general,

$$(ab)^{2\lambda}=2AB(m-n)^{2\lambda}e^{(m+n)x}.$$

If

$$f(x)\equiv\cos x=\frac{e^{\iota x}+e^{-\iota x}}{2},$$

$$(ff)^2=2.\tfrac{1}{2}.\tfrac{1}{2}(2\iota)^2e^{\iota x}e^{-\iota x}=-2$$

and

$$(ff)^{2\lambda}=\tfrac{1}{2}(2\iota)^{2\lambda}=(-1)^{\lambda}2^{2\lambda-1}.$$

Similarly, if $f(x)$ is $\sin x$,

$$(ff)^2=+2 \quad\text{and}\quad (ff)^{2\lambda}=(-1)^{\lambda+1}2^{2\lambda-1}.$$

Also

$$(ab)^{\lambda}(bc)^{\mu}=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^{\lambda}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)^{\mu}f(x)f(y)f(z) \quad (y=z=x),$$

and, if $f=\phi+\psi$,

$$(ab)^{\lambda}(bc)^{\mu}=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^{\lambda}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)^{\mu}\{\phi(x)\phi(y)\phi(z)+\phi(x)\phi(y)\psi(z)+\dots\}.$$

If $\phi(x)=Ae^{mx}$ and $\psi(x)=Be^{nx}$, each term in the above expression vanishes identically except the terms obtained from $\phi(x)\psi(y)\phi(z)$ and $\psi(x)\phi(y)\psi(z)$. These give

$$(-1)^{\mu}A^2B(m-n)^{\lambda+\mu}e^{(2m+n)x} \quad\text{and}\quad (-1)^{\lambda}AB^2(m-n)^{\lambda+\mu}e^{(2n+m)x}$$

respectively. Hence

$$(ab)^{\lambda}(bc)^{\mu}=AB(m-n)^{\lambda+\mu}\left[(-1)^{\mu}Ae^{mx}+(-1)^{\lambda}Be^{nx}\right]e^{(m+n)x},$$

and we may show similarly that

$$(bc)^{\lambda}(ca)^{\mu}(ab)^{\nu}=0.$$

If

$$f=\cos x=\tfrac{1}{2}(e^{\iota x}+e^{-\iota x}),$$

$$(ab)^{\lambda}(bc)^{\mu}=2^{\lambda+\mu-3}\iota^{(\lambda+\mu)}\left[(-1)^{\mu}e^{\iota x}+(-1)^{\lambda}e^{-\iota x}\right].$$

For example, $\quad (ab)^2(bc) = -\iota^3 2\iota \sin x = -2\sin x \quad$ if $f = \cos x$,

and $\quad = 2\cos x \quad$ if $f = \sin x$.

3. Before proceeding further with the calculation of covariants of particular forms, it will be convenient to obtain another expression for any covariant.

We consider $$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^n f(x)\,\phi(y) \qquad (y = x).$$

If we write $x+h$ for x and $y-h$ for y, this becomes

$$(D_1+D_2)^n f(x+h)\,\phi(y-h) \quad (h = 0;\ y = x),$$

where $D_1 = d/dh$ and operates only on f, whilst $D_2 = d/dh$ and operates only on ϕ.

But this is exactly $$\left(\frac{d}{dh}\right)^n f(x+h)\,\phi(x-h) \qquad (h = 0),$$

or the covariant $(ab)^n$ of the two forms $f(x)$, $\phi(x)$ is

$$n!\,\{\text{coefficient of } h^n \text{ in } f(x+h)\,\phi(x-h)\}.$$

Again, take the expression

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^\lambda \left(\frac{\partial}{\partial y}-\frac{\partial}{\partial z}\right)^\mu f(x)\,\phi(y)\,\psi(z) \qquad (y = z = x).$$

This is $$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^\lambda f(x)\left(\frac{d}{dh}\right)^\mu [\phi(y+h)\,\psi(z-h)]$$
$$(h = 0;\ y = z = x),$$

which becomes $$\left(\frac{d}{dk}\right)^\lambda f(x+k)\left(\frac{d}{dh}\right)^\mu [\phi(y+h-k)\,\psi(z-h)] \qquad (y = z = x),$$

or $$\left(\frac{d}{dk}\right)^\lambda \left(\frac{d}{dh}\right)^\mu f(x+k)\,\phi(x+h-k)\,\psi(x-h) \qquad (h = k = 0)$$

Hence the expression is equal to

$$\lambda!\,\mu!\,\{\text{coefficient of } k^\lambda h^\mu \text{ in } f(x+k)\,\phi(x+h-k)\,\psi(x-h)\},$$

and, in general, it is seen that a covariant may be expressed as a certain coefficient in the expansion of a function which may be immediately written down. As an example, consider the covariants of the Weierstrassian σ function. Those of the type $(ab)^\lambda$ are the coefficients of the various powers of h in the expansion of $\sigma(x+h)\,\sigma(x-h)$, each multiplied by an appropriate numerical factor.

Now $$\wp(x)-\wp(h) = -\frac{\sigma(x+h)\,\sigma(x-h)}{\sigma^2(x)\,\sigma^2(h)}.$$

Hence $\sigma(x+h)\sigma(x-h) = \sigma^2(h)\wp(h)\sigma^2(x) - \sigma^2(h)\wp(x)\sigma^2(x)$.

The covariants required are therefore of the type $\sigma^2(x)[A+B\wp(x)]$, where A and B are certain constants, A being the coefficient of h^λ in $\sigma^2(h)\wp h$, and B being the coefficient of h^λ in $-\sigma^2(h)$. This may also be written in the form $C\sigma(x+a)\sigma(x-a)$, where C and a are constants.

In particular, if $\lambda = 2$, the covariant is practically $\sigma^2(x)\wp(x)$, whilst, if $\lambda = 4$, it is $g_2\sigma^2(x) \times$ a numerical factor.

The covariants of the type $(ab)^\lambda(bc)^\mu$ are the various coefficients in the expansion of $\sigma(x+h)\sigma(x+k-h)\sigma(x-k)$.

Now $\dfrac{\sigma(x+h)\sigma(x+k-h)\sigma(x-k)}{\sigma^3(x)}$ is an elliptic function, since

$$h+(k-h)+(-k) = 0.$$

It is easily seen that it is $A\wp(x)+B\wp'(x)+C$, where A, B, C depend on h and k alone, and are readily determined. The covariants of this type are therefore $(a\wp x+b\wp' x+c)\sigma^3(x)$, where a, b, c are constants. These results admit of immediate generalization; for we are always concerned with a product of n σ's, $\sigma(x+a)\sigma(x+\beta)\ldots\sigma(x+\lambda)$, such that $a+\beta+\ldots+\lambda = 0$.

Hence $$\frac{\sigma(x+a)\,\sigma(x+\beta)\ldots\sigma(x+\lambda)}{\sigma^n(x)}$$

is an elliptic function, and it has one irreducible infinity of the n-th order at the origin.

Hence it is $A+B\wp(x)+C\wp'(x)+D\wp''(x)+\ldots+K\wp^{(n-2)}(x)$, where A, B, C, …, K are functions of a, β, … only. The corresponding covariant is therefore

$$[a+b\wp(x)+c\wp'(x)+\ldots+k\wp^{(n-2)}(x)]\sigma^n(x),$$

where a, b, …, k are constants.

The covariants for the product of any number of σ's may be obtained in exactly the same manner.

4. The above is sufficient to show how the covariants of any power series may be calculated. We shall now proceed to find those functions for which the simpler covariants vanish. The corresponding problem for forms of finite order has often been dealt with by Hilbert and others; in the present case the only justification is the curious character of the results. It will be seen, in fact, that the first class, like

$$(ab)^2, \quad (ab)^2(bc),$$

lead to exponential functions, and the second class,

$$(ab)^4, \quad (ab)^4(bc),$$

lead to elliptic functions.

5. In solving the differential equation

$$(ab)^4 = 0,$$

which is
$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)^4 f_x f_y = 0 \quad (y = x),$$

or
$$ff_4 - 4f_1 f_3 + 3f_2^2 = 0,$$

it is convenient to put $f = e^u$.

Now, in virtue of the foregoing, $(ab)^4$ is practically the coefficient of h^4 in the expansion of $f(x+h)$, $f(x-h)$. Hence we want the coefficient of h^4 in

$$e^{u+hu_1+(h^2/2!)u_2\ldots}\, e^{u-hu_1+(h^2/2!)u_2\ldots} = f^2\left\{1+\frac{A_2}{2!}h^2+\frac{A_4}{4!}h^4\ldots\right\},$$

where
$$1+\frac{A_2}{2!}h^2+\frac{A_4}{4!}h^4\ldots = e^{2(u_2h^2/2!+u_4h^4/4!\ldots)};$$

whence $A_2 = 2u_2$, $A_4 = 2(u_4+6u_2^2)$, $A_6 = 2(u_6+30u_2u_4+60u_2^3)$, ...

6. Hence, if $(ab)^2 = 0$, we have

$$u_2 = 0, \quad u = ax+b, \quad \text{and} \quad f = e^{(ax+b)},$$

as before.

If $(ab)^4 = 0$,
$$u_4+6u_2^2 = 0,$$

and, putting $u_2 = w$, this is $w_2+6w^2 = 0$.

Hence
$$w = \wp\,\{\iota(x+c),\ 0,\ g_3\},$$

c and g_3 being arbitrary constants.

Finally,
$$f = e^{ax+b}\,\sigma\,\{x+c,\ 0,\ g_3\}$$

for
$$w = +\frac{\partial^2}{\partial x^2}\{\log f\}.$$

The equation $(ab)^6 = 0$ leads to new functions, and I have not yet studied it completely. It may be mentioned that the subsidiary equation

$$w_4+30ww_2+60w^3 = 0$$

admits a solution of the form

$$\wp(\iota x+\alpha,\ 0,\ g_3)-\wp(x+\beta,\ 0,\ g_3),$$

α, β, g_3 being arbitrary constants; further, any non-essential singularity of w at a finite distance is a pole of the second order. It seems extremely probable that any equation of the form

$$(ab)^{2\lambda} = 0$$

is satisfied only by integral functions.

7. The next form in order of simplicity is $(ab)^{\lambda}(bc)$.

Except for a numerical multiplier this is the coefficient of $h^{\lambda}k$ in

$$f(x+h)\,f(x+k-h)\,f(x-k),$$

which is that of h^{λ} in

$$f(x+h)\,f'(x-h)\,f(x)-f(x+h)\,f(x-h)\,f'(x);$$

or, on putting $f = e^{u}$,

$$e^{u+hu_1+(h^2/2!)u_2\dots}\ e^{u-hu_1+(h^2/2!)u_2\dots}\ e^{u}\left(u_1-hu_2+\frac{h^2}{2}u_3-\dots\right)$$

$$-e^{u+hu_1+(h^2/2!)u_2\dots}\ e^{u-hu_1+(h^2/2!)u_2}\dots e^{u}u_1$$

$$= f^3 e^{2[(h^2/2!)u_2+(h^4/4!)u_4\dots]}\left(-hu_2+\frac{h^2}{2!}u_3-\frac{h^4}{3!}u_4+\dots\right).$$

Writing this $\qquad f^3\left(B_1h+\dfrac{B_2h^2}{2!}+\dfrac{B_3h^3}{3!}-\dots\right),$

we have

$$B_1 = -u_2,\qquad B_2 = u_3,\qquad B_3 = -(u_4+6u_2^2),\qquad B_4 = u_5+12u_2u_3.$$

It is not difficult to prove the general formulæ

$$2B_{2n} = \frac{d}{dx}A_{2n},\qquad 2B_{2n+1} = -A_{2n+2},$$

but they are useless for our present purposes, except as showing, for example, that the equations

$$(ab)^{\lambda} = 0,\qquad (ab)^{\lambda}(bc) = 0$$

lead to similar functions.

Returning to the equations, we see that $(ab)^\lambda(bc) = 0$ leads to nothing new if λ be odd, as is indeed *a priori* obvious.

Further, $\quad (ab)^2(bc) = 0 \quad$ gives $\quad f = e^{ax^2+2bx+c}$,

$$(ab)^4(bc) = 0 \quad \text{gives} \quad u_4 + 6u_2^2 = \text{const.},$$

and thence
$$f = e^u = e^{ax+b}\sigma\{x+c,\ g_2,\ g_3\};$$

so the general σ function is obtained.

Other examples could be easily discussed, *e.g.*,

$$(ab)^2(bc)^2(ca)^2 = 0 \quad \text{gives} \quad f = e^{ax+b} + e^{cx+d},$$

$$(ab)^2(bc)^2(cd)^2 = 0 \quad \text{gives} \quad f = e^{ax+b}\sigma\{x+a,\ g_2,\ 0\}.$$

I propose to return subsequently to the interesting equation

$$(ab)^6 = 0.$$

ON A DEFICIENT MULTINOMIAL EXPANSION

By Major P. A. MACMAHON, R.A., Sc.D., F.R.S.

[Received November 23rd, 1904. — Read December 8th, 1904.]

ART. 1. In *Crelle*, t. I., p. 367, there is given a generalization of the binomial theorem which was restated by Cayley in 1851* in a better form. The statement is as follows:—

"If $\{x_1+x_2+x_3+\ldots\}^p$

denote the expansion of $(x_1+x_2+x_3+\ldots)^p$,

retaining those terms only $Nx_1^{a_1}x_2^{a_2}x_3^{a_3}\ldots$

in which
$$a_2+a_3+a_4+\ldots \not> p-1,$$
$$a_3+a_4+\ldots \not> p-2,$$
$$\ldots \quad \ldots \quad \ldots,$$

then
$$\begin{aligned} X^n = {} & 1 \times && (X+x_1)^n \\ & -\binom{n}{1}\times\{x_1\}^1 && (X+x_1+x_2)^{n-1} \\ & +\binom{n}{2}\times\{x_1+x_2\}^2 && (X+x_1+x_2+x_3)^{n-2} \\ & -\binom{n}{3}\times\{x_1+x_2+x_3\}^3 && (X+x_1+x_2+x_3+x_4)^{n-3} \\ & +\ldots\text{"} \end{aligned}$$

The formula is curious, and, as remarked by Cayley, of some interest by reason of the introduction of the deficient multinomial expansion
$$\{x_1+x_2+x_3+\ldots\}^p.$$
I propose to investigate properties of the expansion by the application of a method which has been found to be valuable in similar questions.

ART. 2. We may first enquire into the number of distinct terms in the development of
$$\{x_1+x_2+\ldots+x_p\}^q \quad (q \geqslant p).$$

* Cayley, *Collected Papers*, Vol. II., p. 102.

A term
$$N x_1^{a_1} x_2^{a_2} \dots x_p^{a_p}$$
arises when
$$\begin{aligned} a_2+a_3+\dots+a_p &\leqslant q-1, \\ a_3+\dots+a_p &\leqslant q-2, \\ \dots \quad & \quad \dots \\ a_p &\leqslant q-p+1, \end{aligned}$$
a number of Diophantine inequalities which may be replaced by the set
$$\begin{aligned} a_1 &\geqslant 1, \\ a_1+a_2 &\geqslant 2, \\ \dots \quad & \quad \dots \\ a_1+a_2+\dots+a_p &\geqslant p, \end{aligned}$$
and, of course,
$$a_1+a_2+\dots+a_p = q.$$

To form a generating function for the number in question consider the case $p = 3$ and $q \geqslant 3$,
$$\begin{aligned} a_1 &\geqslant 1, \\ a_1+a_2 &\geqslant 2, \\ a_1+a_2+a_3 &= q, \end{aligned}$$
and form the expression
$$\underset{\geqslant}{\Omega} \frac{\dfrac{1}{a_1 a_2^2 a_3^3}}{(1-a_1 a_2 a_3 x_1)(1-a_2 a_3 x_2)(1-a_3 x_3)},$$
where $\underset{\geqslant}{\Omega}$ signifies that, in the ascending expansion, all terms involving negative powers of the quantities a are to be rejected, and those quantities then put equal to unity.

From the construction it is clear that x_1 must occur at least once, for otherwise every term would be rejected by the operation Ω; similarly the exponents of x_1 and x_2 must together amount to at least two, and those of x_1, x_2, and x_3 to at least three. The development will contain every product
$$x_1^{a_1} x_2^{a_2} x_3^{a_3}$$
which satisfies the given conditions, and the expansion is therefore a representative generating function.

We pass to the enumerating generating function by putting

$$x_1 = x_2 = x_3 = x,$$

and thence, eliminating a_1, a_2, a_3 in succession, we find

$$\Omega \frac{\dfrac{1}{a_1 a_2^2 a_3^3}}{(1-a_1 a_2 a_3 x)(1-a_2 a_3 x)(1-a_3 x)}$$

$$= \Omega \frac{\dfrac{x}{a_2 a_3^2}}{(1-a_2 a_3 x)^2 (1-a_3 x)}$$

$$= \Omega \left\{ \frac{\dfrac{x^2}{a_3}}{(1-a_3 x)^3} + \frac{\dfrac{x^2}{a_3}}{(1-a_3 x)^2} \right\}$$

$$= \Omega \frac{x^2}{(1-x)^3} + \frac{x^2}{(1-x)^2} - 2x^2$$

$$= \frac{5x^3 - 6x^4 + 2x^5}{(1-x)^3}$$

$$= \Sigma \tfrac{1}{2} (q+2)(q-1)\, x^q \quad (q \geqslant 3).$$

It is thus established that the number of terms in

$$\{x_1 + x_2 + x_3\}^q$$

is
$$\tfrac{1}{2} (q+2)(q-1).$$

Passing to the general case, we have

$$\Omega \frac{\dfrac{1}{a_1 a_2^2 \dots a_p^p}}{(1-a_1 \dots a_p x)(1-a_2 \dots a_p x) \dots (1-a_{p-1} a_p x)(1-a_p x)}.$$

The reduction will be made by employing the formula

$$\binom{s}{1} + \binom{s+1}{2} x + \binom{s+2}{3} x^2 + \dots = \frac{1}{1-x} + \frac{1}{(1-x)^2} + \dots + \frac{1}{(1-x)^s}.$$

The expression is

$$\Omega \frac{\dfrac{x}{a_2 a_3^2 \dots a_p^{p-1}}}{(1-a_2 \dots a_p x)^2 (1-a_3 \dots a_p x) \dots (1-a_{p-1} a_p x)(1-a_p x)}$$

$$= \Omega \frac{\dfrac{x^2}{a_3 a_4^2 \dots a_p^{p-2}} (2 + 3a_3 \dots a_p x + 4a_3^2 \dots a_p^2 x^2 + \dots)}{(1-a_3 \dots a_p x)(1-a_4 \dots a_p x) \dots (1-a_p x)},$$

AUTHORS of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. *d.*	*s.* *d.*	*s.* *d.*
4 pages	Free	3 6	5 0	6 0
8 ,, ...	,,	4 6	6 6	8 0
12 ,, ...	,,	7 0	10 6	13 0
16 ,, ...	,,	8 0	12 0	15 0
32 ,, ...	,,	14 0	21 0	26 0

It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. FRANCIS HODGSON, 89 Farringdon Street, E.C., at the following rates:—

Vol.	I.	Nos.	1–8	... Jan.	1865	to Nov.	1866	...	10*s.*
,,	II.	,,	9–20	... Nov.	1866	,,	1869	...	16*s.*
,,	III.	,,	21–40	... ,,	1869	,,	1871	...	20*s.*
,,	IV.	,,	41–66	... ,,	1871	,,	1873	...	31*s.* 6*d.*
,,	V.	,,	67–75	... ,,	1873	,,	1874	...	15*s.*
,,	VI.	,,	76–88	... ,,	1874	,,	1875	...	21*s.*
,,	VII.	,,	89–103	... ,,	1875	,,	1876	...	21*s.*
,,	VIII.	,,	104–123	... ,,	1876	,,	1877	...	21*s.*
,,	IX.	,,	124–140	... ,,	1877	,,	1878	...	21*s.*
,,	X.	,,	141–155	... ,,	1878	,,	1879	...	18*s.*
,,	XI.	,,	156–165	... ,,	1879	,,	1880	...	12*s.* 6*d.*
,,	XII.	,,	166–179	... ,,	1880	,,	1881	...	16*s.*
,,	XIII.	,,	180–194	... ,,	1881	,,	1882	...	18*s.*
,,	XIV.	,,	195–214	... ,,	1882	,,	1883	...	20*s.*
,,	XV.	,,	215–232	... ,,	1883	,,	1884	...	20*s.*
,,	XVI.	,,	233–252	... ,,	1884	,,	1885	...	20*s.*
,,	XVII.	,,	253–279	... ,,	1885	,,	1886	...	25*s.*
,,	XVIII.	,,	280–304	... ,,	1886	,,	1887	...	25*s.*
,,	XIX.	,,	305–342	... ,,	1887	,,	1888	...	30*s.*
,,	XX.	,,	343–369	... ,,	1888	,,	1889	...	25*s.*
,,	XXI.	,,	370–398	... ,,	1889	,,	1890	...	25*s.*
,,	XXII.	,,	399–429	... ,,	1890	,,	1891	...	27*s.* 6*d.*
,,	XXIII.	,,	430–449	... ,,	1891	,,	1892	...	20*s.*
,,	XXIV.	,,	450–474	... ,,	1892	,,	1893	...	22*s.*
,,	XXV.	,,	475–499	... ,,	1893	,,	1894	...	22*s.*
,,	XXVI.	,,	500–534	... ,,	1894	,,	1895	...	30*s.*
,,	XXVII.	,,	535–574	... ,,	1895	,,	1896	...	35*s.*
,,	XXVIII.	,,	575–611	... ,,	1896	,,	1897	...	32*s.* 6*d.*
,,	XXIX.	,,	612–657	... ,,	1897	,,	1898	Pt. I. / Pt. II.	20*s.* / 20*s.*
,,	XXX.	,,	658–678	... ,,	1898	to Mar.	1899	...	20*s.*
,,	XXXI.	,,	679–709	... April	1899	,, Dec.	1899	...	20*s.*
,,	XXXII.	,,	710–737	... Jan.	1900	,, June	1900	...	20*s.*
,,	XXXIII.	,,	738–763	Nov.	1900	,, Feb.	1901	...	20*s.*
,,	XXXIV.	,,	764–789	... Mar.	1901	,, April	1902	...	20*s.*
,,	XXXV.	,,	790–819	... May	1902	,, Jan.	1903	...	20*s.*

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s.* each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25*s.* each Volume.

The price of cases for binding the Volumes of the Second Series is 1*s.* 3*d.*

Vol. 1 of Series 2 (1904) has been published.

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY.

SERIES 2.—VOL. 2.—PART 7.

CONTENTS.

Issued March 25, 1905.

LONDON:
PRINTED BY C. F. HODGSON & SON,
2 NEWTON STREET, HIGH HOLBORN, W.C.,
AND PUBLISHED FOR THE SOCIETY BY
FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

THE LONDON MATHEMATICAL SOCIETY is instituted for the promotion and extension of Mathematical Knowledge.

It was founded in 1865, and incorporated under Section 23 of the Companies Act 1867 in 1894.

Every Candidate for Membership must be proposed and recommended, according to a form, which the Secretaries will supply, by not less than three Members, of whom one at least, except in special cases to be submitted for the decision of the Council, must certify his personal knowledge of the Candidate.

This form is read at one of the Ordinary or Annual General Meetings, and the Candidate is balloted for at the next ensuing meeting, provided that seven Members are present thereat.

The Candidate, if elected, is informed of his election by one of the Secretaries, and supplied with a copy of the Memorandum and Articles of Association and By-Laws. He must pay the contribution which is due from him within six months after the day of his election, otherwise his election shall be void.

An entrance fee of one guinea is required to be paid by each newly elected Member.

The Annual Subscription to be paid by each Member is one guinea: any Member may compound for his *annual* subscriptions by the payment of ten guineas in one sum.

Every Member is considered liable for his annual subscription until he has signified in writing his desire to resign, and has returned all books and property belonging to the Society.

The affairs of the Society are directed by the Council and Officers.

The Council consists of sixteen Members, including the Officers, and is chosen from among the Ordinary Members of the Society at the Annual General Meeting, held on the second Thursday in November.

The Officers are a President, Vice-Presidents, a Treasurer, and Secretaries.

The Ordinary Meetings of the Society are held at its Rooms, 22 Albemarle Street, and commence at 5.30 o'clock in the evening. The dates of meeting for the year 1904 are the *second* Thursdays in January, February, March, April, May, June, November and December.

At these meetings papers are read and communications made: upon each paper or communication the Chairman invites discussion.

The Council alone decides whether any paper proposed for reading shall or shall not be read.

After a paper has been presented to the Society, it is referred by the Council to two or more Members, who report to the Council on its fitness for publication in the *Proceedings*. After hearing the reports, the Council decides by ballot whether it shall be printed or not.

Authors of Papers intended for communication to the Society are requested to furnish to the Secretaries short abstracts of their Papers, indicating the nature of the methods employed and the character of the results obtained.

Communications for the Secretaries may be forwarded to them at the following addresses:—

London Mathematical Society, 22 Albemarle Street, W. { A. E. H. Love. J. H. Grace.

34 St. Margaret's Road, Oxford.—A. E. H. Love.

Peterhouse, Cambridge.—J. H. Grace.

and by the help of the above formula this is

$$\Omega\,\frac{\dfrac{x^2}{a_3a_4^2\dots a_p^{p-2}}}{(1-a_3\dots a_px)^2(1-a_4\dots a_px)\dots(1-a_px)}$$

$$+\Omega\,\frac{\dfrac{x^2}{a_3a_4^2\dots a_p^{p-2}}}{(1-a_3\dots a_px)^3(1-a_4\dots a_p)\,x\dots(1-a_px)},$$

viz., after eliminating a_2 we have two fractions, and it may be readily seen that after elimination of a_s we will have a sum of s fractions.

Assume the result

$$\Omega\,\frac{\dfrac{(2s-4)!}{(s-2)!\,(s-1)!}\,\dfrac{x^{s-1}}{a_sa_{s+1}^2\dots a_p^{p-s+1}}}{(1-a_s\dots a_px)^2(1-a_{s+1}\dots a_px)\dots(1-a_px)}$$

$$+\Omega\,\frac{\dfrac{(2s-5)!}{(s-3)!\,(s-1)!}\,2\,\dfrac{x^{s-1}}{a_3a_{s+1}^2\dots a_p^{p-s+1}}}{(1-a_s\dots a_px)^3(1-a_{s+1}\dots a_px)\dots(1-a_px)}$$

$$+\Omega\,\frac{\dfrac{(2s-6)!}{(s-4)!\,(s-1)!}\,3\,\dfrac{x^{s-1}}{a_sa_{s+1}^2\dots a_p^{p-s+1}}}{(1-a_s\dots a_px)^4(1-a_{s+1}\dots a_px)\dots(1-a_px)}$$

$$+\dots$$

$$+\Omega\,\frac{\dfrac{x^{s-1}}{a_sa_{s+1}^2\dots a_p^{p-s+1}}}{(1-a_s\dots a_px)^s(1-a_{s+1}\dots a_px)\dots(1-a_px)}$$

to accrue after elimination of a_{s-1}.

It consists of $s-1$ fractions and agrees with the result obtained when $s=2$.

It will be shown that the result when a_s is eliminated may be obtained from the above by writing $s+1$ for s.

The series above may be written

$$\sum_{m=2}^{m=s}\Omega\,\frac{\dfrac{(2s-m-2)!}{(s-m)!\,(s-1)!}\,(m-1)\,\dfrac{x^{s-1}}{a_sa_{s+1}^2\dots a_p^{p-s+1}}}{(1-a_s\dots a_px)^m(1-a_{s+1}\dots a_px)\dots(1-a_px)}.$$

This general term becomes, on elimination of a_s,

$$\Omega\,\frac{\dfrac{(2s-m-2)!}{(s-m)!\,(s-1)!}\,(m-1)\left\{\binom{m}{1}+\binom{m+1}{2}\,a_{s+1}\dots a_p+\dots\right\}\dfrac{x^s}{a_{s+1}a_{s+2}^2\dots a_p^{p-s}}}{(1-a_{s+1}\dots a_px)(1-a_{s+2}\dots a_px)\dots(1-a_px)},$$

and this by the above quoted formula is

$$\sum_{t=2}^{t=m+1} \Omega \frac{\dfrac{(2s-m-2)!}{(s-m)!\,(s-1)!}(m-1)\dfrac{x^s}{a_{s+1}a_{s+2}^2\dots a_p^{p-s}}}{(1-a_{s+1}\dots a_p x)^t\,(1-a_{s+2}\dots a_p x)\dots(1-a_p x)}.$$

Summing this expression from $m=2$ to $m=s$, we find for the coefficient of the fraction whose denominator involves

$$(1-a_{s+1}\dots a_p x)^n$$

the sum

$$\frac{1}{s-1}\left\{(n-2)\binom{2s-n-1}{s-2}+(n-1)\binom{2s-n-2}{s-2}+\dots \text{ to } s-n+2 \text{ terms}\right\},$$

which by elementary algebra is

$$\frac{(2s-n)!}{(s-n+1)!\,s!}(n-1),$$

which is what the coefficient

$$\frac{(2s-n-2)!}{(s-n)!\,(s-1)!}(n-1),$$

which occurs in the assumed formula, becomes on writing $s+1$ for s. Hence the result is established.

Putting therein $s=p$, we obtain

$$\sum_{t=0}^{t=p-2} \Omega \frac{\dfrac{(2p-t-4)!}{(p-t-2)!\,(p-1)!}(t+1)\dfrac{x^{p-1}}{a_p}}{(1-a_p x)^{t+2}}$$

$$= \sum_{t=0}^{t=p-2} \frac{(2p-t-4)!}{(p-t-2)!\,(p-1)!}(t+1)\left\{\frac{1}{1-x}+\frac{1}{(1-x)^2}+\dots+\frac{1}{(1-x)^{t+2}}\right\}x^p$$

$$= \frac{(2p-2)!}{(p-1)!\,p!}\frac{x^p}{1-x}+2\frac{(2p-3)!}{(p-2)!\,p!}\frac{x^p}{(1-x)^2}+\dots+\frac{x^p}{(1-x)^p},$$

after some reductions.

Herein the coefficient of x^q is

$$\frac{(2p-2)!}{(p-1)!\,p!}+2(q-p+1)\frac{(2p-3)!}{(p-2)!\,p!}+3\binom{q-p+2}{2}\frac{(2p-4)!}{(p-3)!\,p!}+\dots$$

to p terms.

This sum by elementary algebra is

$$\frac{q-p+2}{p-1}\binom{q+p-1}{q+1} = (q-p+2)\frac{(q+p-1)!}{(q+1)!\,(p-1)!};$$

which is therefore the number of distinct terms in

$$\{x_1+x_2+\ldots+x_p\}^q \quad \text{for} \quad q \geqslant p.$$

Denoting the number in question by

$$H_{p,q}, \quad q \geqslant p,$$

it is easy to prove the formula

$$H_{p,q} = H_{p-1,p-1}+H_{p-1,p}+\ldots+H_{p-1,q}$$

by considering the coefficient of b^q in

$$\Omega \frac{\dfrac{1}{x_{p-1}^{p-1}x_{p-2}^{p-2}\ldots x_2^2 x_1}}{(1-bx_{p-1})(1-bx_{p-1}x_{p-2})\ldots(1-bx_{p-1}x_{p-2}\ldots x_2x_1)},$$

and then eliminating x_{p-1}.

Art. 3. Consider next the sum of the coefficients in

$$\{x_1+x_2+\ldots+x_p\}^q.$$

If $q = p$, we find from Cayley's formula, by putting

$$X = x_1 = x_2 = \ldots = x_p = 1,$$

$$1 = 2^n - \binom{n}{1} C_{1,1} 3^{n-1} + \binom{n}{2} C_{2,2} 4^{n-2} - \ldots + (-)^n \binom{n}{n} C_{n,n},$$

where $C_{p,p}$ is the sum of the coefficients in

$$\{x_1+x_2+\ldots+x_p\}^p.$$

From this relation it is easy to show that

$$C_{p,p} = (p+1)^{p-1}.$$

In general, consider the expression

$$\Omega \frac{(1+x_{p-1}+x_{p-1}x_{p-2}+\ldots+x_{p-1}x_{p-2}\ldots x_2x_1)^q}{x_{p-1}^{p-1}x_{p-2}^{p-2}\ldots x^2 x_1},$$

wherein Ω as usual operates by rejecting all terms in the development which involve negative powers of the quantities x, and subsequently replaces each of these quantities by unity.

The general term under the operator Ω is

$$\frac{q!}{a_1!\,a_2!\ldots a_{p-1}!}\,\frac{x_1^{a_1}x_2^{a_1+a_2}\ldots x_{p-1}^{a_1+a_2+\ldots+a_{p-1}}}{x_1x_2^2\ldots x_{p-1}^{p-1}},$$

and hence the conditions that the term may be integral are precisely those that define the deficient multinomial expansion.

Hence, $q \geqslant p$, the expression above written is equal to the sum of the coefficients in

$$\{x_1+x_2+\ldots+x_p\}^q,$$

which we will denote by $C_{p,q}$.

Write the expression

$$C_{p,q} = \Omega \frac{X_{p-1}^q}{X'_{p-1}} = \Omega \frac{(1+x_{p-1}X_{p-2})^q}{x_{p-1}^{p-1}X'_{p-2}}$$

$$= \binom{q}{p-1} \Omega \frac{X_{p-2}^{p-1}}{X'_{p-2}} + \binom{q}{p} \Omega \frac{X_{p-2}^{p}}{X'_{p-2}} + \ldots + \binom{q}{q} \Omega \frac{X_{p-2}^{q}}{X'_{p-2}}.$$

Hence the difference equation

$$C_{p,q} = \binom{q}{p-1} C_{p-1,p-1} + \binom{q}{p} C_{p-1,p} + \ldots + \binom{q}{q} C_{p-1,q},$$

from which the quantities $C_{p,q}$

can be calculated.

ART. 4. There is a syzygetic theory of the distinct terms in

$$\{x_1+x_2+\ldots+x_p\}^q, \quad q \geqslant p,$$

of which I give an illustration for $p = 3$. The sum of the terms

$$x_1^{a_1} x_2^{a_2} \ldots x_p^{a_p}$$

has been shown, for $p = 3$, to be

$$\Omega \frac{\dfrac{1}{a_1 a_2^2 a_3^3}}{(1-a_1a_2a_3x_1)(1-a_2a_3x_2)(1-a_3x_3)}$$

which without difficulty may be given the expression

$$\frac{x_1x_2x_3+x_1x_2^2+x_1^2x_2+x_1^2x_3+x_1^3-(x_1x_2^2x_3+2x_1^2x_2x_3+x_1^2x_2^2+x_1^3x_2+x_1^3x_3) + x_1^2x_2^2x_3+x_1^3x_2x_3}{(1-x_1)(1-x_2)(1-x_3)},$$

which shows that every term $x_1^{a_1}x_2^{a_2}x_3^{a_3}$

contains one of the products

$$x_1x_2x_3, \quad x_1x_2^2, \quad x_1^2x_2, \quad x_1^2x_3, \quad x_1^3,$$

which may be called ground products. The other products are formed by

multiplying these into any power products of x_1, x_2, x_3, but the numerator indicates by the terms

$$x_1x_2^2x_3, \quad 2x_1^2x_2x_3, \quad x_1^2x_2^2, \quad x_1^3x_2, \quad x_1^3x_3$$

six syzygies of the first order, viz.,

$$\begin{aligned}
A &= x_3(x_1x_2^2)-x_2(x_1x_2x_3) = 0,\\
B &= x_3(x_1^2x_2)-x_1(x_1x_2x_3) = 0,\\
C &= x_2(x_1^2x_3)-x_1(x_1x_2x_3) = 0,\\
D &= x_1(x_1x_2^2)-x_2(x_1^2x_2) \quad = 0,\\
E &= x_2(x_1^3) \quad -x_1(x_1^2x_2) \quad = 0,\\
F &= x_1(x_1^2x_3)-x_3(x_1^3) \quad = 0,
\end{aligned}$$

and by the terms $\quad x_1^2x_2^2x_3, \quad x_1^3x_2x_3$

two syzygies of the second order, viz.,

$$x_1A-x_2C = x_1C-x_3E = 0.$$

INDEX.

INDEX OF PROCEEDINGS.

[*The numbers refer to pages.*]

INDEX OF PAPERS READ AND COMMUNICATIONS MADE DURING THE SESSION NOVEMBER, 1903—JUNE, 1904.

[*The names of Authors are given and the numbers of the pages on which the titles of the papers, &c., will be found.*]

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[*The numbers refer to pages.*]

INDEX ACCORDING TO SUBJECTS.

[*The numbers refer to pages.*]

Printed by C. F. Hodgson & Son, 2 Newton Street, High Holborn, W.C.

AUTHORS of Papers printed in the *Proceedings* are entitled to receive, free of cost, 25 separate copies of their communications, stitched in a paper cover. They can, however, by previous notice to the Printers, obtain as many copies as they wish, at the following rates:—

Of a Paper not exceeding	25 copies.	50 copies.	75 copies.	100 copies.
		s. *d.*	*s.* *d.*	*s.* *d.*
4 pages	Free	3 6	5 0	6 0
8 "	"	4 6	6 6	8 0
12 "	"	7 0	10 6	13 0
16 "	"	8 0	12 0	15 0
32 "	"	14 0	21 0	26 0

It was ordered by the Council (Feb. 13, 1896), "That separate parts of the *Proceedings* may be obtained from the Publisher at 1*s.* 6*d.* per sheet, when the number in stock exceeds 100."

"That Members be allowed to complete Sets at trade price" [*i.e.*, at 1*s.* 4*d.* per sheet].

It was further ordered by the Council (March 9, 1899), "That the Society allow 5 per cent. extra discount to each purchaser of a *complete* set of *Proceedings up to the last completed volume.*"

Complete Volumes of the *Proceedings*, bound in cloth, 8vo, can be purchased by the public from the Publisher, Mr. FRANCIS HODGSON, 89 Farringdon Street, E.C., at the following rates:—

Vol.	I.	Nos.	1-8	... Jan. 1865	to Nov. 1866	...	10*s.*
"	II.	"	9-20	... Nov. 1866	" 1869	...	16*s.*
"	III.	"	21-40	... " 1869	" 1871	...	20*s.*
"	IV.	"	41-66	... " 1871	" 1873	...	31*s.* 6*d.*
"	V.	"	67-75	... " 1873	" 1874	...	15*s.*
"	VI.	"	76-88	... " 1874	" 1875	...	21*s.*
"	VII.	"	89-103	... " 1875	" 1876	...	21*s.*
"	VIII.	"	104-123	... " 1876	" 1877	...	21*s.*
"	IX.	"	124-140	... " 1877	" 1878	...	21*s.*
"	X.	"	141-155	... " 1878	" 1879	...	18*s.*
"	XI.	"	156-165	... " 1879	" 1880	...	12*s.* 6*d.*
"	XII.	"	166-179	... " 1880	" 1881	...	16*s.*
"	XIII.	"	180-194	... " 1881	" 1882	...	18*s.*
"	XIV.	"	195-214	... " 1882	" 1883	...	20*s.*
"	XV.	"	215-232	... " 1883	" 1884	...	20*s.*
"	XVI.	"	233-252	... " 1884	" 1885	...	20*s.*
"	XVII.	"	253-279	... " 1885	" 1886	...	25*s.*
"	XVIII.	"	280-304	... " 1886	" 1887	...	25*s.*
"	XIX.	"	305-342	... " 1887	" 1888	...	30*s.*
"	XX.	"	343-369	... " 1888	" 1889	...	25*s.*
"	XXI.	"	370-398	... " 1889	" 1890	...	25*s.*
"	XXII.	"	399-429	... " 1890	" 1891	...	27*s.* 6*d.*
"	XXIII.	"	430-449	... " 1891	" 1892	...	20*s.*
"	XXIV.	"	450-474	... " 1892	" 1893	...	22*s.*
"	XXV.	"	475-499	... " 1893	" 1894	...	22*s.*
"	XXVI.	"	500-534	... " 1894	" 1895	...	30*s.*
"	XXVII.	"	535-574	... " 1895	" 1896	...	35*s.*
"	XXVIII.	"	575-611	... " 1896	" 1897	...	32*s.* 6*d.*
"	XXIX.	"	612-657	... " 1897	" 1898	Pt. I. Pt. II.	20*s.* 20*s.*
"	XXX.	"	658-678	... " 1898	to Mar. 1899	...	20*s.*
"	XXXI.	"	679-709	... April 1899	" Dec. 1899	...	20*s.*
"	XXXII.	"	710-737	... Jan. 1900	" June 1900	...	20*s.*
"	XXXIII.	"	738-763	... Nov. 1900	" Feb. 1901	...	20*s.*
"	XXXIV.	"	764-789	... Mar. 1901	" April 1902	...	20*s.*
"	XXXV.	"	790-819	... May 1902	" Jan. 1903	...	20*s.*

Cases for binding the above Volumes can also be procured from the Publisher, price 1*s.* each.

A complete Index of all the papers printed in Vols. I. to XXX. of the *Proceedings* of the Society (112 pp.) and a List of Members of the Society from the foundation (1865) to Nov. 9th, 1899 (16 pp.), can be obtained gratis from the Society's Publisher, at the above address.

The price of the Second Series is 25*s.* each Volume.

The price of cases for binding the Volumes of the Second Series is 1*s.* 3*d.*

Vol. 1 of Series 2 (1904) has been published.